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Characterization of $\mathrm{SL}(2, q)$ by its Non-commuting Graph*

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Abstract. Let G be a non-abelian group and $Z(G)$ be its center. The non-commuting graph $\mathcal{A}_G$ of G is the graph whose vertex set is $G \backslash Z(G)$ and two vertices are joined by an edge if they do not commute. Let $\mathrm{SL}(2, q)$ be the special linear group of degree 2 over the finite field of order q. In this paper we prove that if G is a group such that $\mathcal{A}_G \cong \mathcal{A}_{\mathrm{SL}(2,q)}$ for some prime power $q \geq 2$, then $G \cong \mathrm{SL}(2, q)$.
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1. Introduction and results

Let G be a non-abelian group and $Z(G)$ be its center. One can associate with G a graph whose vertex set is $G \backslash Z(G)$ and two vertices are joined by an edge whenever they do not commute. We call this graph the non-commuting graph of G and it will be denoted by $\mathcal{A}_G$. The non-commuting graph $\mathcal{A}_G$ was first introduced by Paul Erdös [4] to formulate the following question: If every complete subgraph of $\mathcal{A}_G$ is finite, is there a finite bound on the cardinalities of complete subgraphs of $\mathcal{A}_G$?

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Neumann [4] answered positively Erdös question by proving that $|G : Z(G)| = n$ is finite and n is obviously the requested finite bound.

The non-commuting graph has been studied by many people (see e.g., [1], [3] and [5]). It is proved in [7] (resp. in [8]) that if G is a finite group with $\mathcal{A}_G \cong \mathcal{A}_{\mathrm{PSL}(2,q)}$ (resp. $\mathcal{A}_G \cong \mathcal{A}_{A_{10}}$), then $G \cong \mathrm{PSL}(2,q)$ (resp., $G \cong A_{10}$). For any prime power q, let $\mathrm{GL}(2,q)$ (resp. $\mathrm{SL}(2,q)$) be the general (resp. special) linear group of degree 2 over the finite field of order q. In this paper we study the groups whose non-commuting graphs are isomorphic to either $\mathrm{GL}(2,q)$ or $\mathrm{SL}(2,q)$. Our main results are the following.

Theorem 1.1. *Let G be a group such that $\mathcal{A}_G \cong \mathcal{A}_{\mathrm{GL}(2,q)}$ for some prime power $q > 3$. Then $G/Z(G) \cong \mathrm{PGL}(2,q)$, $G' \cong \mathrm{SL}(2,q)$ and $Z(G)$ is of order $q-1$. In particular, if q is even, then $G = G' \times Z(G)$.*

Theorem 1.2. *Let G be a group such that $\mathcal{A}_G \cong \mathcal{A}_{\mathrm{SL}(2,q)}$ for some prime power $q \geq 2$. Then $G \cong \mathrm{SL}(2,q)$.*

For any prime power q, we denote by $\mathrm{PGL}(2,q)$ (resp. $\mathrm{PSL}(2,q)$) the projective general (resp. special) linear group of degree 2 over the finite field of order q.

2. Proofs

Here for convenience, we remind some of the properties of non-commuting graphs and common properties of groups with isomorphic non-commuting graphs.

Let G and H be two non-abelian groups such that $\mathcal{A}_G \cong \mathcal{A}_H$. By Lemma 3.1 of [1], if one of G or H is finite, then so is the other. The order of $\mathcal{A}_G$ is $|G| - |Z(G)|$ and so $|G| - |Z(G)| = |H| - |Z(H)|$. The degree of a vertex x in $\mathcal{A}_G$ is equal to $|G| - |C_G(x)|$. Thus the multisets of degrees of vertices of two graphs $\mathcal{A}_G$ and $\mathcal{A}_H$ are the same.

A non-abelian group G is called an *AC*-group, if the centralizer $C_G(x)$ of every non-central element x of G is abelian.

Recall that a non-empty subset X of the vertices of a simple graph Γ is called independent if every two distinct vertices of X are not joint by an edge in Γ. Thus an independent set S of the non-commuting graph of a group is a set of pairwise commuting non-central elements of the group.

Lemma 2.1. *Let G and H be two finite non-abelian groups with $\mathcal{A}_G \cong \mathcal{A}_H$.*

(1) *If $|G| = |H|$, then the multisets (sets with multiplicities) $\{|C_G(g)| : g \in G\backslash Z(G)\}$ and $\{|C_H(h)| : h \in H\backslash Z(H)\}$ are equal.*

(2) *If G is an AC-group, then H is also an AC-group.*

Proof. (1) It is straightforward, if we note that the set of non-adjacent vertices to a vertex x in the non-commuting graph H is $C_H(x)\backslash Z(H)$, and note that from $|G| = |H|$ we also have $|Z(G)| = |Z(H)|$, since $|H| - |Z(H)| = |G| - |Z(G)|$.

(2) Note that a subgroup S of a non-abelian group K is abelian if and only if either $S\backslash Z(S)$ is empty or $S\backslash Z(S)$ is an independent set in the non-commuting

graph $\mathcal{A}_K$. Let ϕ be a graph isomorphism from $\mathcal{A}_H$ onto $\mathcal{A}_G$. Then it is easy to see that for each $h \in H \backslash Z(H)$,

$$C_H(h) \backslash Z(H) = \phi^{-1}\big(C_G(\phi(h)) \backslash Z(G)\big). \qquad (*)$$

Now since G is an AC-group, $C_G(g)$ is abelian for all $g \in G \backslash Z(G)$ and so it follows from $(*)$ and the remark above that $C_H(h)$ is abelian. Hence H is also an AC-group. □

Finite non-nilpotent AC-groups were completely characterized by Schmidt [6]. We use the following results in our proofs.

Theorem 2.2. ([6, Satz 5.9.]) *Let G be a finite non-solvable group. Then G is an AC-group if and only if G satisfies one of the following conditions:*

1. $G/Z(G) \cong \mathrm{PSL}(2, p^n)$ *and* $G' \cong \mathrm{SL}(2, p^n)$, *where p is a prime and $p^n > 3$.*
2. $G/Z(G) \cong \mathrm{PGL}(2, p^n)$ *and* $G' \cong \mathrm{SL}(2, p^n)$, *where p is a prime and $p^n > 3$.*
3. $G/Z(G) \cong \mathrm{PSL}(2, 9)$ *and G' is a covering group of A_6. In particular, G' is isomorphic to*

$$\begin{aligned} \mathcal{A} \cong < c_1, c_2, c_3, c_4, k \mid c_1^3 = c_2^2 = c_3^2 = c_4^2 = (c_1c_2)^3 = (c_1c_3)^2 = \\ = (c_2c_3)^3 = (c_3c_4)^3 = k^3, (c_1c_4)^2 = k, \\ c_2c_4 = k^3c_4c_2, kc_i = c_ik(i = 1, \ldots, 4), k^6 = 1 > . \end{aligned}$$

4. $G/Z(G) \cong \mathrm{PGL}(2, 9)$ *and* $G' \cong \mathcal{A}$.

For a finite simple graph Γ, we denote by $\omega(\Gamma)$ the maximum size of a complete subgraph of Γ. So $\omega(\mathcal{A}_G)$ is the maximum number of pairwise non-commuting elements in a finite non-abelian group G.

Theorem 2.3. (Satz 5.12 of [6]) *Let G be a finite non-abelian solvable group. Then G is an AC-group if and only if G satisfies one of the following properties:*

1. *G is non-nilpotent and it has an abelian normal subgroup N of prime index and* $\omega(\mathcal{A}_G) = |N : Z(G)| + 1$.
2. *$G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively and F and K are abelian subgroups of G; and* $\omega(\mathcal{A}_G) = |F : Z(G)| + 1$.
3. *$G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively; and K is an abelian subgroup of G, $Z(F) = Z(G)$, and $F/Z(G)$ is of prime power order; and* $\omega(\mathcal{A}_G) = |F : Z(G)| + \omega(\mathcal{A}_F)$.
4. $G/Z(G) \cong S_4$ *and V is a non-abelian subgroup of G such that $V/Z(G)$ is the Klein 4-group of $G/Z(G)$; and* $\omega(\mathcal{A}_G) = 13$.
5. $G = A \times P$, *where A is an abelian subgroup and P is an AC-subgroup of prime power order.*

Proof of Theorems 1.1 and 1.2. Let $q_1 = p_1^{n_1} > 3$ and $q_2 = p_2^{n_2} \geq 2$, where p_1 and p_2 are two prime numbers. Let $M_1 = \mathrm{GL}(2,q_1)$ and $M_2 = \mathrm{SL}(2,q_2)$ and suppose that G_1 and G_2 are two groups such that $\mathcal{A}_{G_i} \cong \mathcal{A}_{M_i}$ for $i = 1,2$.
If $q_1 = 2$, then $M_2 \cong S_3$ is the symmetric group of degree 3 and so by Proposition 3.2 of [1], $G_2 \cong M_2$. If $q_2 = 3$, then M_2 is a group of order 24 and its center has order 2. As there is some element g with $|C_{G_2}(g)| = 6$, we see that there is no normal Sylow 3-subgroup in G_2. Hence $G_2/Z(G_2) \cong A_4$. So either $G_2 \cong M_2$ or $\mathbb{Z}_2 \times A_4$. But as there are elements $h \in G_2$ with $|C_{G_2}(h)| = 4$, we have $G_2 \cong M_2$.

Now let $q_2 > 3$. If q_2 is even, then $\mathrm{PSL}(2,q_2) \cong M_2$ and so $\mathcal{A}_{G_2} \cong \mathcal{A}_{\mathrm{PSL}(2,q_2)}$. Then by Corollary 5.3 of [1], $G_2 \cong \mathrm{PSL}(2,q_2) \cong M_2$. Therefore we may assume that $q_2 \geq 5$ is odd.

By Proposition 4.3 of [1], $|G_i| = |M_i|$ for $i = 1,2$. By Lemma 3.5 of [1], M_i's are AC-groups and so by Lemma 2.1(2) G_i's are also AC-groups. Now since $\mathcal{A}_{G_i} \cong \mathcal{A}_{M_i}$ and $|G_i| = |M_i|$, by Lemma 2.1 we have the following equality between multisets

$$W_i = \{|C_{G_i}(x)| \mid x \in G_i \backslash Z(G_i)\} = \{|C_{M_i}(g)| \mid g \in M_i \backslash Z(M_i)\}, \quad i = 1,2.$$

Also, since the order of two graphs $\mathcal{A}_{G_i}$ and $\mathcal{A}_{M_i}$ are the same, we have that $|G_i| - |Z(G_i)| = |M_i| - |Z(M_i)|$ and so $|Z(G_i)| = |Z(M_i)|$ $(i = 1,2)$. Therefore, it follows from Propositions 3.14 and 3.26 of [1] that the multiset W_1 (resp. W_2) consists of three distinct integers $(q_1-1)^2$ (resp. $(q_2-1)/2$), q_1^2-1 (resp. $(q_2+1)/2$) and $q_1(q_1-1)$ (resp. q_2) with multiplicities $\frac{q_i(q_i+1)}{2}$, $\frac{q_i(q_i-1)}{2}$ and q_i+1, respectively.
We claim that both groups G_1 and G_2 are not nilpotent. Suppose, for a contradiction, that G_i is nilpotent, then so is $G_i/Z(G_i)$. Therefore $G_i/Z(G_i)$ has only one Sylow p_i-subgroup. Since W_1 (resp., W_2) contains $q_i + 1$ elements all equal to $q_1(q_1-1)$ (resp., q_2), there exist two non-central elements x_1 and y_1 in G_1 (resp., x_2 and y_2 in G_2) such that $C_{G_1}(x_1) \neq C_{G_1}(y_1)$ and $|C_{G_1}(x_1)| = |C_{G_1}(y_1)| = q_1(q_1-1)$ (resp., $C_{G_2}(x_2) \neq C_{G_2}(y_2)$ and $|C_{G_2}(x_2)| = |C_{G_2}(y_2)| = 2q_2$). Since $C_{G_i}(x_i)/Z(G_i)$ and $C_{G_i}(y_i)/Z(G_i)$ are of the same order q_i, they are Sylow p_i-subgroups of $G_i/Z(G_i)$. It follows that $C_{G_i}(x_i)/Z(G_i) = C_{G_i}(y_i)/Z(G_i)$ and so $C_{G_i}(x_i) = C_{G_i}(y_i)$, a contradiction.
Now we prove that both G_1 and G_2 cannot be solvable. Suppose, for a contradiction, that G_i's are solvable. Then since G_i are not nilpotent, it follows from Theorem 2.3 that G_i's satisfy one of properties (1)–(4) in Theorem 2.3. Since $q_i > 3$ is a prime power and q_2 is odd, both of $|G_1/Z(G_1)| = q_1(q_1^2-1)$ and $|G_2/Z(G_2)| = \frac{q_2(q_2^2-1)}{2}$ cannot equal to $|S_4| = 24$. Therefore G_i's do not satisfy (4). If G_i satisfies either (1) or (2), then W_i contains only two distinct elements, since in the case (1), if $x \in N\backslash Z(G_i)$, then $C_{G_i}(x) = N$; and if $x \in G\backslash N$ then $C_N(x) = Z(G_i)$; so $|C_{G_i}(x)| \in \{|G_i : N||Z(G_i)|, |N|\}$ for every non-central element $x \in G_i$, and in the case (3), $|C_{G_i}(x)| \in \{|K|, |F|\}$. This is not possible, since W_i has exactly three distinct elements.
Finally, suppose that G_i satisfies (3). Note that $C_{G_i}(x) = C_F(x)$ for every non-central element $x \in F$ and $C_{G_i}(x)$ is equal to the conjugate of K which contains the non-central element x. It follows that the three distinct elements of the multiset

$W_i' = \{w/|Z(G)| \mid w \in W_i\}$ are $|K/Z(G_i)|, r^k, r^\ell$, where $|F/Z(G)| = r^m$ and r is a prime number. This is impossible, since no two of the numbers q_1, q_1+1 or q_1-1 (resp., q_2, $(q_2+1)/2$ or $(q_2-1)/2$) can simultaneously be powers of the same prime.

Hence G_i's are finite non-solvable AC-groups. By Theorem 2.2, G_i's satisfy one of the conditions (1)–(4) stated in Theorem 2.2. If G_i satisfies (3), then as A_6 has self-centralizing elements of order 4 and 5, G_i contains two elements x_i, y_i such that $|\frac{C_{G_i}(x_i)}{Z(G_i)}| = 4$ and $|\frac{C_{G_i}(y_i)}{Z(G_i)}| = 5$. This implies that $q_1 \in \{4,5\}$ and $q_2 = 9$. Therefore $|G_1/Z(G_1)| = 4\cdot(4^2-1)$ or $5\cdot(5^2-1)$, which is impossible, since $|G_1/Z(G_1)| = |\mathrm{PSL}(2,9)| = \frac{9\cdot(9^2-1)}{2}$. Since $M_2 = \mathrm{SL}(2,9)$, $|Z(M_2)| = 2$. But 3 divides $Z(G_2)$ by Theorem 2.2, a contradiction.

If G_i satisfies (4), then as PGL$(2,9)$ contains self-centralizing elements of order 8 and 10, G_i contains two elements t_i and s_i such that $|\frac{C_{G_i}(t_i)}{Z(G_i)}| = 8$ and $|\frac{C_{G_i}(s_i)}{Z(G_i)}| = 10$. It follows that $\{8,10\} \subset \{q_2, \frac{q_2-1}{2}, \frac{q_2-1}{2}\}$, which is a contradiction as q_2 is a prime power; and for $i=1$, it follows that $q_1 = 9$. Hence $|Z(M_1)| = 8$. But 3 divides $|Z(G_1)|$, a contradiction. Thus G_i does not satisfy both (3) and (4).

Now suppose that G_i satisfies either (1) or (2). The group PGL$(2,r^m)$ (resp. PSL$(2,r^m)$) has a partition $\mathcal{P}$ consisting of r^m+1 Sylow r-subgroups, $\frac{(r^m+1)r^m}{2}$ cyclic subgroups of order r^m-1 (resp. $\frac{r^m-1}{\gcd(2,r^m-1)}$) and $\frac{(r^m-1)r^m}{2}$ cyclic subgroups of order r^m+1 (resp. $\frac{r^m+1}{\gcd(2,r^m-1)}$) (see pp. 185–187 and p. 193 of [2]). Now [6, (5.3.3) in p. 112] states that if $x \in G_i \backslash Z(G_i)$, then $C_{G_i}(x_i)/Z(G_i)$ belongs to $\mathcal{P}$. Suppose that $G_i/Z(G_i) \cong \mathrm{PGL}(2,r^m)$ (resp. PSL$(2,r^m)$). Thus there exist elements $g_{i1}, g_{i2}, g_{i3} \in G_i\backslash Z(G_i)$ such that $|C_{G_i}(g_{i1})|/|Z(G_i)| = r^m$, $|C_{G_i}(g_{i2})|/|Z(G_i)| = r^m-1$ (resp. $\frac{r^m-1}{\gcd(2,r^m-1)}$), $|C_{G_i}(g_{i3})|/|Z(G_i)| = r^m+1$ (resp. $\frac{r^m+1}{\gcd(2,r^m-1)}$).

Therefore, if $G_i/Z(G_i) \cong \mathrm{PGL}(2,r^m)$ (resp. PSL$(2,r^m)$), then $\{q_1-1, q_1, q_1+1\} = \{r^m-1, r^m, r^m+1\}$ (resp. $\{\frac{r^m-1}{\gcd(2,r^m-1)}, r^m, \frac{r^m+1}{\gcd(2,r^m-1)}\}$ and $\{\frac{q_2-1}{2}, \frac{q_2+1}{2}, q_2\} = \{r^m-1, r^m, r^m+1\}$ (resp. $\{\frac{r^m-1}{\gcd(2,r^m-1)}, r^m, \frac{r^m+1}{\gcd(2,r^m-1)}\}$).

It follows that, if $G_2/Z(G_2) \cong \mathrm{PGL}(2,r^m)$ then $q_2 = r^m+1$, $\frac{q_2+1}{2} = r^m$ and $\frac{q_2-1}{2} = r^m-1$. Since $q_2 \geq 5$, we have a contradiction as $3 \leq q_2 - \frac{q_2-1}{2} = r^m+1-r^m+1 = 2$. Hence $G_2/Z(G_2) \cong \mathrm{PSL}(2,r^m)$, $G_2' \cong \mathrm{SL}(2,r^m)$ and $r^m = q_2$. Now since $|G_2'| = |G_2| = |M_2|$, we have that $G_2 \cong M_2 = \mathrm{SL}(2,q_2)$. This completes the proof of Theorem 1.2.

Now if $G_1/Z(G_1) \cong \mathrm{PGL}(2,r^m)$ (resp. PSL$(2,r^m)$), it follows that $q_1 = r^m$ (resp. $q_1 = 2^m$). Since PSL$(2,2^m) \cong$ PGL$(2,2^m)$, we have if G_1 satisfies either (1) or (2), then $G_1/Z(G_1) \cong \mathrm{PGL}(2,q_1)$ and $G_1' \cong \mathrm{SL}(2,q_1)$.

Therefore G_1 is a group satisfying the following conditions:

$$G_1/Z(G_1) \cong \mathrm{PGL}(2,q_1) \ \ (\bullet), \ \ G_1' \cong \mathrm{SL}(2,q_1) \quad \text{and} \ \ |Z(G_1)| = q_1 - 1.$$

If $q_1 = 2^m$ for some integer $m > 1$, then SL$(2,q_1) \cong$ PGL$(2,q_1) \cong$ PSL$(2,q_1)$. Thus as PSL$(2,q_1)$ is a non-abelian simple group, it follows from $(\bullet)$ that $G_1 =$

$G_1'Z(G_1)$; and since G_1' is also non-abelian simple, $G_1' \cap Z(G_1) = 1$. Therefore $G_1 = G_1' \times Z(G_1)$. This completes the proof of Theorem 1.1. □

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References

[1] Abdollahi, A.; Akbari, S.; Maimani, H. R.: *Non-commuting graph of a group*. J. Algbera **298** (2006), 468–492. Zbl 1105.20016

[2] Huppert, B.: *Endliche Gruppen. I.* Springer-Verlag, Berlin 1967. Zbl 0217.07201

[3] Moghaddamfar, A. R.; Shi, W. J.; Zhou, W.; Zokayi, A. R.: *On the non-commuting graph associated with a finite group*. Sib. Math. J. **46**(2) (2005), 325–332. Zbl 1096.20027

[4] Neumann, B. H.: *A problem of Paul Erdős on groups*. J. Aust. Math. Soc., Ser. A **21** (1976), 467–472. Zbl 0333.05110

[5] Pyber, L.: *The number of pairwise non-commuting elements and the index of the centre in a finite group*. J. Lond. Math. Soc., II. Ser. **35**(2) (1987), 287–295. Zbl 0588.20016

[6] Schmidt, R.: *Zentralisatorverbände endlicher Gruppen*. Rend. Sem. Mat. Univ. Padova **44** (1970), 97–131. Zbl 0243.20039

[7] Wang, L.; Shi, W.: *A new characterization of $L_2(q)$ by its noncommuting graph*. Front. Math. China **2**(1) (2007), 143–148.

[8] Wang, L.; Shi, W.: *A new characterization of A_{10} by its noncommuting graph*. Commun. Algebra **36**(2) (2008), 523–528. Zbl pre05261424

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