How far from f can $B_n(f)$ be in $LIP([0,1])$?

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Abstract

In this paper we show that all Bernstein polynomials $B_n(f)$ of a Lipschitz function $f : [0,1] \longrightarrow \mathbb{R}$ stay inside the ball, in the Banach space $LIP([0,1])$, centered in f and having the radius $2\|f\|$.

1 INTRODUCTION.

In [1], on page 401, the Banach space of Lipschitz functions is considered. It is mentioned that this space plays an important role in connection with the study of certain singular integral operators (particularly those which arise in the theory of partial differential operators) and that very little seems to be known about its properties.

Meanwhile a lot of papers have appeared yielding much information about isomorphic classification (see [2], [3], [8]), weak* sequential convergence (see [2]) and duals (see [4], [10]) of these spaces. Recently, even a book was dedicated to the study of these spaces (see [11]).

On the other hand, the Bernstein polynomials are very important in approximation theory since a continuous function defined on a compact interval is the uniform limit of the sequence of its Bernstein polynomials and because they mimic the behavior of the function to a remarkable degree. It took more than twenty years before results concerning the rate of convergence of $B_n(f)$ to f appeared due to Popoviciu (see [9]) and Kac (see [6]). Recently H. Gzyl & J.L. Palacios (see [5]) and P. Mahté (see [7]) obtained results about the rates of convergence via probabilistic arguments.

For example, one can show the following:

Theorem 1 *If the function $f:[0,1]\to\mathbb{R}$ is Hölder continuous with exponent α and constant L then*

$$|f(x)-B_n(f)(x)|\leq L\left(\frac{x(1-x)}{n}\right)^{\alpha/2}$$

for all $n\in\mathbb{N}$ and all $x\in[0,1]$.

Therefore it is natural to study what is the relation between $B_n(f)$ and f as elements of $LIP([0,1])$.

More precisely, the following questions arise:

1. How far from f can $B_n(f)$ be in $LIP([0,1])$?

In this paper we give an answer for this question, namely we show that all Bernstein polynomials $B_n(f)$ of a Lipschitz function $f:[0,1]\to\mathbb{R}$ stay inside the ball, in the Banach space $LIP([0,1])$, centered in f and having the radius $2\|f\|$.

Since it seems that the radius that we obtained, $2\|f\|$, is not the optimal one, would be of interest to study what is the smallest radius of a ball centered into f that contains all $B_n(f)$ and eventually to answer the following two questions:

2. Is it true that $B_n(f)$ converges to f in $LIP([0,1])$, for all $f\in LIP([0,1])$?

3. If the answer for 2 is no, describe the functions for which the convergence is valid.

2 PRELIMINARIES.

Definition 2 *Given two metric space, (X,d) and (Y,d'), a function $f:X\to Y$ is said to be Hölder continuous with exponent α, for some $\alpha\in(0,1]$, and constant $L>0$ if*

$$d'(f(x),f(y))\leq Ld(x,y)^{\alpha},$$

for all $x,y\in[0,1]$. When $\alpha=1$, f is said to be Lipschitz.

Definition 3 *Given a metric space, $Lip(X,d)$ denotes the Banach space of Lipschitz bounded real valued functions on X with norm given by*

$$\|f\|=\max\{\|f\|_{\infty},\|f\|_d\},$$

where

$$\|f\|_{\infty}=\sup\{|f(x)|:\ x\in X\}$$

and

$$\|f\|_d=\sup\{\frac{|f(x)-f(y)|}{d(x,y)}:\ x,y\in X,x\neq y\}.$$

Remark 4 For $(X,d) = ([0,1], |\,|)$, we denote $Lip([0,1], |\,|)$ by $LIP([0,1])$.

Definition 5 *Let us consider $f : [0,1] \to \mathbb{R}$. The n^{th} $(n \in \mathbb{N}^*)$ Bernstein polynomial of f is given by*

$$B_n(f)(x) = \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

3 THE RESULT.

Theorem 6 *Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function. Then*

$$\|f - B_n(f)\| \leq 2\,\|f\|,$$

for every $n \in \mathbb{N}^*$.

Proof. First we show that:

$$\|f - B_n(f)\|_\infty \leq 2\,\|f\|_\infty, \tag{1}$$

for every $n \in \mathbb{N}^*$.

Indeed, for every $x \in [0,1]$, we have

$$|(f - B_n(f))(x)| = \left| f(x) - \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k} \right| =$$

$$= \left| f(x) \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} - \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k} \right| =$$

$$= \left| \sum_{k=0}^{n} [f(x) - f(\frac{k}{n})] C_n^k x^k (1-x)^{n-k} \right| \leq$$

$$\leq \sum_{k=0}^{n} \left| f(x) - f(\frac{k}{n}) \right| C_n^k x^k (1-x)^{n-k} \leq$$

$$\leq \sum_{k=0}^{n} (|f(x)| + \left| f(\frac{k}{n}) \right|) C_n^k x^k (1-x)^{n-k} \leq$$

$$\leq \sum_{k=0}^{n} 2\left\|f\right\|_{\infty} C_n^k x^k (1-x)^{n-k} = 2\left\|f\right\|_{\infty} (x+1-x)^n = 2\left\|f\right\|_{\infty},$$

for every $n \in \mathbb{N}^*$.

Now we prove that

$$\left\|f - B_n(f)\right\|_d \leq 2\left\|f\right\|_d \tag{2}$$

for every $n \in \mathbb{N}^*$.

We will use the following two obvious relations:

$$\sum_{k=0}^{n}\sum_{l=0}^{k} a_{l,k} = \sum_{l=0}^{n}\sum_{j=0}^{n-l} a_{l,l+j}. \tag{3}$$

$$\sum_{l=0}^{n}\sum_{j=0}^{n-l} a_{l,j} = \sum_{j=0}^{n}\sum_{l=0}^{n-j} a_{l,j}. \tag{4}$$

For $x, y \in [0,1]$, $x \neq y$, we have:

$$\frac{|(f - B_n(f))(y) - (f - B_n(f))(x)|}{|y-x|} \leq \frac{|f(y) - f(x)| + |B_n(f)(y) - B_n(f)(x)|}{|y-x|} \leq$$

$$\leq \left\|f\right\|_d + \frac{\left|\sum\limits_{k=0}^{n} C_n^k f(\frac{k}{n}) y^k (1-y)^{n-k} - \sum\limits_{l=0}^{n} C_n^l f(\frac{l}{n}) x^l (1-x)^{n-l}\right|}{|y-x|} =$$

$$= \left\|f\right\|_d + \frac{1}{|y-x|} \left| \sum_{k=0}^{n} C_n^k f(\frac{k}{n})(x+y-x)^k (1-y)^{n-k} - \right.$$
$$\left. - \sum_{l=0}^{n} C_n^l f(\frac{l}{n}) x^l ((y-x)+(1-y))^{n-l} \right| =$$

$$= \left\|f\right\|_d + \frac{1}{|y-x|} \left| \sum_{k=0}^{n} C_n^k f(\frac{k}{n})(1-y)^{n-k} \sum_{l=0}^{k} C_k^l x^l (y-x)^{k-l} - \right.$$
$$\left. - \sum_{l=0}^{n} C_n^l f(\frac{l}{n}) x^l \sum_{j=0}^{n-l} C_{n-l}^j (y-x)^j (1-y)^{n-l-j} \right| =$$

$$= \|f\|_d + \frac{1}{|y-x|}\Bigg|\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{n!}{k!(n-k)!}\frac{k!}{l!(k-l)!}f(\frac{k}{n})x^l(1-y)^{n-k}(y-x)^{k-l} -$$

$$-\sum_{l=0}^{n}\sum_{j=0}^{n-l}\frac{n!}{l!(n-l)!}\frac{(n-l)!}{j!(n-l-j)!}f(\frac{l}{n})x^l(1-y)^{n-l-j}(y-x)^j\Bigg| = \text{(using (3))}$$

$$= \|f\|_d + \frac{1}{|y-x|}\Bigg|\sum_{l=0}^{n}\sum_{j=0}^{n-l}\frac{n!}{(n-l-j)!}\frac{1}{l!}\frac{1}{j!}f(\frac{l+j}{n})x^l(1-y)^{n-l-j}(y-x)^j -$$

$$-\sum_{l=0}^{n}\sum_{j=0}^{n-l}\frac{n!}{l!}\frac{1}{j!}\frac{1}{(n-l-j)!}f(\frac{l}{n})x^l(1-y)^{n-l-j}(y-x)^j\Bigg| =$$

$$= \|f\|_d + \frac{1}{|y-x|}\Bigg|\sum_{l=0}^{n}\sum_{j=0}^{n-l}\frac{n!}{(n-l-j)!}\frac{1}{l!}\frac{1}{j!}f(\frac{l+j}{n})x^l(1-y)^{n-l-j}(y-x)^j -$$

$$-\sum_{l=0}^{n}\sum_{j=0}^{n-l}\frac{n!}{l!}\frac{1}{j!}\frac{1}{(n-l-j)!}f(\frac{l}{n})x^l(1-y)^{n-l-j}(y-x)^j\Bigg| =$$

$$= \|f\|_d + \frac{\left|\sum_{l=0}^{n}\sum_{j=0}^{n-l}\frac{n!}{(n-l-j)!}\frac{1}{l!}\frac{1}{j!}x^l(1-y)^{n-l-j}(y-x)^j(f(\frac{l+j}{n})-f(\frac{l}{n}))\right|}{|y-x|} \leq$$

$$\leq \|f\|_d + \|f\|_d\,\frac{\left|\sum_{l=0}^{n}\sum_{j=0}^{n-l}\frac{n!}{(n-l-j)!}\frac{1}{l!}\frac{1}{j!}x^l(1-y)^{n-l-j}(y-x)^j(\frac{j}{n})\right|}{|y-x|} = \text{(using (4))}$$

$$= \|f\|_d + \|f\|_d\,\frac{\left|\sum_{j=0}^{n}\sum_{l=0}^{n-j}\frac{n!}{(n-l-j)!}\frac{1}{l!}\frac{1}{j!}x^l(1-y)^{n-l-j}(y-x)^j(\frac{j}{n})\right|}{|y-x|} =$$

$$= \|f\|_d + \|f\|_d\,\frac{\left|\sum_{j=0}^{n}\frac{j}{n}\frac{n!}{(n-j)!}\frac{1}{j!}(y-x)^j\sum_{l=0}^{n-j}\frac{1}{l!}\frac{(n-j)!}{(n-l-j)!}x^l(1-y)^{n-l-j}\right|}{|y-x|} =$$

$$= \|f\|_d + \frac{\|f\|_d}{|y-x|} \left| \sum_{j=0}^{n} \frac{j}{n} C_n^j (y-x)^j (1-(y-x))^{n-j} \right| =$$

$$= \|f\|_d + \frac{\|f\|_d}{|y-x|} \left| (y-x) \sum_{j=1}^{n} C_{n-1}^{j-1} (y-x)^{j-1} (1-(y-x))^{(n-1)-(j-1)} \right| =$$

$$= \|f\|_d + \frac{\|f\|_d}{|y-x|} |y-x| \left[y-x + (1-(y-x)) \right]^{n-1} = \|f\|_d + \|f\|_d = 2 \|f\|_d .$$

From (1) and (2) we obtain our conclusion. ■

Remark 7 Let x_0 be a fixed element of $[0,1]$. Then

$$LIP_0([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is Lipschitz and } f(x_0) = 0\}$$

endowed with $\| \|_d$ is a Banach spaces. The above result is true if $LIP([0,1])$ is replaced by $LIP_0([0,1])$.

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