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Helly-type Theorems for Infinite and for Finite Intersections of Sets Starshaped via Staircase Paths

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Abstract. Let d be a fixed integer, $0 \leq d \leq 2$, and let $\mathcal{K}$ be a family of simply connected sets in the plane. For every countable subfamily $\{K_n : n \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_n : n \geq 1\}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cap\{K : K \text{ in } \mathcal{K}\}$ has these properties as well. For the finite case, define function f on $\{0,1\}$ by $f(0) = 3, f(1) = 4$. Let $\mathcal{K} = \{K_i : 1 \leq i \leq n\}$ be a finite family of compact sets in the plane, each having connected complement. For d fixed, $d \,\epsilon\, \{0,1\}$, and for every $f(d)$ members of $\mathcal{K}$, assume that the corresponding intersection is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cap\{K_i : 1 \leq i \leq n\}$ has these properties, also. There is no analogous Helly number for the case in which $d = 2$.
Each of the results above is best possible.

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1. Introduction

We begin with some definitions from [2] and [4]–[6]. Let S be a nonempty set in the plane. Set S is called an *orthogonal polygon* if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. Set S is *horizontally convex* if and only if for each pair x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. Set S is *orthogonally convex* if and only if S is both horizontally and vertically convex.

Let λ be a simple polygonal path in the plane whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Path λ is a *staircase path* if and only if the associated vectors alternate in direction. That is, for an appropriate labeling, for i odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for i even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction. Edge $[v_{i-1}, v_i]$ will be called north, south, east, or west according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points. For $n \geq 1$, if the staircase path λ is a union of at most n edges, then λ is called a *staircase n-path.*

Let $S \subseteq \mathbb{R}^2$. For points x and y in set S, we say x *sees* y (x is *visible* from y) via staircase paths if and only if there is a staircase path in S that contains both x and y. Set S is called *convex via staircase paths (staircase convex)* if and only if for every x, y in S, x sees y via staircase paths. Similarly, set S is *starshaped via staircase paths (staircase starshaped)* if and only if for some point p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S, denoted $Ker\, S$. Observe that a staircase starshaped set cannot be empty.

A familiar theorem by Victor Klee [13] establishes the following Helly-type theorem for countable intersections of convex sets: Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^d$. If every countable subfamily of $\mathcal{C}$ has nonempty intersection, then $\cap\{C : C \text{ in } \mathcal{C}\}$ is nonempty as well. Moreover, results in [3] provide the following analogue of Klee's theorem for sets that are starshaped via segments: Let k and d be fixed integers, $0 \leq k \leq d$, and let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$, if every countable subfamily of $\mathcal{K}$ has as its intersection a starshaped set whose kernel is at least k-dimensional, then all members of $\mathcal{K}$ have such an intersection.

Many theorems in convexity that involve the more conventional notion of visibility via straight line segments have interesting analogues that employ the idea of visibility via staircase paths. (See [2], [4]–[6].) Thus it is reasonable to pursue staircase versions of the results above, and we establish a staircase analogue for infinite families of simply connected sets in the plane. Further, we settle a question on the existence of related Helly numbers for finite families of compact planar sets, each having connected complement.

Throughout the paper, we will use the following terminology and notation. We say that a planar set S is *simply connected* if and only if for every simple closed curve $\delta \subseteq S$, the bounded region determined by δ lies in S. If λ is a simple path containing points x and y, then $\lambda(x, y)$ will denote the subpath of λ from x to y (ordered from x to y). For any set S, *int* S will denote its interior. Readers may refer to Valentine [15], to Lay [14], to Danzer, Grünbaum, Klee [8], and to Eckhoff [9] for discussions concerning Helly-type theorems, visibility via straight line segments, and starshaped sets.

2. Results for infinite intersections

We begin with two easy propositions.

Proposition 1. *Let $\mathcal{K}$ be any family of sets in $\mathbb{R}^d$. If every countable intersection of members of $\mathcal{K}$ has a nonempty interior, then $\cap\{K : K \text{ in} \mathcal{K}\}$ has a nonempty interior as well.*

Proof. We use a contrapositive argument. Let $\mathcal{N} = \{N_j : j = 1, 2, \ldots\}$ denote the countable family of spherical neighborhoods having rational centers and rational radii in $\mathbb{R}^d$. If $\cap\{K : K \text{ in} \mathcal{K}\}$ has empty interior, then for every N_j in $\mathcal{N}$ there is an associated K_j in $\mathcal{K}$ not containing N_j. Hence $\cap\{K_j : j = 1, 2, \ldots\}$ contains no spherical neighborhood at all and therefore has empty interior. The contrapositive statement establishes the result.

Observe that an analogous result holds in any second countable topological space.

Proposition 2. *Assume that K is a simply connected set in the plane and int $K = \phi$. If K is starshaped via staircase paths, then for each pair of distinct points x, y in K there is a unique simple path λ in K joining x to y. Moreover, λ is either a staircase path or a union of two staircase paths.*

Proof. Let $q \,\epsilon\, Ker\, K$. Then K contains staircase paths $\mu_x(q, x), \mu_y(q, y)$ joining q to x, q to y, respectively, and ordered from q to x, from q to y. Let z denote the last point shared by μ_x, μ_y relative to this order. Reversing the order on $\mu_x, \lambda \equiv \mu_x(x, z) \cup \mu_y(z, y)$ is a simple $x - y$ path in K. By our hypotheses for K, clearly λ is unique (up to order), and λ satisfies the proposition.

The following lemma will be helpful.

Lemma 1. *Let $\mathcal{K}$ be a family of simply connected sets in the plane, and let $p, s \,\epsilon \cap \{K : K \text{ in} \mathcal{K}\}$. Let n be a fixed integer, $n \geq 1$. If every countable intersection of members of $\mathcal{K}$ contains a staircase n-path from p to s, then $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path as well.*

Proof. To establish the result, we use induction on n. If $n = 1$, the result is immediate. If $n = 2$, just two distinct staircase 2-paths from p to s exist, so clearly one of these lies in $\cap\{K : K \text{ in} \mathcal{K}\}$. Inductively, assume that the result is true for natural members $j, 2 \leq j < k$, to prove for k. For convenience, assume that s is strictly northeast of p. Let $\mathcal{J}$ denote the family of all countable intersections of members of $\mathcal{K}$. Then $\mathcal{J}$ satisfies our hypotheses, too. Moreover, for at least one direction north or east, say east, every countable intersection of members of $\mathcal{J}$ contains a staircase k-path from p to s whose first (nontrivial) segment is east. For each J in $\mathcal{J}$, there is in J an associated family of staircase k-paths λ from p to s having first segment east. To each λ in our family let $[p, t_\lambda], t_\lambda \neq p$, be the corresponding first segment, and let T_J denote the collection of points t_λ. An easy geometric argument shows that T_J is convex. (Of course, $p \notin T_J$.)

Using our comments above, every countable subfamily of $\{T_J : J \,\text{in}\, \mathcal{J}\}$ has a nonempty intersection, so by Klee's theorem [13], $\cap\{T_J : J \,\text{in}\, \mathcal{J}\} \neq \phi$. Select $t_0 \,\epsilon\, \cap \{T_J : J \,\text{in}\, \mathcal{J}\}$. Then for every J in $\mathcal{J}$, J contains a staircase k-path from p to s having first segment $[p, t_0]$. Thus each J in $\mathcal{J}$, and hence each countable intersection of members of $\mathcal{K}$, contains a staircase $(k-1)$-path from t_0 to s. By our induction hypothesis, $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains a staircase $(k-1)$-path $\mu(t_0, s)$ from t_0 to s, and $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains the staircase k-path $[p, t_0] \cup \mu(t_0, s)$ from p to s. Therefore, the result holds for k and by induction holds for every integer $n \geq 1$, finishing the proof.

Corollary. *Let $\mathcal{K}$ be a family of simply connected sets in the plane, and let $p, s \,\epsilon\, \cap \{K : K \,\text{in}\, \mathcal{K}\}$. If every countable intersection of members of $\mathcal{K}$ contains a staircase path from p to s, then $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains such a path as well.*

Proof. We use a contrapositive argument. Suppose that $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains no staircase path from p to s. Then for every integer $n \geq 1, \cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains no staircase n-path from p to s. By the lemma, there exists a countable subfamily $\mathcal{K}_n$ of $\mathcal{K}$ for which $\cap\{K : K \,\text{in}\, \mathcal{K}_n\}$ contains no staircase n-path from p to s. Then $\cap\{K : K \,\text{in some}\, \mathcal{K}_n, n \geq 1\}$ is a countable intersection of members of $\mathcal{K}$ containing no staircase $p - s$ path at all. The contrapositive statement establishes the corollary.

Theorem 1. *Let d be a fixed integer, $0 \leq d \leq 2$, and let $\mathcal{K}$ be a family of simply connected sets in the plane. For every countable subfamily $\{K_n : n \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_n : n \geq 1\}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ also is starshaped via staircase paths, and its staircase kernel contains a convex set of dimension d.*

Proof. There are two parts to the proof.

Part 1. In this part of the argument, we will show that $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains a convex subset of dimension at least d. If every countable intersection of members of $\mathcal{K}$ has a nonempty interior, then by Proposition 1 $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ has nonempty interior as well, and thus $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains a convex set of dimension at least d. Therefore, it suffices to consider the case in which, for some countable subfamily $\{K_n : n \geq 1\}$ of $\mathcal{K}, G \equiv \cap\{K_n : n \geq 1\}$ has empty interior. Of course, this implies that $0 \leq d \leq 1$. Let $q \,\epsilon\, Ker\, G \neq \phi$. If q is the only point in G, then $d = 0, q$ belongs to every K in $\mathcal{K}$, and $\{q\} = \cap\{K : K \,\text{in}\, \mathcal{K}\}$. Again $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains a convex set of dimension d. Thus we assume that G is nontrivial. For future reference, observe that G and its staircase starshaped subsets satisfy Proposition 2.

Without loss of generality, assume that q is the origin, and let Q_1, Q_2, Q_3, Q_4 denote the standard four closed quadrants of the plane at q. For each $i, 1 \leq i \leq 4$, let A_{1i} denote the set of points in $G \cap Q_i$ district from q and visible from q via a staircase 1-path (horizontal or vertical segment) in G. If $A_{1i} \neq \phi$, then each of its points lies in a maximal connected subset s of A_{1i}, and s is either a segment with endpoint q or a ray emanating from q. Define E_{1i} to be the associated collection of sets $s \cup \{q\}$. If $A_{1i} = \phi$, then $E_{1i} = \phi$. Finally, define $E_1 = \cup\{E_{1i} : 1 \leq i \leq 4\}$. Since G is nontrivial, E_1 contains at least one set, and clearly E_1 contains at most four distinct sets.

Inductively, for $k > 1$ assume that E_{k-1} is defined. For $1 \leq i \leq 4$, let A_{ki} denote the set of points in Q_i visible from q via a staircase k-path in G but not via a staircase $(k-1)$-path in G. If $A_{ki} \neq \phi$, then each of its points lies in a maximal connected subset s of A_{ki}, and s is either a segment or a ray. Moreover, s has exactly one endpoint x_s in a member of E_{k-1}. Define E_{ki} to be the associated collection of sets $s \cup \{x_s\}$. If $A_{ki} = \phi$, then $E_{ki} = \phi$. Finally, let $E_k = \cup\{E_{ki} : 1 \leq i \leq 4\}$. By induction, E_k is defined for every integer $k \geq 1$. Clearly $\cup\{e : e \,\text{in}\, E_k \text{ for some } k\} = G$.

For future reference, notice that for any $x \,\epsilon\, G \backslash \{q\}$ and for $\lambda(q, x)$ the associated (unique) $q - x$ staircase path in G, each segment of λ is a subset of a corresponding member of $\cup\{E_k : k \geq 1\}$. If λ consists of exactly k_0 segments, then for $1 \leq j \leq k_0$ the jth segment of λ lies in a corresponding member of E_j.

Let $\mathcal{J}$ denote the family of all countable intersections of members of $\{K \cap G : K \,\text{in}\, \mathcal{K}\}$. Of course, $\mathcal{J}$ satisfies our hypotheses. The following propositions will be useful.

Proposition 3. *For some $k \geq 1$, let $e \,\epsilon\, E_k$, and let $J \,\epsilon\, \mathcal{J}$. If points x, y belong to $J \cap e$, then $[x, y] \subseteq J \cap e$. That is, $J \cap e$ is convex.*

Proof of Proposition 3. Since e is a segment or a ray in G, certainly $[x, y] \subseteq e$. By Proposition 2, $[x, y]$ is the only simple $x - y$ path in G, hence the only simple $x - y$ path available to lie in the staircase starshaped subset J. Thus $[x, y] \subseteq J$ as well.

Proposition 4. *For some $k \geq 1$, let $e_x, e_y \,\epsilon\, E_{k+1}$, and let $e \,\epsilon\, E_k$. Assume that each set e_x, e_y contains a point of e. Let $J \,\epsilon\, \mathcal{J}$. If $x \,\epsilon\, J \cap e_x$ and $y \,\epsilon\, J \cap e_y$, then $J \cap e \neq \phi$.*

Proof of Proposition 4. Set G necessarily contains the simple path from point x along e_x to e, then along e to e_y, then along e_y to point y. Again using Proposition 2, this path lies in J as well, so $J \cap e \neq \phi$.

Proposition 5. *For some $k \geq 2$ and for $J \,\epsilon\, \mathcal{J}$, assume that points x, y of J belong to distinct members e_x, e_y of E_k. Then for some e in $E_{k-1}, e \cap e_x \cap J \neq \phi$. (A parallel statement holds for e_y.)*

Proof of Proposition 5. Let $\mu_x(q, x), \mu_y(q, y)$ denote the staircase paths in G from q to x, q to y, respectively, and let z be the last point the paths share. By the argument in Proposition 2, $\lambda \equiv \mu_x(x, z) \cup \mu_y(z, y)$ is the unique simple $x - y$ path in G, so $\lambda \subseteq J \cap G$. Of course, by previous comments z belongs to a member of E_j for some $1 \leq j \leq k$. If $j < k$, then $\mu_x(z, x)$ will meet a segment or ray from each set $E_j, E_{j+1}, \ldots, E_k$. If $\mu_x(z, x)$ meets edge e of E_{k-1}, then the last point of $\mu_x(z, x) \cap e$ satisfies the proposition.

We will show that this must occur. Suppose on the contrary that z is not on a member of E_j for any $j \leq k-1$. Then z is in a member of E_k (only). This implies that $\mu_x(q,x)$ and $\mu_y(q,y)$ share points from a member of E_1, from a member of $E_2, \ldots$, from a member of E_k. But then z, x, y must all lie on the same member of E_k, contradicting our hypothesis. This finishes the argument and establishes Proposition 5.

We will prove that $\cap\{J : J \,\text{in}\, \mathcal{J}\} \neq \phi$ and hence $\cap\{K : K \,\text{in}\, \mathcal{K}\} \neq \phi$. There are two cases to consider.

Case 1. Suppose that for each $k \geq 1$ there is some J_k in $\mathcal{J}$ meeting at most countably many segments and rays from E_k. Then $J_0 \equiv \cap\{J_k : k \geq 1\}$ is a member of $\mathcal{J}$ that meets at most countably many members e of $\cup\{E_k : k \geq 1\}$. For every such e, $J_0 \cap e$ is convex by Proposition 3. Thus J_0 is a countable union of segments and/or rays f, each a maximal convex subset of an associated member e of $\cup\{E_k : k \geq 1\}$. Label these f sets by $\{f_m : m \geq 1\}$.

We assert that for some m_0, every member J of $\mathcal{J}$ meets f_{m_0} : Suppose, on the contrary, that for every m there is some J'_m with $J'_m \cap f_m = \phi$. Then $\cap\{J_0 \cap J'_m : m \geq 1\} = \phi$, contradicting our hypothesis for countable intersections of members of $\mathcal{K}$. This proves the assertion.

Therefore, for some $m_0 \geq 1$, every member of $\mathcal{J}$ meets f_{m_0}. Since any countable intersection of members of $\{J \cap J_0 : J \,\text{in}\, \mathcal{J}\}$ is in $\mathcal{J}$, any such intersection meets f_{m_0}. That is, every countable subfamily of $\{J \cap J_0 \cap f_{m_0} : J \,\text{in}\, \mathcal{J}\}$ has a nonempty intersection in f_{m_0}. Since the sets $J \cap J_0 \cap f_{m_0}$ are convex, by Klee's theorem [13] it follows that $\cap\{J \cap J_0 \cap f_{m_0} : J \,\text{in}\, \mathcal{J}\} \neq \phi$. Of course, this nonempty intersection lies in $\cap\{K : K \,\text{in}\, \mathcal{K}\}$, so $\cap \{K : K \,\text{in}\, \mathcal{K}\} \neq \phi$, the desired result. This finished Case 1.

Case 2. Suppose that for some $k_0 \geq 1$ there is no corresponding J_{k_0} in $\mathcal{J}$ meeting at most countably many segments and rays from E_{k_0}. That is, every J in $\mathcal{J}$ meets uncountably many members of E_{k_0}. Without loss of generality, assume that k_0 is as small as possible. Observe that $k_0 \geq 2$ since E_1 has at most four members. Then for $E_i, 1 \leq i \leq k_0 - 1$, there is some J_i in $\mathcal{J}$ meeting at most countably many segments and rays from E_i.

Fix J in $\mathcal{J}$ and consider the set $J_1 \cap \cdots \cap J_{k_0-1} \cap J$. This member of $\mathcal{J}$ meets uncountably many members of E_{k_0} but just countably many members of $E_1 \cup \cdots \cup E_{k_0-1}$. By Proposition 5, if points x, y in $J_1 \cap \cdots \cap J_{k_0-1} \cap J$ belong to distinct members of E_k, then $J_1 \cap \cdots \cap J_{k-1} \cap J$ contains points from at least one member of E_{k_0-1}. Since only countably many members of E_{k_0-1} are available, points from uncountably many members of E_{k_0} must correspond to the same member of E_{k_0-1}. That is, uncountably many E_{k_0} members meet the same E_{k_0-1} member at points of $J_1 \cap \cdots \cap J_{k_0-1} \cap J$. We define $\mathcal{E}(J)$ to be the collection of all members e of E_{k_0-1} for which $J_1 \cap \cdots \cap J_{k_0-1} \cap J$ contains points from uncountably many members of E_{k_0} along e. Of course, if $e \,\epsilon\, \mathcal{E}(J)$, then $e \cap J_{k_0-1} \neq \phi$.

We assert that some member of E_{k_0-1} belongs to every set $\mathcal{E}(J)$: Otherwise, for every e in E_{k-1}, there would be an associated J_e in $\mathcal{J}$ for which $e \notin \mathcal{E}(J_e)$. But then $J_0 \equiv \cap\{J_1 \cap \cdots \cap J_{k-1} \cap J_e : e \text{ in } E_{k-1}, e \cap J_{k-1} \neq \phi\}$ would be a countable intersection of members of $\mathcal{J}$ for which $\mathcal{E}(J_0) = \phi$. That is, $J_1 \cap \cdots \cap J_{k_0-1} \cap J_0 \equiv J_0$ would belong to $\mathcal{J}$ yet meet only countably many members of E_{k_0}, contradicting our choice of k_0. Our assertion is established.

Select $e_0 \,\epsilon \cap \{\mathcal{E}(J) : J \text{ in } \mathcal{J}\} \neq \phi$. For each J in $\mathcal{J}$, define $e_J = J_1 \cap \cdots \cap J_{k_0-1} \cap J \cap e_0$. By our choice of e_0, each set e_J contains points from uncountably many members of E_{k_0}, so $e_J \neq \phi$. By Proposition 3, e_J is convex as well. It is easy to see that every countable subfamily of $\{e_J : J \,\text{in}\, \mathcal{J}\}$ has a nonempty intersection, so $\cap\{e_J : J \,\text{in}\, \mathcal{J}\} \neq \phi$ by Klee's theorem [13]. Since $\cap\{e_J : J \,\text{in}\, \mathcal{J}\} \subseteq \cap\{K : K \,\text{in}\, \mathcal{K}\}, \cap\{K : K \,\text{in}\, \mathcal{K}\} \neq \phi$. Again we have the desired result. This finishes Case 2.

We conclude that $\cap\{K : K \,\text{in}\, \mathcal{K}\} \neq \phi$. Select point p in this intersection. If $d = 0$, the argument in Part 1 is finished. If $d = 1$, then every countable intersection of members of $\mathcal{K}$ is starshaped via staircase paths, is nontrivial, and contains p. Thus every such intersection contains at p a nondegenerate segment that is either north, south, east, or west of p. It is easy to see that for at least one of these four directions, say north, every countable intersection of members of $\mathcal{K}$ contains at p a nondegenerate segment north of p. Moreover, a simple contrapositive argument shows that $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ necessarily contains such a segment as well. That is, $\cap\{K : K \,\text{in}\, \mathcal{K}\}$ contains a convex subset of dimension one. This finishes Part 1 in the proof.

Part 2. In this part of the proof, we will show that the nonempty set $S \equiv \cap\{K : K \,\text{in}\, \mathcal{K}\}$ is starshaped via staircase paths and that its staircase kernel contains a convex subset of dimension d. Let $\mathcal{J}$ denote the family of all countable intersections of members of $\mathcal{K}$. Clearly $\mathcal{J}$ satisfies our hypotheses for Theorem 1.

We will use a strategy employed in [2], [3], and [7]. Adapting an argument by Bobylev [1], for each J_α in $\mathcal{J}$, we define $M_\alpha = \{x : x \,\text{in}\, J_\alpha, x \text{ sees via staircase paths in } J_\alpha \text{ each point of } S\}$. Let $\mathcal{M}$ denote the family of all the M_α sets. We will show that $\mathcal{M}$ satisfies the hypotheses for Theorem 1.

It is easy to see that each set M_α is simply connected: Let γ be any simple closed curve in M_α, with point b in the bounded region B determined by γ. Clearly $B \subseteq J_\alpha$. Select points v, w in γ such that $b \,\epsilon\, (v, w) \subseteq B \subseteq J_\alpha$ and $[v, w]$ is vertical. For every point s in S, both v and w see s via staircase paths in J_α. The simply connected region bounded by these paths and $[v, w]$ lies in J_α. Moreover, by [6, Lemma 2] this region is an orthogonally convex (and staircase convex) polygon, and hence b sees s via a staircase path in J_α. Since this is true for every s in $S, b \,\epsilon\, M_\alpha$. That is, M_α is simply connected.

We will show that each countable intersection of members of $\mathcal{M}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. In particular, for any countable subfamily $\{M_n : n \geq 1\}$ of $\mathcal{M}$ and associated subfamily $\{J_n : n \geq 1\}$ of $\mathcal{J}$, we will show that $Ker \cap \{J_n : n \geq 1\} \subseteq Ker \cap \{M_n : n \geq 1\}$.

To begin, notice that for any point z in $Ker \cap \{J_n : n \geq 1\} \neq \phi$, since $S \subseteq \cap\{J_n : n \geq 1\}, z$ sees each point of S via staircase paths in $\cap\{J_n : n \geq 1\}$. Hence z sees each point of S via staircase paths in J_n for each $n \geq 1$, so $z \,\epsilon \cap \{M_n :$

$n \geq 1\} \neq \phi$.

Let $z \epsilon Ker \cap \{J_n : n \geq 1\} \neq \phi$ and let $w \epsilon \cap \{M_n : n \geq 1\} \neq \phi$ to show that z sees w via staircase paths in $\cap\{M_n : n \geq 1\}$. Since $\cap\{M_n : n \geq 1\} \subseteq \cap\{J_n : n \geq 1\}$, certainly z sees w via a staircase path $\lambda \equiv \lambda(z,w)$ in $\cap\{J_n : n \geq 1\}$. We will show that $\lambda \subseteq \cap\{M_n : n \geq 1\}$. For convenience of notation let $n = 1$ to show that $\lambda \subseteq M_1$. Fix $s \epsilon S$. By comments above, since $z \epsilon Ker \cap \{J_n : n \geq 1\}, z \epsilon \cap \{M_n : n \geq 1\}$, so $z \epsilon M_1$. Hence z sees s via a staircase path $\mu(z,s)$ in J_1. Similarly, since $w \epsilon M_1, w$ sees s via a staircase path $\delta(w,s)$ in J_1. By [6, Lemma 2], the bounded region determined by $\lambda(z,w) \cup \mu(z,s) \cup \delta(w,s)$ is an orthogonally convex (and staircase convex) polygon, and this region lies in J_1 by hypothesis. We conclude that each point of $\lambda(z,w)$ sees s via a staircase path in J_1. Since this is true for every s in $S, \lambda(z,w) \subseteq M_1$. A parallel argument holds for each integer $n \geq 1$, so $\lambda(z,w) \subseteq \cap\{M_n : n \geq 1\}$, the desired result. We have shown that, for every w in $\cap\{M_n : n \geq 1\}, z$ sees w via a staircase path in $\cap\{M_n : n \geq 1\}$. Hence $z \epsilon Ker \cap \{M_n : n \geq 1\}$, and $Ker \cap \{J_n : n \geq 1\} \subseteq Ker \cap \{M_n : n \geq 1\}$. This implies that $\cap\{M_n : n \geq 1\}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d.

Since $\mathcal{M}$ satisfies the hypotheses of Theorem 1, we may apply the argument in Part 1 above to conclude that $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\}$ is nonempty and contains a convex subset of dimension at least d. Let p belong to $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\}$. Then for every J_α in $\mathcal{J}$ and for every s in S, p sees s via a staircase path $\lambda_\alpha(p,s)$ in J_α. That is, every countable intersection of members of $\mathcal{K}$ contains a staircase path from p to s. By the corollary to Lemma 1, $\cap\{K : K \text{ in } \mathcal{K}\} \equiv S$ contains such a path as well. Since this holds for every s in S, $p \epsilon KerS$. Thus $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\} \subseteq KerS$, and in fact it is easy to see that the sets are equal. Thus $KerS$ is nonempty and contains a convex set of dimension at least d, finishing Part 2 and establishing the theorem.

The result in Theorem 1 fails without the simple connectedness condition, as Example 1 demonstrates.

Example 1. For each real number r, let $(r,0)$ be the associated point on the x axis and let $K_r = \mathbb{R}^2 \backslash \{(r,0)\}$. Define $\mathcal{K} = \{K_r : r \,\text{real}\}$. It is easy to see that countable intersections of members of $\mathcal{K}$ are starshaped via staircase paths: For any countable collection $\{r_n : n \geq 1\}$ of real numbers, choose $r_0 \notin \{r_n : n \geq 1\}$. Then for every real number $s \neq 0, (r_0, s) \epsilon Ker \cap \{K_{r_n} : n \geq 1\}$. However, $\cap\{K_r : r \,\text{real}\}$ is not connected and certainly is not starshaped via staircase paths.

Theorem 1 yields the following result for convex sets.

Corollary 1. *Let d be a fixed integer, $0 \leq d \leq 2$, and let $\mathcal{K}$ be a family of sets in the plane. For every countable subfamily $\{K_n : n \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_n : n \geq 1\}$ is convex via staircase paths and contains a convex set of dimension at least d. Then $\cap\{K : K \,in \mathcal{K}\}$ has these properties as well.*

Proof. Clearly staircase convex sets are simply connected and hence $\mathcal{K}$ satisfies the hypotheses of Theorem 1. Using the notation from the proof of Theorem 1, Part 2, let $S = \cap\{K : K \,\text{in}\, \mathcal{K}\}$. Let $\mathcal{J}$ denote the family of all countable intersections of members of $\mathcal{K}$ and define the associated family of sets $\mathcal{M}$. Observe that since each member J_α of $\mathcal{J}$ is convex via staircase paths, $J_\alpha = M_\alpha$. Then by the proof of Theorem 1, $S = \cap\{J_\alpha : J_\alpha \,\text{in}\, \mathcal{J}\} = \cap\{M_\alpha : \mathcal{M}_\alpha \,\text{in}\, \mathcal{M}\} = KerS$ contains a convex set of dimension at least d. Moreover, using [6, Lemma 2], it is easy to see that $KerS$ is convex via staircase paths, so $S = KerS$ has the required properties.

It is easy to show that Theorem 1 and Corollary 1 fail if we replace countable with finite in the hypothesis. For example, consider the family $\mathcal{K} = \{K_n : n \geq 1\}$, where either $K_n = \{(x, y) : x \geq n\}$ or $K_n = \{(x, y) : 0 < x < \frac{1}{n}, 0 < y < \frac{1}{n}\}$. In each case, finite intersections of members of $\mathcal{K}$ are starshaped (in fact, convex) via staircase paths, and associated kernels are fully two-dimensional. However, $\cap\{K_n : n \geq 1\} = \phi$.

More interesting is the situation in which we require the sets in $\mathcal{K}$ to be compact. Even then we cannot replace countable with finite in Theorem 1 and its corollary, as Example 2 illustrates.

Example 2. In the plane let $T = \{(x, y) : 0 \leq x \leq 4 \text{ and } 0 \leq y \leq 2\}$. For every real number $r, 0 < r < 4$, let $T_r = \{(x, y) : r < x \leq 4 \text{ and } 0 \leq y < \sqrt{r}\,\}$, and let $T'_r = \{(x, y) : 0 \leq x < r \,\text{and}\, \sqrt{r} < y \leq 2\}$.

Define $K_r = T \backslash (T_r \cup T'_r)$. (Set $K_1 \cap K_2$ is illustrated in Figure 1.) It is easy to see that $\{K_r : 0 < r < 4\}$ is a family of simply connected orthogonal polygons. Moreover, every finite subfamily of $\{K_r : 0 < r < 4\}$ has an intersection that is starshaped (in fact, convex) via staircase paths, and the associated convex kernel contains convex sets of dimension two. However, $\cap\{K_r : 0 < r < 4\} \equiv \{(x, y) : y = \sqrt{x}, 0 \leq x \leq 4\}$ is not staircase starshaped. Thus countable cannot be replaced by finite in Theorem 1, even when the sets are compact (so that the associated intersection is nonempty).

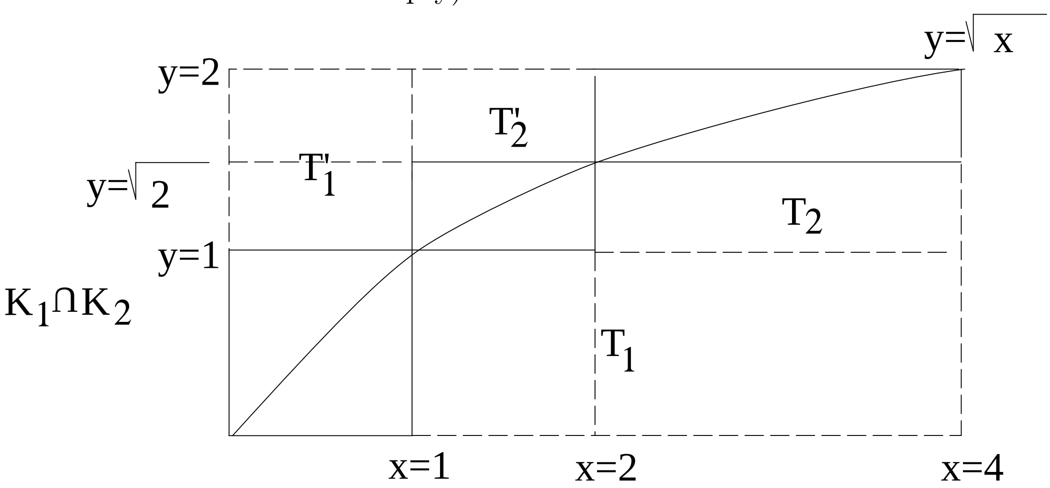

Figure 1

3. Results for finite intersections

Example 2 above shows that no finite Helly number exists to ensure that infinite intersection of simply connected compact sets will be starshaped via staircase paths, even when the sets are orthogonal polygons. However, for finite intersections, we have the following result from [5, Theorem 1 s]: Let $\mathcal{F}$ be a finite family of compact sets in the plane, each having connected complement. If every three (not necessarily distinct) members of $\mathcal{F}$ have a nonempty intersection that is starshaped via staircase paths, then all members of $\mathcal{F}$ have such an intersection.

Because there are Helly-type theorems to predict the dimension of the kernel in a finite intersection of starshaped sets (see [7]), it is natural to seek staircase analogues of these results as well. Perhaps surprisingly, there exists a finite Helly number to guarantee that the staircase kernel contain a one-dimensional convex subset, yet no similar number exists for the two-dimensional case. We have the following result.

Theorem 2. *Define function f on $\{0,1\}$ by $f(0) = 3, f(1) = 4$. Let $\mathcal{K} = \{K_i : 1 \leq i \leq n\}$ be a finite family of compact sets in the plane, each having connected complement. For d fixed, $d \epsilon \{0,1\}$, and for every $f(d)$ members of $\mathcal{K}$, assume that the corresponding intersection is starshaped via staircase paths and that the associated staircase kernel contains a convex set of dimension at least d. Then all members of f have such an intersection.*

The number $f(d)$ is best possible in each case. There is no analogous Helly number for the case in which $d = 2$.

Proof. If $d = 0$, the result follows immediately from [5, Theorem 1 s], so we restrict our attention to the case in which $d = 1$. Again by [5, Theorem 1 s], set $S \equiv \cap\{K_i : 1 \leq i \leq n\}$ is nonempty (and starshaped via staircase paths). As in the proof of Theorem 1, Part 2, for each $i, 1 \leq i \leq n$, define set $M_i \equiv \{x : x \,\text{in}\, K_i, x$ sees via staircase paths in K_i each point of $S\}$, and let $\mathcal{M} \equiv \{M_i : 1 \leq i \leq n\}$. By arguments like those in [5, Theorem 1 s], $\mathcal{M}$ is a family of compact simply connected sets. Moreover, by arguments like those in the proof of Theorem 1, Part 2, above, for any four members M_i of $\mathcal{M}, 1 \leq i \leq 4, Ker\cap\{K_i : 1 \leq i \leq 4\} \subseteq Ker\cap \{M_i : 1 \leq i \leq 4\}$. Then certainly every two members of $\mathcal{M}$ have a path connected intersection and every three have a nonempty intersection. Using a version of Molár's theorem by Karimov, Repovš, and Željko [11], $\cap\{M_i : 1 \leq i \leq n\} \neq \phi$. Of course, this intersection is exactly $KerS$. Select $p \epsilon \cap \{M_i : 1 \leq i \leq n\}$. Using our hypothesis and comments above, for every four members M_i of $\mathcal{M}, 1 \leq i \leq 4, \cap\{M_i : 1 \leq i \leq 4\}$ is nontrivial, is starshaped via staircase paths, and contains p. Thus every four members of $\mathcal{M}$ contain at p a nondegenerate segment that is either north, south, east, or west of p. As in the proof of Theorem 1, Part 1, for at least one of these four directions, say north, every member of $\mathcal{M}$ contains at p a nondegenerate segment north of p. Since $\mathcal{M}$ is finite, $\cap\{M_i : 1 \leq i \leq n\} \equiv KerS$ contains such a segment as well, finishing the proof of the first statement in the theorem.

Examples 1 and 2 in [2] show that the number $f(0) = 3$ above is best, while the following easy example shows that $f(1) = 4$ is best as well.

Example 3. In the plane, let $s_j, 1 \le j \le 4$, denote the four horizontal or vertical unit segments having one endpoint at the origin. For $1 \le i \le 4$, let $K_i = \cup\{s_j : 1 \le j \le 4, j \ne i\}$. Every three of the K_i sets intersect in a staircase starshaped (staircase convex) set whose corresponding staircase kernel contains a nondegenerate segment. However, $\cap\{K_i : 1 \le i \le 4\}$ contains only the origin.

Example 4 demonstrates that there is no corresponding Helly number for $d = 2$.

Example 4. In the plane, for each integer $k \ge 0$, let A_k denote the orthogonal square having vertices (k, k) and $(k + 1, k + 1)$, and let λ_k represent the east - north 2-staircase from (k, k) to $(k + 1, k + 1)$. Fix the integer $n \ge 1$, and let $\lambda = \cup\{\lambda_k : 0 \le k \le n\}$. For each $j, 0 \le j \le n$, define $K_j = \cup\{\lambda \cup A_k : 0 \le k \le n, k \ne j\}$, and let $\mathcal{K} = \{K_j : 0 \le j \le n\}$. (Figure 2 illustrates K_1 for $n = 4$.) Then every n members of $\mathcal{K}$ intersect in a simply connected orthogonal polygon that is starshaped (in fact, convex) via staircase paths and whose staircase kernel contains a two-dimensional convex subset. However, the staircase starshaped (staircase convex) set $\cap\{K_j : 0 \le j \le n\} \equiv \lambda$ has empty interior. Since n may be as large as we like, this example reveals that no finite Helly number exists to guarantee that the staircase kernel of our intersection contain a two-dimensional convex subset.

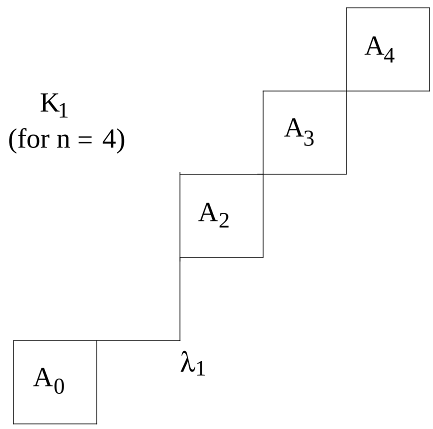

Figure 2

It is interesting to notice that the Helly numbers in Theorem 2 for $d = 0$ and $d = 1$ agree with the corresponding Helly numbers for convex sets in [10] and [12] and for starshaped sets in [7]. However, the analogy breaks down for $d = 2$, as Example 4 has revealed.

Finally we turn to the convex case. Although [4, Example 4] demonstrates that there is no finite Helly number for intersections of staircase convex polygons, with the additional hypothesis that appropriate intersections again be staircase convex, we have this result.

Corollary 2. *Define function f on $\{0, 1\}$ by $f(0) = 3, f(1) = 4$. Let $\mathcal{K} = \{K_i : 1 \le i \le n\}$ be a finite family of compact sets in the plane. For d fixed, $d \in \{0, 1\}$,* *and for every $f(d)$ members of $\mathcal{K}$, assume that the corresponding intersection is convex via staircase paths and contains a convex set of dimension at least d. Then all members of $\mathcal{K}$ have such an intersection. The number $f(d)$ is best possible in each case. There is no analogous Helly number when $d = 2$.*

Proof. The argument resembles the proof of Corollary 1 above. Observe that each member K of $\mathcal{K}$ has a connected complement: Otherwise, $\mathbb{R}^2 \backslash K$ would have a bounded component B. For $b \in B$ and H the horizontal line at b, H would meet $bdry B \subseteq K$ at a_1, a_2 with $b \in (a_1, a_2)$, impossible since K is horizontally convex. Hence members of $\mathcal{K}$ satisfy the hypotheses of Theorem 2. As in the proof of that theorem, let $S = \cap\{K_i : 1 \leq i \leq n\}$. By [5, Theorem 1 s], $S \neq \phi$, and we define the associated family of sets $\mathcal{M} = \{M_i : 1 \leq i \leq n\}$. Observe that $K_i = M_i, 1 \leq i \leq n$, and therefore $S = \cap\{K_i : 1 \leq i \leq n\} = \cap\{M_i : 1 \leq i \leq n\} = Ker S$, which is staircase convex. If $d = 0$, the argument is finished. If $d = 1$, then by Theorem 2 above, $Ker S$ (and hence S) contains a convex set of dimension at least one, and again the proof is complete.

Example 1 in [2] shows that $f(0) = 3$ is best, while Example 3 above indicates that $f(1) = 4$ is best. Of course, Example 4 above settles the case for $d = 2$.

It is interesting to compare the Helly numbers above to the Krasnosel'skii numbers for the dimension of the staircase kernel of an orthogonal polygon in [4].

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