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Certain Equalities and Inequalities concerning Polygons in $\mathbb{R}^2$

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Abstract. This article can be considered as an appendix to the article [3]. Here we mainly deal with k-outscribed polygons, where we use the definition of such polygons as it is given in [3]. The aim and purpose of the article is to find and investigate certain equalities and inequalities concerning k-outscribed polygons.

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1. Introduction

The definition of the determinant of a rectangular matrix has been introduced in the article [1]. The determinant of an $m \times n$ matrix $A, m \leq n$, with columns $A_1, \ldots, A_n$ is the sum

$$\sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} (-1)^{r+s} |A_{j_1}, \ldots, A_{j_m}|,$$

where $r = 1 + \cdots + m$, $s = j_1 + \cdots + j_m$.

It is clear that every real $m \times n$ matrix $A = [A_1, \ldots, A_n]$ determines a polygon in $\mathbb{R}^m$(the columns of the matrix correspond to the vertices of the polygons) and vice versa. The polygon which corresponds to the given matrix $[A_1, \ldots, A_n]$ will be denoted by $A_1 \ldots A_n$.

Here, and in what follows, a special case of the above definition will be used when $m = 2$, that is,

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{vmatrix} = \sum_{1\le i<j\le n} (-1)^{1+2+(i+j)} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}. \tag{1.1}$$

Also, we will make use of some results given in [2] and [3]. First, we list those given in [2], keeping back the same numeration of cited results as there.

Theorem 3. *Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$. Then*

$$2 \text{ area of } A_1 \dots A_n = |A_1 + A_2, A_2 + A_3, \dots, A_{n-1} + A_n, A_n + A_1|. \tag{1.2}$$

Corollary 6.1. *If n is odd, then for every point X in $\mathbb{R}^2$, we have*

$$|A_1 + X, \dots, A_n + X| = |A_1, \dots, A_n|. \tag{1.3}$$

Theorem 7. *Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ and let n be an even integer. Then, for every point X in $\mathbb{R}^2$, it follows that*

$$|A_1 + X, \dots, A_n + X| = |A_1, \dots, A_n| \tag{1.4}$$

when $\sum_{i=1}^n (-1)^i A_i = 0$.

Theorem 8. *Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ and let $\sum_{i=1}^n (-1)^i A_i = 0$. Then*

$$|A_1, \dots, A_n| = |A_1, \dots, A_{n-1}|. \tag{1.5}$$

Corollary 10.1. *For n odd, for each $i \in \{1, \dots, n\}$ we have the cyclic permutation property*

$$|A_i, \dots, A_n, A_1, \dots, A_{i-1}| = |A_1, \dots, A_n|. \tag{1.6}$$

Moreover, for n even and $\sum_{i=1}^n (-1)^i A_i = 0$, (1.6) remains valid.

We will apply the following results from [3], keeping back the original numeration.

Theorem 1. *Let $A_1 \dots A_n$ be a given polygon in $\mathbb{R}^2$ and let k be a positive integer such that $k < n$ and $GCD(k, n) = 1$. Then, there exists a unique k-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ such that*

$$2 \text{ area of } P_1 \cdots P_n = k^2 |B_1 + B_2, B_2 + B_3, \dots, B_n + B_1|, \tag{1.7}$$

where

$$B_i = A_i + A_{i+k} + \cdots + A_{i+(x_k-1)k} \qquad \big(i = 1, \dots, n\big) \tag{1.8}$$

and x_k is the least positive integer x satisfying

$$kx = 1 \ (\text{mod } n). \tag{1.9}$$

Theorem 4. *Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ and let k be an integer such that $1 < k < n$ and $GCD(k, n) = d > 1$. Then, only one of the following two assertions is true:*

(i) *There is no k-outscribed polygon to the polygon $A_1 \dots A_n$.*

(ii) *There are infinitely many k-outscribed polygons to the polygon $A_1 \dots A_n$.*

The second statement (ii) appears only if for each $i = 1, \dots, d$ there holds (E_k) (existence for k-outscribed)

$$A_1 + A_{1+k} + \cdots + A_{1+(\hat{x}-1)k} = A_i + A_{i+k} + \cdots + A_{i+(\hat{x}-1)k} \qquad (E_k)$$

where $\hat{x}$ is the least positive integer solution of the equation

$$kx = 0 \ (\mathrm{mod}\ n). \qquad (1.10)$$

Theorem 10. *Let $\begin{bmatrix} A_1, \dots, A_n \end{bmatrix}$ be any given real $2 \times n$ matrix. Then*

$$\begin{aligned} |A_1, \dots, A_n| = |A_1, A_2, A_3| + |A_1 - A_2 + A_3, A_4, A_5| + \\ |A_1 - A_2 + A_3 - A_4 + A_5, A_6, A_7| + \cdots + L, \end{aligned} \qquad (1.11)$$

where

$$L = \begin{cases} \left| \sum_{i=1}^{n-2} (-1)^{i+1} A_i, A_{n-1}, A_n \right| & n \ odd \\ \left| \sum_{i=1}^{n-1} (-1)^{i+1} A_i, A_n \right| & n \ even\ . \end{cases}$$

2. Certain equalities and inequalities concerning some polygons in $\mathbb{R}^2$

Theorem 1. *Let $A_1 \dots A_n$ be a polygon i3n $\mathbb{R}^2$ and let k be an integer $1 < k < n$. Let $GCD(k, n) = d > 1$ and let (E_k) be fulfilled, that is*

$$A_i + A_{i+k} + A_{i+2k} + \cdot + A_{i+(\hat{x}-1)k} = \frac{S}{d}, \qquad \big(i = 1, \dots, d\big) \qquad (\mathrm{E}_k)$$

where

$$\hat{x} = \frac{n}{d}, \quad S = \sum_{i=1}^{n} A_i. \qquad (2.1)$$

Then for arbitrary points $P_1, \dots, P_{d-1}$ from $\mathbb{R}^2$ there exists k-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ such that

$$\frac{n}{d} P_d = S - \frac{n}{d}(P_1 + \cdots + P_{d-1}) - \sum_{i=1}^{d} [S_{i+k} + \cdots + S_{i+(\hat{x}-1)k}], \qquad (2.2)$$

$$P_{i+jk} = P_i + S_{i+jk}, \qquad \big(i = 1, \dots, d\big) \ \ and \ \ \big(j = 1, \dots, \hat{x} - 1\big) \qquad (2.3)$$

where S_{i+jk}, $i = 1, \dots, d$ and $j = 1, \dots, \hat{x} - 1$ are the sums of certain vertices $A_1, \dots, A_n$.

Proof. The system

$$P_i + P_{i+1} + \cdots + P_{i+k-1} = kA_i \qquad (i = 1, \ldots, n) \tag{2.4}$$

can be rewritten as

$$P_i - P_{i+k} = k(A_i - A_{i+1}) \qquad (i = 1, \ldots, n). \tag{2.5}$$

It is easy to see that

$$P_{i+k} = P_i - k(A_i - A_{i+1}), \tag{2.6}$$
$$P_{i+2k} = P_{i+k} - k(A_{i+k} - A_{i+1+k}), \tag{2.7}$$
$$\vdots$$
$$P_{i+(\hat{x}-1)k} = P_{i+(\hat{x}-2)k} - k(A_{i+(\hat{x}-2)k} - A_{i+1+(\hat{x}-2)k}), \tag{2.8}$$

for all $i = 1, \ldots, d$. The relations (2.3) can be obtained from (2.6)–(2.8). So, by (2.6) we see that

$$P_{i+k} = P_i + S_{i+k},$$

where $S_{i+k} = -k(A_i - A_{i+1})$. Then, by (2.6) and (2.7) we deduce

$$P_{i+2k} = P_i - k(A_i - A_{i+1}) - k(A_{i+k} - A_{i+1+k}),$$

such that can be written as

$$P_{i+2k} = P_i + S_{i+2k},$$

where

$$S_{i+2k} = -k(A_i - A_{i+1}) - k(A_{i+k} - A_{i+1+k}).$$

In the same way it can be seen that

$$P_{i+3k} = P_i + S_{i+3k},$$

where

$$S_{i+3k} = -k(A_i - A_{i+1}) - k(A_{i+k} - A_{i+1+k}) - k(A_{i+2k} - A_{i+1+2k}).$$

Finally, we get

$$P_{i+(\hat{x}-1)k} = P_i + S_{i+(\hat{x}-1)k},$$

where

$$S_{i+(\hat{x}-1)k} = -k(A_i - A_{i+1}) - k(A_{i+k} - A_{i+1+k}) - \cdots - k(A_{i+(\hat{x}-2)k} - A_{i+1+(\hat{x}-2)k}).$$

Now from (2.4) it is easy to see that

$$\sum_{i=1}^{n} P_i = S, \tag{2.9}$$

where S is given by (2.1). Thus,

$$[P_1 + P_{1+k} + \cdots + P_{1+(\hat{x}-1)k}] + \cdots + [P_d + P_{d+k} + \cdots + P_{d+(\hat{x}-1)k}] = S$$

or

$$P_1 + \cdots + P_d + [P_{1+k} + \cdots + P_{1+(\hat{x}-1)k}] + \cdots + [P_{d+k} + \cdots + P_{d+(\hat{x}-1)k}] = S,$$

such that is equivalent to (2.2). This proves Theorem 1. □

The following result is proved in [3, Theorem 4] in another way using vector spaces techniques.

Corollary 1.1. *Denote M the matrix of the system* (2.4). *Then*

$$\text{rank of } M = n - d + 1.$$

Corollary 1.2. *Let $GCD(k,n) = 2$. The area of $P_1 \ldots P_n$ has the form*

$$2 \text{ area of } P_1 \ldots P_n = |S_3, S_3 + S_4, S_4 + S_5, \ldots, S_{n-1} + S_n, S_n|. \tag{2.10}$$

Proof. Since

$$\begin{aligned} P_{2i+1} &= P_1 + S_{2i+1}, \\ P_{2i+2} &= P_2 + S_{2i+2}, \qquad \left(i = 1, \ldots, \frac{n}{2} - 1\right) \\ P_2 &= -P_1 + \frac{2}{n}\Big[S - \sum_{i=1}^{n/2-1} (S_{2i+1} + S_{2i+2})\Big], \end{aligned}$$

we can write

$$\begin{aligned} 2 \text{ area of } P_1 \ldots P_n &= |P_1 + P_2, P_2 + P_3, \ldots, P_n + P_1| \\ &= |T, T + S_3, T + S_3 + S_4, \ldots, T + S_{n-1} + S_n, T + S_n|, \end{aligned}$$

where

$$T = \frac{2}{n}\Big[S - \sum_{i=1}^{n/2-1} (S_{2i+1} + S_{2i+2})\Big].$$

The last display can be rewritten into the one like (2.10), having in mind (1.4). □

Using [3, Theorem 10] (see introduction) the area of $P_1 \ldots P_n$ can also be expressed like

$$\begin{aligned} \text{area of } P_1 \ldots P_n = |S_3, S_3 + S_4, S_4 + S_5| + |S_5, S_5 + S_6, S_6 + S_7| + \cdots + \\ |S_{n-3}, S_{n-3} + S_{n-2}, S_{n-2} + S_{n-1}| + |S_{n-1}, S_n|. \end{aligned}$$

Let us remark that

$$\begin{aligned} &S_3 - (S_3 + S_4) + (S_4 + S_5) = S_5 \\ &S_3 - (S_3 + S_4) + (S_4 + S_5) - (S_5 + S_6) + (S_6 + S_7) = S_7 \text{ and so on.} \end{aligned}$$

Corollary 1.3. *Let $GCD(k,n)=1$. Then there are $S_2,\dots,S_n$ so that*

$$P_{1+i}=P_1+S_{1+i} \qquad \big(i=1,\dots,n-1.\big) \tag{2.11}$$

Proof. Since $P_1\dots P_n$ is unique by [3, Theorem 1] (consult introduction), there are $S_2,\dots,S_n$ so that

$$P_2=P_1+S_2,\dots,P_n=P_1+S_n.$$

Thus, when $d=1$, there is only one P_1 such that (2.11) is valid. □

Corollary 1.4. *Let $GCD(k,n)=d>2$. For the sake of simplicity take $d=3$. Then*

$$2 \text{ area of } P_1\dots P_n=|P_1+P_2,P_2+P_3,P_3+P_4,\dots,P_n+P_1|, \tag{2.12}$$

where $P_4=P_1+S_4$, $P_5=P_2+S_5$, $P_6=P_3+S_6$, $P_7=P_1+S_7$ etc. Thus, the area of $P_1\dots P_n$ is 4-parametric, since $P_1(\alpha_1,\beta_1)$ and $P_2(\alpha_2,\beta_2)$ can be taken arbitrarily in $\mathbb{R}^2$ and they cannot be eliminated from (2.12). Therefore, k-outscribed polygons to the polygon $A_1\dots A_n$ have different areas when $GCD(k,n)>2$ (cf. [2, Theorem 11]).

Corollary 1.5. *Let $GCD(k,n)=d=3$, where n is odd. Then*

$$|P_1+P_2+P_3,P_2+P_3+P_4,\dots,P_n+P_1+P_2|=constant, \tag{2.13}$$

where $P_4=P_1+S_4$, $P_5=P_2+S_5$, $P_6=P_3+S_6$, $P_7=P_1+S_7$ etc.

Proof. By (1.3) the determinant on the left-hand side of (2.13) can be expressed in the form

$$|0,S_4,S_4+S_5,S_4+S_5+S_6,\dots,S_{n-2}+S_{n-1}+S_n,S_{n-1}+S_n,S_n|$$

or

$$|S_4,S_4+S_5,\dots,S_{n-1}+S_n,S_n|.$$ □

Corollary 1.6. *Let $A_1\dots A_n$ be a polygon in $\mathbb{R}^2$, $k\in\mathbb{N}$ such that $GCD(k,n)=2$ and $\sum_{i=1}^{n}(-1)^iA_i=0$, which can be written as $S=2\sum_{i=1}^{n/2-1}A_{2i-1}=2\sum_{i=1}^{n/2-1}A_{2i+2}$. Then the polygon $Q_1\dots Q_n$ given by*

$$nQ_1=S-2(S_3+S_5+\dots+S_{n-1}), \tag{2.14}$$
$$nQ_2=S-2(S_4+S_6+\dots+S_n), \tag{2.15}$$
$$nQ_{2i+1}=Q_1+S_{2i+1}, \tag{2.16}$$
$$nQ_{2i+2}=Q_2+S_{2i+2}, \qquad \left(i=1,\dots,\frac{n}{2}-1\right) \tag{2.17}$$

is the only one k-outscribed to the polygon $A_1\dots A_n$ and in the same time it can be k-outscribed.

Proof. The condition (E_k) if fulfilled, that is, $S = 2\sum_{i=1}^{n/2-1} Q_{2i-1} = 2\sum_{i=1}^{n/2-1} Q_{2i}$, if Q_1 and Q_2 are chosen so, that

$$\frac{S}{2} = Q_1 + (Q_1 + S_3) + \cdots + (Q_1 + S_{n-1}),$$
$$\frac{S}{2} = Q_2 + (Q_2 + S_4) + \cdots + (Q_2 + S_n).$$

Namely, in that case (2.14) and (2.15) are fulfilled. □

Corollary 1.7. *Let $A_1 \ldots A_n$ be any given polygon in $\mathbb{R}^2$, $k \in \mathbb{R}^2$, such that $GCD(n,k) = d > 2$ and let (E_k) be fulfilled. Then the polygon $Q_1 \ldots Q_n$ given by*

$$nQ_i = S - d(S_{i+k} + S_{i+2k} + \cdots + S_{i+(\hat{x}-1)k}), \qquad \big(i = 1, \ldots, d\big) \tag{2.18}$$

and by

$$\begin{aligned} Q_{1+ik} &= Q_1 + S_{1+ik}, \\ Q_{2+ik} &= Q_2 + S_{2+ik}, \\ &\vdots \\ Q_{d+ik} &= Q_d + S_{d+ik}, \qquad \big(i = 1, \ldots, \hat{x} - 1\big) \end{aligned} \tag{2.19}$$

is the only one k-outscribed to the polygon $A_1 \ldots A_n$ and has the property that it can be k-outscribed.

Proof. Analogously to the case when $d = 2$, we choose $Q_1, \ldots, Q_d$ so that

$$\frac{S}{d} = Q_i + (Q_i + S_{i+k}) + \cdots + (Q_i + S_{i+(\hat{x}-1)k}), \qquad \big(i = 1, \ldots, d\big). \tag{E$_k$}$$

□

The following theorems refer to an inequality concerning k-outscribed polygons. Let us introduce the notation which will be used in the sequel.

Let $P_i(p_i, q_i)$ and $P_{i+1}(p_{i+1}, q_{i+1})$ be points in $\mathbb{R}^2$. Then

$$|P_i - P_{i+1}|^2 = (p_i - p_{i+1})^2 + (q_i - q_{i+1})^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}p_i}|P_i - P_{i+1}|^2 = 2(p_i - p_{i+1}), \quad \frac{\mathrm{d}}{\mathrm{d}q_i}|P_i - P_{i+1}|^2 = 2(q_i - q_{i+1}).$$

Remark 1. For sake of simplicity we write $\frac{\mathrm{d}}{\mathrm{d}P_i}|P_i - P_{i+1}|^2$, whose meaning is $\frac{\mathrm{d}}{\mathrm{d}P_i}|P_i - P_{i+1}|^2 = (2(p_i - p_{i+1}), 2(q_i - q_{i+1}))$ or $\frac{\mathrm{d}}{\mathrm{d}P_i}|P_i - P_{i+1}|^2 = 2(P_i - P_{i+1})$. Of course, $\frac{\mathrm{d}}{\mathrm{d}P_{i+1}}|P_i - P_{i+1}|^2 = -2(P_i - P_{i+1})$.

Theorem 2. *Let $A_1 \dots A_n$ be any given polygon in $\mathbb{R}^2$ such that n is even, $d = 2$ and $\sum_{i=1}^{n}(-1)^i A_i = 0$. Then for every 2-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ we have*

$$\sum_{i=1}^{n} |P_i - P_{i+1}|^2 \geq \sum_{i=1}^{n} |Q_i - Q_{i+1}|^2,$$

where $Q_1 \dots Q_n$ is 2-outscribed to $A_1 \dots A_n$ and it can be 2-outscribed.

Proof. First, let us remark that from

$$\sum_{i=1}^{n}(-1)^i Q_i = 0 \text{ and } Q_i + Q_{i+1} = 2A_i, \qquad \big(i = 1, \dots, n\big) \tag{2.20}$$

one gets

$$\begin{aligned}
\frac{n}{2}\, Q_1 &= (n-1)A_1 - (n-2)A_2 + (n-3)A_3 - \dots + A_{n-1}, \\
\frac{n}{2}\, Q_2 &= (n-1)A_2 - (n-2)A_3 + (n-3)A_4 - \dots + A_n, \\
&\vdots \\
\frac{n}{2}\, Q_n &= (n-1)A_n - (n-2)A_1 + (n-3)A_2 - \dots + A_{n-2}\,.
\end{aligned} \tag{2.21}$$

Since by $P_i + P_{i+1} = 2A_i,\ i = 1, \dots, n$, it follows

$$\begin{aligned}
P_2 &= 2A_1 - P_1, \\
P_3 &= 2A_2 - 2A_1 + P_1, \\
&\vdots \\
P_n &= 2A_{n-1} - 2A_{n-2} + 2A_{n-3} - \dots + 2A_1 - P_1,
\end{aligned} \tag{2.22}$$

it is clear that the sum $\sum_{i=1}^{n} |P_i - P_{i+1}|^2$ depends only on P_1; the equation $\frac{\mathrm{d}}{\mathrm{d}P_i} \sum_{i=1}^{n} |P_i - P_{i+1}|^2 = 0$ can be written as

$$(2P_1 - 2A_1) - (-2P_1 - 2A_2 + 4A_1) + \dots - (2A_n - 2P_1) = 0,$$

so, that we conclude

$$\frac{n}{2}\, P_1 = (n-1)A_1 - (n-2)A_2 + (n-3)A_3 - \dots + A_{n-1}.$$

Now, from (2.22) it follows

$$\begin{aligned}
\frac{n}{2}\, P_2 &= (n-1)A_2 - (n-2)A_3 + (n-3)A_4 - \dots + A_n, \\
&\vdots \\
\frac{n}{2}\, P_n &= (n-1)A_n - (n-2)A_1 + (n-3)A_2 - \dots + A_{n-2}.
\end{aligned}$$

Thus, $P_i = Q_i$, $i = 1, \ldots, n$, where Q_i are given by (2.21). Indeed, it is not difficult to see that $\frac{\mathrm{d}^2}{\mathrm{d}P_1^2} \sum_{i=1}^n |P_i - P_{i+1}|^2 = 2n > 0$. Consequently, $\sum_{i=1}^n |P_i - P_{i+1}|^2$ takes its minimal value for $P_i = Q_i,\ i = 1, \ldots, n$. □

Although this theorem is a corollary of the following one, its proof is interesting in itself. The following theorem concerns Corollary 1.6.

Theorem 3. *Let $A_1 \ldots A_n$ and $Q_1 \ldots Q_n$ be as in Corollary* 1.6. *Then for any k-outscribed polygon $P_1 \ldots P_n$ to the polygon $A_1 \ldots A_n$ we have*

$$\sum_{i=1}^n |P_i - P_{i+1}|^2 \geq \sum_{i=1}^n |Q_i - Q_{i+1}|^2,$$

that is, $P_i = Q_i$, $i = 1, \ldots, n$ minimizes $\sum_{i=1}^n |P_i - P_{i+1}|^2$.

Proof. Since $d = 2$, one rewrites $\sum_{i=1}^n P_i = S$ into

$$P_1 + P_2 + (P_1 + S_3) + (P_2 + S_4) + \cdots + (P_1 + S_{n-1}) + (P_2 + S_n) = S,$$

from which it follows

$$P_2 = L - P_1, \tag{2.23}$$

where

$$L = \frac{2}{n} S - \frac{2}{n}(W_1 + W_2), \quad W_1 = S_3 + S_5 + \cdots + S_{n-1}, \quad W_2 = S_4 + S_6 + \cdots + S_n.$$

The equation

$$\frac{\mathrm{d}}{\mathrm{d}P_1} \sum_{i=1}^n |P_i - P_{i+1}|^2 = 0 \tag{2.24}$$

becomes

$$P_1 - (L - P_1) - (L - P_1) - (P_1 + S_3) + (P_1 + S_3) - (L - P_1 + S_4) + \cdots + \\ (P_1 + S_{n-1}) - (L - P_1 + S_n) - (L - P_1 + S_n) - P_1 = 0,$$

from which it follows

$$2nP_1 - nL + 2W_1 - 2W_2 = 0,$$

and finally

$$nP_1 = S - 2W_1.$$

Now, by (2.23) and $P_3 = P_1 + S_3, \ldots, P_n = P_2 + S_n$, we have

$$nP_2 = S - 2W_2,$$
$$P_{2i+1} = P_1 + S_{2i+1}, \quad P_{2i+2} = P_2 + S_{2i+2}, \qquad \big(i = 1, \ldots, n/2 - 1\big)$$

(cf. Corollary 1.6). It is easy to see that the second derivative equals $2n > 0$).

This proves Theorem 3. □

The following theorem concerns Corollary 1.7.

Theorem 4. *Let $A_1 \dots A_n$ be any given k-outscribed polygon in $\mathbb{R}^2$, where $1 < k < n$ and $GCD(k,n) = d > 2$. In other words, let* (E_k) *be fulfilled, that is*

$$A_i + A_{i+k} + A_{i+2k} + \cdots + A_{i+(\hat{x}-1)k} = \frac{S}{d}, \qquad \big(i = 1, \dots, d\big) \tag{2.25}$$

where

$$\hat{x} = \frac{n}{d}, \quad S = \sum_{i=1}^{n} A_i. \tag{2.26}$$

Then, for every k-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ we have that

$$\sum_{i=1}^{n} |P_i - P_{i+1}|^2 \geq \sum_{i=1}^{n} |Q_i - Q_{i+1}|^2, \tag{2.27}$$

where $Q_1 \dots Q_n$ is k-outscribed to $A_1 \dots A_n$ and it can be k-outscribed too.

In other words, $P_i = Q_i$, $i = 1, \dots, n$ minimizes $\sum_{i=1}^{n} |P_i - P_{i+1}|^2$.

Proof. The following lemma will be used in the proving procedure.

Lemma 1. *For all $i = 1, \dots, d$ we have*

$$\{A_i, A_{i+k}, A_{i+2k}, \dots, A_{i+(\hat{x}-1)k}\} = \{A_i, A_{i+d}, A_{i+2d}, \dots, A_{i+(\hat{x}-1)d}\}, \tag{2.28}$$

calculating the indices mod n, *that is,*

$$\{i, i+k, i+2k, \dots, i+(\hat{x}-1)k\} = \{i, i+d, i+2d, \dots, i+(\hat{x}-1)d\}. \tag{2.29}$$

Proof of Lemma 1. Let q be the integer with $k = qd$. By (2.25) it is clear that for each $i = 1, \dots, d$ there are no two integers in the sequence

$$i, i+qd, i+2qd, \dots, i+(\hat{x}-1)qd \tag{2.30}$$

equal modulo n. Also, it is clear that for each $i = 1, \dots, d$, between integers

$$i, i+d, i+2d, \dots, i+(\hat{x}-1)d \tag{2.31}$$

we cannot find two identical ones modulo n. But, since d is a divisor of n, that is $\hat{x}d = n$, we conclude (2.29). This proves Lemma 1. □

Let us proceed with the proof of Theorem 4. Let $P_1 \dots P_n$ be a k-outscribed to $A_1 \dots A_n$, that is, Theorem 1 ensures the existence of points $P_1, \dots, P_{d-1}$ such that

$$P_{i+jd} = P_i + S_{i+jd}, \qquad \big(i = 1, \dots, d\big) \quad \text{and} \quad \big(j = 1, \dots, \hat{x}-1\big) \tag{2.32}$$

where P_d is given by

$$P_d = \frac{d}{n} S - \frac{d}{n}(W_1 + \cdots + W_d) - (P_1 + \cdots + P_{d-1}), \tag{2.33}$$

and

$$W_i = S_{i+d} + S_{i+2d} + \cdots + S_{i+(\hat{x}-1)d}, \qquad \big(i = 1, \ldots, d\big). \tag{2.34}$$

It will be shown that the equations

$$\frac{\mathrm{d}}{\mathrm{d}P_j} \sum_{i=1}^{n} |P_i - P_{i+1}|^2 = 0, \qquad \big(j = 1, \ldots, d-1\big)$$

are fulfilled with $P_1, \ldots, P_d$ given by

$$nP_i = S - d(S_{i+d} + S_{i+2d} + \cdots + S_{i+(\hat{x}-1)d}), \qquad \big(i = 1, \ldots, d\big) \tag{2.35}$$

that means, if $P_i = Q_i,\ i = 1, \ldots, n$ (cf. Corollary 1.7).

First, put $j = 1$. Then, $\frac{\mathrm{d}}{\mathrm{d}P_1} \sum_{i=1}^{n} |P_i - P_{i+1}|^2$ can be written as

$$\begin{aligned}
&2(P_1 - P_2) + 0 + \cdots + 0 + 2(P_{d-1} - P_d) - 4(P_d - P_{1+d}) + \\
&2(P_{1+d} - P_{2+d}) + 0 + \cdots + 0 + 2(P_{2d-1} - P_{2d}) - 4(P_{2d} - P_{1+2d}) + \\
&\qquad\vdots \\
&2(P_{1+(\hat{x}-1)d} - P_{2+(\hat{x}-1)d}) + 0 + \cdots + 0 + 2(P_{n-1} - P_n) - 4(P_n - P_1) = 0,
\end{aligned} \tag{2.36}$$

where $d - 1 + (\hat{x} - 1)d = n - 1$, i.e. $\hat{x}d = n$.

Using the relations (2.35) transformed into:

$$P_i = \frac{S}{d} - (P_{i+d} + P_{i+2d} + \cdots + P_{i+(\hat{x}-1)d}), \qquad \big(i = 1, \ldots, d\big) \tag{2.37}$$

it is not difficult to see that the equation (2.36) can be written as

$$\begin{aligned}
&6P_1 - 2P_2 + 2P_{d-1} - 6P_d + \\
&\quad 6(P_{1+d} + P_{1+2d} + \cdots + P_{1+(\hat{x}-1)d}) - \\
&\quad 2(P_{2+d} + P_{2+2d} + \cdots + P_{2+(\hat{x}-1)d}) + \\
&\quad 2(P_{d-1+d} + P_{d-1+2d} + \cdots + P_{d-1+(\hat{x}-1)d}) - \\
&\quad 6(P_{d+d} + P_{d+2d} + \cdots + P_{d+(\hat{x}-1)d}) = 0.
\end{aligned}$$

Thus, the equation (2.36) is satisfied for $P_1, \ldots, P_d$ such that are given by (2.37) when $P_i = Q_i,\ i = 1, \ldots, n$.

Generally, setting some integer $j \in \{2, \ldots, d-1\}$, the equation $\frac{\mathrm{d}}{\mathrm{d}P_j} \sum_{i=1}^{n} |P_i - P_{i+1}|^2 = 0$ will take the form

$$\begin{aligned}
&-2(P_{j-1} - P_j) + 2(P_j - P_{j+1}) + 0 + \cdots + 0 + 2(P_{d-1} - P_d) - 2(P_d - P_{1+d}) - \\
&\quad\vdots \\
&-2(P_{j-1+w} - P_{j+w}) + 2(P_{j+w} - P_{j+1+w}) + 0 + \cdots \\
&\qquad\qquad \cdots + 0 + 2(P_{d-1+w} - P_{\hat{x}d}) - 2(P_{\hat{x}d} - P_1) = 0,
\end{aligned}$$

where we write $w := (\hat{x} - 1)d$.

Using relations (2.37), it is easy to see that the above equation can be rewritten into

$$\begin{aligned}
2P_1 - 2P_{j-1} &+ 4P_j - 2P_{j+1} + 2P_{d-1} - 4P_d \\
&+ 2(P_{1+d} + P_{1+2d} + \cdots + P_{1+(\hat{x}-1)d}) \\
&- 2(P_{j-1+d} + P_{j-1+2d} + \cdots + P_{j-1+(\hat{x}-1)d}) \\
&+ 4(P_{j+d} + P_{j+2d} + \cdots + P_{j+(\hat{x}-1)d}) \\
&- 2(P_{j+1+d} + P_{j+1+2d} + \cdots + P_{j+1+(\hat{x}-1)d}) \\
&+ 2(P_{d-1+d} + P_{d-1+2d} + \cdots + P_{d-1+(\hat{x}-1)d}) \\
&- 4(P_{d+d} + P_{d+2d} + \cdots + P_{d+(\hat{x}-1)d}) = 0.
\end{aligned}$$

As it can be easily seen, all the second derivatives with respect to $P_1, \dots, P_{d-1}$ are positive. So, by (2.36) it follows that $\frac{\mathrm{d}}{\mathrm{d}P_1}\Big(\frac{\mathrm{d}}{\mathrm{d}P_1}\sum_{i=1}^{n}|P_i - P_{i+1}|^2\Big) = 2d + 6d > 0$, since for example, $P_{1+d} = P_1 + S_{1+d}$, $P_{d+d} = P_d + S_{d+d}$ and (2.33) holds.

Thus, $\sum_{i=1}^{n}|P_i - P_{i+1}|^2$ attains its minimum when $P_i = Q_i$, $i = 1, \dots, n$. □

As a corollary of Theorems 1 and 4 we have the following result concerning a special polygon in the set of all k-outscribed polygons to $A_1 \dots A_n$.

Theorem 5. *Let $A_1 \dots A_n$ be as in Theorem* 4. *Then, there exists only one k-outscribed polygon to $A_1 \dots A_n$ and has the property that it can be k-outscribed.*

Proof. The validity of this theorem can be easily seen by Corollary 1.7. Namely, let $Q_1 \dots Q_n$ be a given k-outscribed polygon to $A_1 \dots A_n$ and which one has the property that it can be k-outscribed; therefore (E_k) (in Corollary 1.7) is fulfilled. By this fact and by the relations (2.2), (2.3) it follows that (2.18), (2.19) have to be valid. Thus, $Q_1 \dots Q_n$ must be the same as that one in Corollary 1.7. □

Theorem 6. *Let $Q_1 \dots Q_n$ be as in Theorem* 5. *Then*

$$2 \text{ area of } Q_1 \dots Q_n = |Q_1 + Q_2, \dots, Q_{n-1} + Q_n, Q_n + Q_1|,$$

where

$$\begin{aligned}
Q_i &= \frac{S}{d} - (S_{i+d} + S_{i+2d} + \cdots + S_{i+(\hat{x}-1)d}), \qquad \big(i = 1, \dots, d\big) \\
Q_{i+jd} &= Q_i + S_{i+jd}, \qquad \big(i = 1, \dots, d\big) \quad \text{and} \quad \big(j = 1, \dots, \hat{x} - 1\big).
\end{aligned}$$

The proof of Theorem 6 is analogous to the proof of Corollary 1.2.

The following question arises immediately: "What will be happen when we replace the power 2 in $\sum_{i=1}^{n}|P_i - P_{i+1}|^2$ with $2\alpha \in \mathbb{R}_+$?" We present an answer only in the case of quadrilaterals, $n = 4$.

Theorem 7. *Let $A_1A_2A_3A_4$ be a 2-outscribed quadrilateral, that is $A_1 - A_2 + A_3 - A_4 = 0$. Let $Q_1Q_2Q_3Q_4$ be a 2-outscribed quadrilateral to $A_1A_2A_3A_4$ and it can be 2-outscribed, that is*

$$\begin{aligned} 2Q_1 &= 3A_1 - 2A_2 + A_3, \\ 2Q_2 &= 3A_2 - 2A_3 + A_4, \\ 2Q_3 &= 3A_3 - 2A_4 + A_1, \\ 2Q_4 &= 3A_4 - 2A_1 + A_2. \end{aligned}$$

Then for every 2-outscribed $P_1P_2P_3P_4$ to the $A_1A_2A_3A_4$ and for every real number $\alpha > 0$ we have

$$\sum_{i=1}^{4} |P_i - P_{i+1}|^{2\alpha} \geq \sum_{i=1}^{4} |Q_i - Q_{i+1}|^{2\alpha}.$$

Proof. Since P_1 can be arbitrary and $P_i + P_{i+1} = 2A_i$, $i = 1, 2, 3, 4$, we have

$$\begin{aligned} &P_1(x, y) \text{ arbitrary} \\ &P_2 = 2A_1 - P_1 \\ &P_3 = 2A_2 - 2A_1 + P_1 \\ &P_4 = 2A_3 - 2A_2 + 2A_1 - P_1 \end{aligned}$$

where $A_i(a_i, b_i)$, $i = 1, 2, 3, 4$. Thus

$$\begin{aligned} P_1 - P_2 &= -2A_1 + 2P_1 \\ P_2 - P_3 &= -2A_2 + 4A_1 - 2P_1 \\ P_3 - P_4 &= -2A_3 + 4A_2 - 4A_1 + 2P_1 \\ P_4 - P_1 &= 2A_3 - 2A_2 + 2A_1 - 2P_1 = 2A_4 - 2P_1. \end{aligned}$$

The equation $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\sum_{i=1}^{4} |P_i - P_{i+1}|^{2\alpha} = 0$ becomes

$$F + G + H + K = 0,$$

where

$$\begin{aligned} F &= 2\alpha[(2x - 2a_1)^2 + (2y - 2b_1)^2]^{\alpha-1}(2x - 2a_1), \\ G &= 2\alpha[(-2x - 2a_2 + 4a_1)^2 + (-2y - 2b_2 + 4b_1)^2]^{\alpha-1}(2x + 2a_2 - 4a_1), \\ H &= 2\alpha[(2x - 2a_3 + 4a_2 - 4a_1)^2 \\ &\quad + (2y - 2b_3 + 4b_2 - 4b_1)^2]^{\alpha-1}(2x - 2a_3 + 4a_2 - 4a_1), \\ K &= 2\alpha[(-2x + 2a_4)^2 + (-2y + 2b_4)^2]^{\alpha-1}(2x - 2a_4). \end{aligned}$$

It is not difficult to see that

$$F + G = H + K = 0 \quad \text{if} \quad 2P_1 = 3A_1 - 2A_2 + A_3,$$

that means:

$$\begin{aligned}(2x-2a_1)+(2x-2a_3+4a_2-4a_1)=0 &\iff 2x=3a_1-2a_2+a_3,\\ (2x+2a_2-4a_1)+(2x-2a_4)=0 &\iff 2x=3a_1-2a_2+a_3,\end{aligned}$$

and also

$$\begin{aligned}(2x-2a_1)^2&=(2x-2a_3+4a_2-4a_1)^2,\\ (-2x-2a_2+4a_1)^2&=(-2x+2a_4)^2.\end{aligned}$$

Now, we have $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big(\dfrac{\mathrm{d}}{\mathrm{d}x}\sum_{i=1}^{4}|P_i-P_{i+1}|^{2\alpha}\Big)>0$ when $2x=3a_1-2a_2+a_3$.

Analogously, the same result can be proved for y in $P(x,y)$. This finishes the proof of Theorem 7. □

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