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A. D. Alexandrov's Uniqueness Theorem for Convex Polytopes and its Refinements

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Abstract. In 1937, A. D. Alexandrov proved that if no parallel faces of two 3-dimensional convex polytopes can be placed strictly one into another via a translation, then the polytopes are translates of one another.
The theory of hyperbolic virtual polytopes elucidates this theorem and suggests natural ways of its refinement.
Namely, we present an example of two different 3-dimensional polytopes such that, for each pair of their parallel faces, there exists at most one translation placing one of the faces into another.
Another refinement: given two polytopes, if for any pair of parallel faces, there exists at most one translation placing the face of the first polytope strictly in the face of the second one, and there exists no translation placing the face of the second polytope strictly in the face of the first one, then the polytopes are translates of one another.
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1. Introduction

In 1939, A. D. Alexandrov formulated the following uniqueness conjecture and proved it for analytic surfaces.

Uniqueness conjecture for smooth convex surfaces. [2] *Let* $K_1 \in \mathbb{R}^3$ *and* $K_2 \in \mathbb{R}^3$ *be smooth closed convex surfaces of positive curvature. If the second differential of the function* $H = H_1 - H_2$ *is an alternating or zero form, then the surfaces* F_1 *and* F_2 *are translates of each other. (*H_1 *and* H_2 *are the support functions of* K_1 *and* K_2*.)*

In the same paper, he claims that there is an analogous assertion for convex polytopes:

Theorem 1.1. (Uniqueness theorem for convex polytopes [1]) *Let* K, M *be* 3-*dimensional convex polytopes. If, for any pair of their parallel faces, no face can be placed strictly into another via a translation, then the polytopes coincide up to a translation.* □

Much later, it turned out that the above conjecture for smooth surfaces is wrong. The first counterexample was presented by Y. Martinez-Maure in 2001 [5]. Some later, the author of the paper presented a series of new counterexamples based on the theory of hyperbolic virtual polytopes [9], [10].

In the paper, we show that Theorem 1.1 admits a natural interpretation in terms of the theory of hyperbolic polytopes. Moreover, a natural question arises in this framework: what happens if we replace the condition of Theorem 1.1 by a milder one. Namely, we allow at most one translation which places one of the parallel faces into another (such a translation is called a *rigid insertion*).

Theorem 5.1 (a positive-type refinement) shows that under this condition, the polytopes K and L do not necessarily coincide. We present such an example of two different polytopes, each of them with 56 faces.

The example is far from trivial. For its construction, we need a hyperbolic polytope H with the following additional property: the fan of H admits a regular triangulation without Steiner points.

It seems that none of hyperbolic polytopes known before (see [6], [9], [10]) possesses this property, so we need some advanced technique to construct hyperbolic polytopes.

To construct such a polytope H, we first construct the dual object, namely, the graph of its support function which is a (spherically) saddle surface in the 3-dimensional sphere, spanned by some special linkage of 8 great semicircles.

Theorem 6.1 (a negative-type refinement) asserts that, if we allow only one-side rigid insertions, then the polytopes are translates of one another.

At the end of the paper, we formulate two open problems.

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2. Virtual polytopes, hyperbolic polytopes

Virtual polytopes (introduced by A. Pukhlikov and A. Khovanskii [4], appeared also in a natural way in P. McMullen's polytope algebra [7]) can be represented in more than four different but equivalent ways.

In the paper, we make use of the first three of them.

- Virtual polytopes are elements of the Grothendieck group of the semigroup of convex polytopes $\mathcal{P}$ equipped with the Minkowski addition $\otimes$; i.e., they are formal expressions of type $K \otimes L^{-1}$, where $K, L \in \mathcal{P}$.
- Virtual polytopes are *polytopal functions* [4], [7], i.e., finite linear combinations of indicator functions of convex polytopes. So it makes sense to speak of the *value* of a polytope K at a point $x \in \mathbb{R}^n$.
- Virtual polytopes are defined by their *support functions*, i.e., piecewise linear positively homogeneous functions defined on $\mathbb{R}^n$ (see [4]).
- A virtual polytope is a pair of type (a closed polytopal surface in $\mathbb{R}^n$ with cooriented facets; a spherical fan) (see [9], [10]).

Now we give detailed explanations of the items, bounding from now on by dimension 3.

Denote by $\mathcal{P}$ the set of all compact convex polytopes in $\mathbb{R}^3$ (degenerate polytopes are also included). $\mathcal{P}$ is a semigroup with respect to the Minkowski addition $\otimes$.

Denote by $\mathcal{P}^*$ the Grothendieck group of $\mathcal{P}$. The element of $\mathcal{P}^*$ that is inverse to K is denoted by K^{-1}.

A function $F : \mathbb{R}^3 \to \mathbb{Z}$ is *polytopal* if it admits a representation of the form

$$F = \sum_i a_i I_{K_i},$$

where $a_i \in \mathbb{Z}$, $K_i \in \mathcal{P}$, and I_{K_i} is the indicator function of the polytope K_i:

$$I_{K_i}(x) = \begin{cases} 1 & \text{if } x \in K_i, \\ 0 & \text{otherwise.} \end{cases}$$

The *affine hull aff*(F) of a polytopal function F is the affine hull of the support of F. The *dimension* $\dim(F)$ of a polytopal function F is the dimension of its affine hull. The set of all polytopal functions $\mathcal{M}$ is endowed with two ring operations. The role of addition is played by the pointwise addition, denoted by $+$. The multiplication is generated by $\otimes$ and is denoted by the same symbol.

The unit element of the ring $\mathcal{M}$ is obviously the function $E = I_{\{O\}}$.

Identifying convex compact polytopes with their indicator functions, we get an inclusion $\pi : \mathcal{P} \subset \mathcal{M}$. Keeping this identification in mind, we write K instead of I_K for convenience.

Due to the following theorem, all elements of the semigroup $\pi(\mathcal{P})$ are invertible in $\mathcal{M}$.

Theorem 2.1. (On Minkowski inversion [4]) *For any convex polytope K, we have*

$$(-1)^{\dim K} I_{Relint(sK)} \otimes K = E,$$

where s is the central symmetry mapping (with respect to the origin O), Relint(sK) is the relative interior of the polytope sK (i.e., the interior taken in the affine hull of K). □

Hence the inclusion $\pi : \mathcal{P} \subset \mathcal{M}$ induces an inclusion $\mathcal{P}^* \subset \mathcal{M}$.

Definition 2.2. *The image of the latter inclusion is called the group of virtual polytopes. We denote it by the same letter $\mathcal{P}^*$ for convenience.*

Definition 2.3. *Let $K = L \otimes M^{-1}$ be a virtual polytope. The support function h_K of K is defined to be the pointwise difference of the support functions of L and M:*

$$h_K = h_L - h_M.$$

Remark 2.4. Similarly to the convex case, the support function of a virtual polytope is piecewise linear with respect to some fan. By a *fan* we mean (as usual) a splitting of $\mathbb{R}^3$ in a union of polytopal cones with the common apex at O. But in the sequel, we sometimes speak of (and draw) the intersection of the fan with the unit sphere S^2 centered at O. Thus the cones correspond to the spherical polytopes (spherical cells).

Definition 2.5. [8] *Let $K = \sum_i a_i K_i$ with $K_i \in \mathcal{P}$. Let $e_i(\xi)$ be a support plane to K_i with the outer normal vector ξ. The polytope $K_i^\xi = K_i \cap e_i(\xi)$ is called the face of the polytope K_i with the normal vector ξ, while the polytopal function $K^\xi = \sum_i a_i K_i^\xi$ is called the face of the polytopal function K with the normal vector ξ.*

A face of a virtual polytope is a virtual polytope as well. The 0-dimensional, 1-dimensional and 2-dimensional faces are called *vertices, edges and facets* respectively.

Similarly to faces of convex polytopes, virtual faces behave linearly with respect to the Minkowski addition:

Theorem 2.6. [8] *In the above notation,*

$$K_1^\xi \otimes K_2^\xi = (K_1 \otimes K_2)^\xi. \qquad \square$$

Definition 2.7. [10] *A point X is called a boundary point of a virtual polytope K, if $x \in cl(supp(K^\xi))$ for some $\xi \in S^2$ such that ξ is not orthogonal to aff(K). (cl denotes the closure.)*

Theorem 2.8. [4] *For two convex polytopes K and L, and a point $x \in \mathbb{R}^3$,*

$$K \otimes L^{-1}(x) = \chi(K \cap t_x(Relint L))(-1)^{\dim L},$$

where χ stands for the Euler characteristic, t_x is the translation by x. $\square$

Corollary 2.9. *Let M_1, M_2 be some convex polygons lying in a plane. Put $M = M_1 \otimes M_2^{-1}$ (it is a virtual polygon, i.e., a 2-dimensional virtual polytope.) The following two assertions are valid :*

(1) *M admits a positive value at a point x if and only if $t_x M_2 \subset M_1$ or $M_1 \subset Relint(t_x M_1)$.*

(2) *Assume that M admits positive values only at its boundary points. If M has no parallel edges (i.e.,* $\dim M^\xi = 1 \Rightarrow \dim M^{-\xi} = 0$*), then it admits a positive value at no more than one point.*

Proof. (1) follows easily from Theorem 2.8.

(2) Suppose that $M(x) = M(y) = 1$ for $x \neq y$. Then for each point v of the segment $[xy]$, we have $M(v)$=1. Indeed, the inclusions $t_x M_2 \subset M_1$ and $t_y M_2 \subset M_1$ imply $t_v M_2 \subset M_1$. This means that the segment $[xy]$ lies on edges M^ξ and $M^{-\xi}$, where ξ is orthogonal to $[xy]$. A contradiction. □

The fan of a virtual polytope is defined below analogously to the classical definition of the outer normal fan.

Definition 2.10. *For a virtual polytope $K \in \mathcal{P}^*$, its fan Σ_K is the collection of sets $\{\Sigma_K(\nu)\}$, where ν ranges over the set of faces of K, and*

$$\Sigma_K(\nu) = cl(\{\xi | K^\xi = \nu\}).$$

These polytopal sets (all of them are finite unions of some spherical polytopes) are called *cells* of a fan. Similarly to the convex case, the support function of K is linear on each cell of Σ_K. Moreover, the fan of a virtual polytope K can be defined as the minimal fan for which h_K is linear on each cell. In addition, we have the usual combinatorial duality: k-dimensional cells of Σ_K correspond to $(3-k-1)$-dimensional faces of K.

The 0-dimensional cells are called the *vertices* of the fan.

3. Spherical graph of support function

It makes sense to draw the graph of the support function of a virtual polytope on the 3-dimensional sphere. Fix an embedding of the 3-dimensional real space $\mathbb{R}^3$ in $\mathbb{R}^4$. The unit sphere centered at O in $\mathbb{R}^3$ (respectively, in $\mathbb{R}^4$) is denoted by S^2 (respectively, by S^3). Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a positively homogeneous continuous function. For a point $\xi \in S^2$, denote by $e(\xi)$ the 2-plane in $\mathbb{R}^3$ tangent to S^2 at ξ. Denote by $h|_e$ the restriction of h on the plane $e = e(\xi)$.

Consider the *affine graph* of the restriction $h|_e$, namely,

$$\Gamma_{\mathit{aff}}(h, e) := \{(x, y, z, t) \in \mathbb{R}^4 \mid (x, y, z) \in e;\ t = h(x, y, z)\}$$

and its image $\Gamma_{\mathit{sph}}(h, e)$ on S^3 under the central projection ϕ with the center O (see Figure 3.1).

The union of all these images $\Gamma_{\mathit{sph}}(h) := \cup_{\xi \in S^2} \Gamma_{\mathit{sph}}(h, e(\xi))$ is called the *spherical graph* of the function h.

It is a 2-dimensional submanifold of S^3. The spherical central projection $\pi : S^3 \setminus \{(0,0,0,1), (0,0,0,-1)\} \to S^2$ maps $\Gamma_{\mathit{sph}}(h)$ one-to-one on S^2.

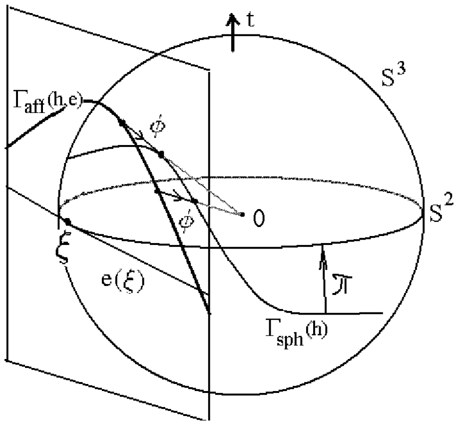


Figure 3.1

Definition 3.2. [3] *A surface $F \subset \mathbb{R}^3$ is called a saddle surface if there is no plane cutting a bounded connected component off F.*

Equivalently, a surface F is saddle if no plane intersects F locally at just one point.

A surface $F \subset S^3$ is called a spherically saddle surface if no great 2-dimensional sphere intersects F locally at just one point.

A surface $F \subset S^3$ is called spherically convex if each its chord lies (non-strictly) above the surface ("above" refers to the direction of t-axes).

Definition 3.3. [9] *A virtual polytope K is called hyperbolic if $\Gamma_{aff}(h, e(\xi))$ is a saddle surface for every $\xi \in S^2$. In the sequel, we call such virtual polytopes for short hyperbolic polytopes.*

Theorem 3.4. [10] *K is a hyperbolic polytope if and only if all non-boundary values of its facets are non-positive.* □

Proposition 3.5. *In the above notation,*

- *h is a convex function if and only if its spherical graph $\Gamma_{sph}(h)$ is a spherically convex surface;*
- *h is a linear function (i.e., the support function of a point) if and only if $\Gamma_{sph}(h)$ is a great 2-dimensional sphere;*
- *h is the support function of a hyperbolic polytope if and only if $\Gamma_{sph}(h)$ is a spherically saddle piecewise linear surface.*

Proof. It suffices to observe that the central projection ϕ does not change the convexity type because it maps planes to great 2-dimensional spheres on S^3. □

4. A. D. Alexandrov's theorem from the viewpoint of hyperbolic polytopes

Consider two conditions for 3-dimensional convex polytopes K and L.

(∗) For each $\xi \in S^2$ such that either $\dim(K^\xi) = 2$ or $\dim(L^\xi) = 2$, there exists no translation t such that either $tK^\xi \subset L^\xi, tK^\xi \neq L^\xi$ or $tK^\xi \supset L^\xi, tK^\xi \neq L^\xi$.

(∗∗) For each $\xi \in S^2$ such that either $\dim(K^\xi) = 2$ or $\dim(L^\xi) = 2$, there exists at most one translation t such that either $tK^\xi \subset L^\xi, tK^\xi \neq L^\xi$ or $tK^\xi \supset L^\xi, tK^\xi \neq L^\xi$. (If such a translation exists, we say that K^ξ can be *rigidly inserted* in L^ξ.)

Theorem 4.1. (A. D. Alexandrov [1]) *The condition* (∗) *implies that the polytopes K and L are translates of one another.* □

The conditions can be reformulated in terms of hyperbolic polytopes.

Proposition 4.2. (1) *Polytopes K and L satisfy the condition* (∗) *if and only if the following two conditions are valid.*

- *The virtual polytope $H = K \otimes L^{-1}$ is hyperbolic.*
- *None of its 2-dimensional faces H^ξ and their inverses $(H^\xi)^{-1}$ (considered as the polytopal functions) admit positive values.*

(2) *If polytopes K and L satisfy the condition* (∗∗), *we have*

- *The virtual polytope $H = K \otimes L^{-1}$ is hyperbolic.*
- *The edges of the fan Σ_H and edges of the fan Σ_L have no common inner points.*
- *No vertex of Σ_L is an inner point of an edge of Σ_H.*
- *A vertex ξ of Σ_H is an inner point of an edge of Σ_L implies* $\dim K^\xi = \dim L^\xi = 1$.

Proof. (1) and the first assertion of (2) follow directly from Theorem 3.4 and Corollary 2.9.

Suppose an edge of Σ_H and an edge of Σ_L have a common inner point ξ. This means that $K^\xi = H^\xi \otimes L^\xi$ is a parallelogram, whereas L^ξ equals one of its edges. A contradiction to (∗∗).

Let ξ be a vertex of Σ_L which is an inner point of an edge of Σ_H. Then $\dim L^\xi = 2$, and $K^\xi \otimes (L^\xi)^{-1}$ is a virtual segment. It means that either $K^\xi \otimes (L^\xi)^{-1}$ or $L^\xi \otimes (K^\xi)^{-1}$ is a convex segment, and therefore admits positive values at all its points. A contradiction. □

5. The main example (a positive-type refinement)

Theorem 5.1. *There exist two different 3-dimensional convex polytopes K, L satisfying the condition* (∗∗).

Proof. Consider the polytopes H and L from the below Lemma 5.2. We can assume that the Minkowski sum $K = H \otimes L$ is convex. (If it is not, replace L by CL for a sufficiently big constant C.) Show that the pair K, L satisfies the condition (∗∗).

By Lemma 5.2 (1), $\dim K^\xi = 2 \Leftrightarrow \dim H^\xi = 2 \Leftrightarrow \dim L^\xi = 2$.

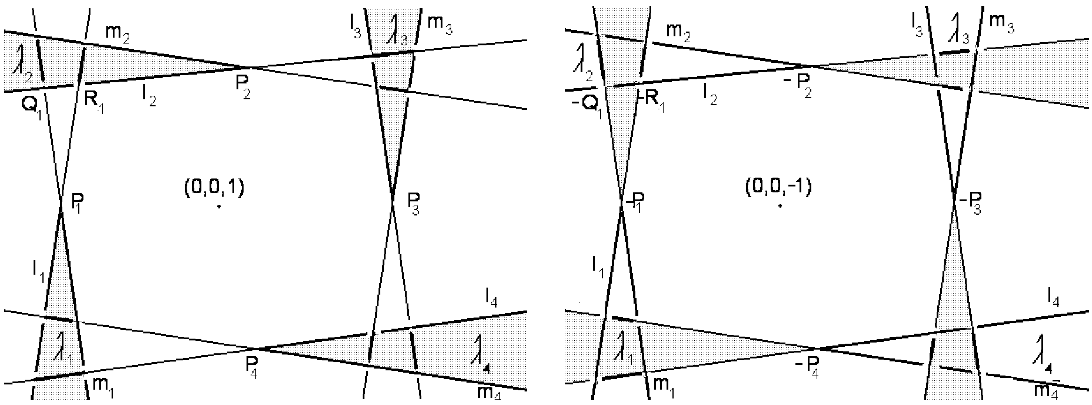
H is a hyperbolic polytope and its facets have no parallel edges. For each ξ, the polytopal functions H^ξ and $(H^\xi)^{-1}$ admit positive values at most at one point (Theorem 3.4 and Corollary 2.9). This means (Corollary 2.9) that there exists at most one translation which places one of the polygons K^ξ, L^ξ into another. $\square$

Lemma 5.2. *There exist a hyperbolic polytope H and a convex polytope L such that*

1. *The fan Σ_L is a refinement of Σ_H without Steiner points. (I.e., Σ_L refines regularly Σ_H without adding new vertices.)*
2. *Facets of H have no parallel edges.*

Proof of the lemma. We shall construct the spherical graph of the support function h_H, keeping in mind the construction from Section 3.

Step 1. Fix the positive and negative hemispheres $S^2_\pm$ with the poles $P = (0, 0, 1)$ and $-P = (0, 0, -1)$. Consider eight geodesic lines (i.e., great circles) in S^3 forming a linkage as is shown in Figure 5.3. This means that each pair of lines l_i, m_i has two common points P_i and $-P_i$. No other pair of lines has intersections. Figure 5.3 depicts the planar diagram of the linkage (i.e., its images under the projection π on the positive and negative hemispheres with indicated "passes"). In particular, the line l_1 passes over l_2 above S^2_+, the line l_1 passes under m_2 above S^2_+ ("under" and "over" refer to the direction of the t-axes). Denote by λ_i the spherical 2-gon with edges lying on l_i and m_i, assuming that its image is marked grey in Figure 5.3. Each of these 2-gons has two vertices, namely, P_i and $-P_i$. The 2-gons $\Lambda_i = \pi(\lambda_i)$ form a disconnected polytopal complex Λ embedded in S^2.



Projection of the linkage on S^2_+ Projection of the linkage on S^2_-

Figure 5.3

For $i = 1, \dots, 4$, let

$$Q_i \in l_{i+1}, \quad \pi(Q_i) \in \pi(m_i) \cap S^2_+;$$

$$R_i \in l_{i+1}, \quad \pi(Q_i) \in \pi(l_i) \cap S^2_+.$$

(We assume here that $l_5 = l_1, m_5 = m_1$.)

The tiling θ of S^2 generated by the collection of lines $\pi(m_i)$ and $\pi(l_i)$ is obviously regular. Each of its vertices lies on the boundary of Λ_i for some i. Fix its regular triangulation Θ.

Step 2. Next, move somewhat the vertices of the triangulation Θ (and denote the new points by the same letters with primes) to satisfy the following conditions:

- All the movements are small.
- The new triangulation Θ' of S^2 is regular.
- Each of Λ_i is replaced by a spherical polytope Λ' bounded by two convex broken lines (see Figure 5.4). The lines are broken at each vertex of the triangulation.
- No two edges of Θ' adjacent to a vertex are parallel. (This is to satisfy the condition (2) of Lemma 5.2.)

Apply the synchronized changes to λ_i. Namely, let λ_i' be 2-dimensional spherical polygons lying close to λ_i such that $\pi(\lambda_i') = \Lambda_i'$. In addition, we claim that the prolongations of the edges of λ_i' adjacent to P_i' (and $(-P_i)'$) and the boundary (broken) lines of λ_i' form the same linkage as the original lines l_i, m_i.

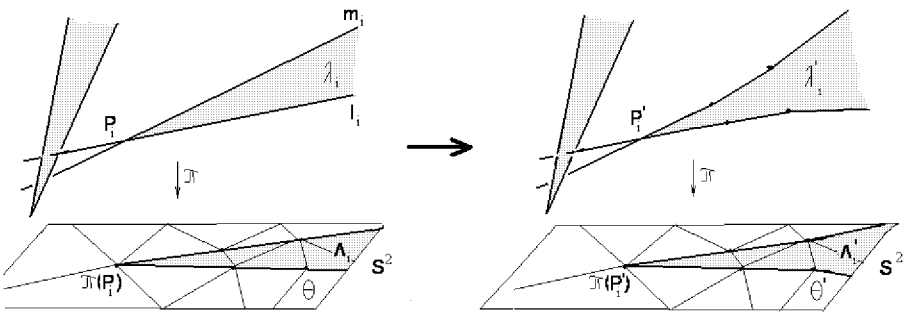


Figure 5.4

Step 3. There exists a unique piecewise linear function h such that

1. The function h is linear on each triangle of Θ' and on each Λ_i' (to be precise, h is linear on each cone in $\mathbb{R}^3$ based on these spherical polygons, see Remark 2.4).

2. The polytopes λ_i' lie on its spherical graph $\Gamma_{sph}(h)$.

Prove that the surface $\Gamma_{sph}(h)$ is saddle at each vertex A.

If A is not one of P_i' or $(-P_i)'$, it is a vertex of λ_i' for some i, and the angle of λ_i' at A is greater than π. This means the required.

Consider now the vertex P_i'. By construction, the surface in question contains four segments with an endpoint at P_i': the two adjacent edges of λ_i' and the segments $P_i'Q_i'$ and $P_i'R_i'$. Due to the linking type, each hemisphere whose boundary passes through P_i' contains at least one of the segments. Therefore P_i' is a saddle vertex as well. The vertices $(-P_i)'$ are treated similarly.

Step 4. Let H be the virtual polytope with the support function h. Let L be a convex polytope with the fan generated by the triangulation θ. □

Remark 5.4. Each of the polytopes K and L has 56 faces. The faces corresponds to pairwise intersection points of the eight lines $\pi(l_i)$ and $\pi(m_i)$. For $\xi = \pi(P_i')$ or $\pi((-P_i)')$, the faces K^ξ and L^ξ are noninsertable in each other. The other 48 pairs of parallel faces admit rigid insertions.

6. A negative-type refinement and some open problems

We show that there is no an analogous example with one-side rigid insertions. Namely, the following theorem holds.

Theorem 6.1. *Let K, M be 3-dimensional convex polytopes. Suppose that for each $\xi \in S^2$ such that either $\dim(K^\xi) = 2$ or $\dim(L^\xi) = 2$, the two assertions are valid:*

1. *There exists at most one translation t such that $tK^\xi \subset L^\xi, tK^\xi \neq L^\xi$.*
2. *There exists no translation t such that $tL^\xi \subset K^\xi, tL^\xi \neq K^\xi$.*

Then the polytopes coincide up to a translation.

Prove the following lemma first.

Lemma 6.2. *Let K be a virtual polytope, $\Gamma = \Gamma_{sph}(h_K)$ be the spherical graph of its support function, $\overline{\Gamma}$ be its subgraph.*

For a point $\xi \in S^2$ and a point E in aff(K^ξ), we have

$$K^\xi(E) = 1 + \chi(e \cap \overline{\Gamma} \cap U(\Xi)),$$

where χ stands for the Euler characteristic; Ξ is the point on Γ such that $\pi(\Xi) = \xi$; e is the spherical graph of the point E (i.e., a 2-dimensional sphere on S^3); $U(\Xi) = \{\eta \in S^3 |\ 0 \neq |\eta, \Xi| < \epsilon\}$ is a deleted neighborhood of Ξ.

Proof of the lemma. Without loss of generality we may assume that $\dim K = 2$ and $K = K^\xi$. Indeed, the spherical graphs of K and K^ξ coincide in a neighborhood of Ξ.

$$-\chi(e \cap \overline{\Gamma} \cap U(\Xi))$$

by duality,

$$= (\#\{l | l \subset \textit{aff}\,(K) \textit{ is an oriented line, } E \in l;\ l \textit{ is a support line to } K\})/2$$

$$= 1 - K(E).$$

The latter equality is well-known for convex polygons. Owing to linearity, it also is valid for virtual polygons. □

Proof of Theorem 6.1. Suppose the contrary and consider the hyperbolic polytope $H = K \otimes L^{-1}$. It is proved in [10] that the fan of a hyperbolic polytope has at least 4 cells (say, $\alpha_1, \ldots, \alpha_4$) possessing the following property. Denote by (α) the great sphere in S^3 containing the face of $\Gamma_{sph}(h_H)$ which corresponds to one of these cells. The spherical polytope α is bounded by two convex (piecewise geodesic) broken lines (say, $L_1$and L_2) such that their convexity directions look like in Figure 6.3.

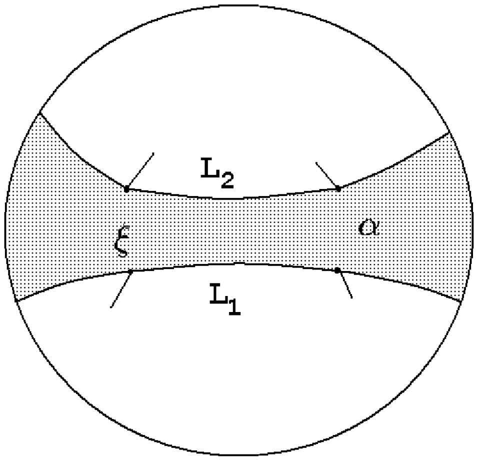


Figure 6.3

Each of L_1, L_2 has inner vertices. Indeed, if L_1 has no inner vertices, it is a geodesic segment longer or equal than π. This means that Σ_H has no regular refinement without Steiner points.

Near inner points of one of these lines (say, L_1), the graph $\Gamma_{sph}(h_H)$ lies (non-strictly) upon (α), whereas in a neighborhood of any inner point of the other line, the graph $\Gamma_{sph}(h_H)$ lies below (α).

By Lemma 6.2, for any inner vertex ξ of the line L_1, $H^\xi(A) = 1$. Therefore, the facet L^ξ can be inserted in K^ξ. □

Open problems

Problem 1 What is the minimal number of facets of two polytopes satisfying the condition $(**)$?

Problem 2 Do there exist 3-dimensional polytopes K and L satisfying the following intermediate condition?
$(***)$ For each $\xi \in S^2$, such that either $\dim(K^\xi) = 2$ or $\dim(L^\xi) = 2$, there exists a unique translation t such that either $tK^\xi \subset L^\xi, tK^\xi \neq L^\xi$ or $tK^\xi \supset L^\xi, tK^\xi \neq L^\xi$.

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