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Graphical introduction to classical Lie algebras

Rafael Díaz & Eddy Pariguan

Abstract

We develop a graphical notation to introduce classical Lie algebras. Although this paper deals with well-known results, our pictorial point of view is slightly different to the traditional one. Our graphical notation is elementary and easy to handle, thus it provides an effective tool for computations with classical Lie algebras. Moreover, it may be regarded as a first and foundational step in the process of uncovering the categorical meaning of Lie algebras.

1 Introduction

A first step in the study of an arbitrary category $\mathcal{C}$ is to define the set $\mathcal{S}(\mathcal{C})$ of isomorphisms classes of *simple* objects in $\mathcal{C}$. For example in **Set** the category of sets, the simple objects $S(\mathbf{Set})$ are the empty set $\emptyset$ and $\{\emptyset\}$. In **Top** the category of topological spaces, the set $S(\mathbf{Top})$ are the homeomorphism classes of connected topological spaces. An object y of an abelian category $\mathcal{C}$ is said to be simple if in any exact sequence

$$0 \to x \to y \to z \to 0,$$

either x is isomorphic to 0 or z is isomorphic to 0, see [7]. It is a remarkable fact that non-equivalent categories may very well have equivalent sets of simple objects. Let us introduce a list of categories that at first seem to be utterly unrelated and yet the corresponding sets of simple objects are deeply connected. We denote by **Group** the category whose objects are groups and whose morphisms are group homomorphisms. We let **LieGroup**, (see Section 2) denote the subcategory of **Group** whose objects are finite dimensional complex Lie groups. Morphism in **LieGroup** are smooth group homomorphisms. We define **FinGroup** to be the full subcategory of **Group** whose objects are finite groups. **WeylGroup** denotes the set of isomorphisms classes of Weyl groups, which can be taken to be $A_n = S_{n+1}$, $B_n = \mathbb{Z}_2^n \rtimes S_n$, $D_n = \mathbb{Z}_2^{n-1} \rtimes S_n$, where S_n is the group of permutations in n letters, and E_6, E_7, E_8, F_4, G_2 are the so called exceptional Weyl groups.

We proceed to introduce the corresponding sets of simple objects. $S(\mathbf{Group})$ denotes the set of isomorphisms classes of groups having no proper normal subgroups. The classification of finite simple groups is a notoriously difficult problem with a fascinating history. It was established in 1981 that every simple finite group is isomorphic to one of the following list: A cyclic group of prime order. An alternating group $A_n \subset S_n$ for $n \geq 5$. A finite group of Lie type (finite analogues of the classical Lie group). A list of 26 sporadic simple groups. The largest sporadic group is called the Monster and appears naturally as the automorphism group of a vertex algebra. $S(\mathbf{LieGroup})$ denotes the set of isomorphisms classes of Lie groups which are simple as groups and also are connected and simply connected. Consider de $\mathbb{C}$-vector space $\mathbb{C}^n$. $Gl_n(\mathbb{C})$ is the group of linear automorphism of $\mathbb{C}^n$ as is defined by $Gl_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) |\ \det(A) \neq 0\}$. The subgroup $SL_n(\mathbb{C})$ of $Gl_n(\mathbb{C})$ consists of volume preserving automorphisms of $\mathbb{C}^n$,

$$SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) |\ \det(A) = 1\}.$$

The groups $SO_n(\mathbb{C}) \subset SL_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C}) \subset SL_{2n}(\mathbb{C})$ are determined by fixing a symmetric $\langle\ ,\ \rangle$ and a skew-symmetric ω non-degenerated bilineal form on $\mathbb{C}^n$, respectively. Then

$$SO_n(\mathbb{C}) = \{A \in SL_n(\mathbb{C}) |\ \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{C}^n\}$$

and

$$Sp_{2n}(\mathbb{C}) = \{A \in SL_{2n}(\mathbb{C}) |\ \omega(Ax, Ay) = \omega(x, y) \text{ for all } x, y \in \mathbb{C}^n\}.$$

$S(\mathbf{FinGroup})$ denotes the set of isomorphisms classes of finite simple groups.

LieAlg denotes the category whose objects are finite dimensional complex Lie algebras (see Section 2), morphism are Lie algebra homomorphism. $S(\mathbf{LieAlg})$ is the set of isomorphisms classes of simple Lie algebras, i.e., Lie algebras having no proper ideals.

Root denotes the category of root systems. Objects in **Root** are triples $(V, \langle\ ,\ \rangle, \Phi)$ such that

- $(V, \langle\ ,\ \rangle)$ is an Euclidean space.
- $\Phi \subset V$ is a finite set generating V.
- If $\alpha \in \Phi$ then $-\alpha \in \Phi$, but $k\alpha \notin \Phi$ for any real number $k \pm 1$. Element of Φ are called roots.
- For $\alpha \in \Phi$ the reflection S_α in the hyperplane $\alpha^\perp$ orthogonal to α given by $\alpha^\perp = \{x \in V : \langle x, \alpha\rangle = 0\}$ maps Φ to itself.
- For $\alpha, \beta \in \Phi$, $A_{\alpha,\beta} = 2\dfrac{\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z}$.

A morphism in **Root** from $(V_1, \langle\ ,\ \rangle_1, \Phi_1)$ to $(V_2, \langle\ ,\ \rangle_2, \Phi_2)$ is a linear transformations $T : V_1 \longrightarrow V_2$ such that $\langle T(x), T(y)\rangle_2 = \langle x, y\rangle_1$ for all $x, y \in V_1$, and $T(\Phi_1) \subset \Phi_2$.

The direct sum of root systems is defined as follows: suppose that $(V_i, \langle\ ,\ \rangle_{V_i}, \Phi_i)$, $i = 1, \ldots, n$ are root systems, then the Euclidean space of the direct sum is $V = \bigoplus_{i=1}^{n} V_i$, with inner product

$$\langle\ ,\ \rangle_V = \sum_{i=1}^{n} \langle\ ,\ \rangle_{V_i}.$$

The roots of the direct sum are $\Phi = \bigsqcup_{i=1}^{n} \Phi_i$. The triple $(V, \langle\ ,\ \rangle_V, \Phi)$ is a root system. $S(\textbf{Root})$ is the set of isomorphisms classes of simple root systems, i.e., root systems which are not isomorphic to the direct sum of two non-vanishing root systems.

Dynkin denotes the category of Dynkin diagrams. Objects in **Dynkin** are called Dynkin diagrams and are non-directed graphs Δ with the following properties

- The set V_Δ of vertices of Δ is equal to $\{1, \ldots, n\}$ for some $n \geq 1$.
- The number of edges joining two vertices in Δ is $0, 1, 2$ or 3.
- If vertices i and j are joined by 2 or 3 edges, then an arrow is chosen pointing either from i to j, or from j to i.
- The quadratic form

  $$Q(x_1, x_2, \cdots, x_n) = 2\sum_{i=1}^{n} x_i^2 - \sum_{i \neq j} \sqrt{n_{ij}}\, x_i x_j$$

  is positive definite where (n_{ij}) is the adjacency matrix of Δ, i.e., n_{ij} equal the number of edges from vertex i to vertex j.

Morphism in **Dynkin** from diagram Δ_1 to diagram Δ_2 consists of maps
$\rho : V_{\Delta_1} \to V_{\Delta_2}$ such that $Q_2(x_{\rho(1)}, x_{\rho(2)}, \cdots, x_{\rho(n)}) = Q_1(x_1, x_2, \cdots, x_n)$.
$S(\textbf{Dynkin})$ denotes the set of of isomorphisms classes of connected Dynkin diagrams. Next theorem gives an explicit characterization of $S(\textbf{Dynkin})$.

Theorem 1. $S(\textbf{Dynkin})$ *consists of the Dynkin diagrams included in the following list*

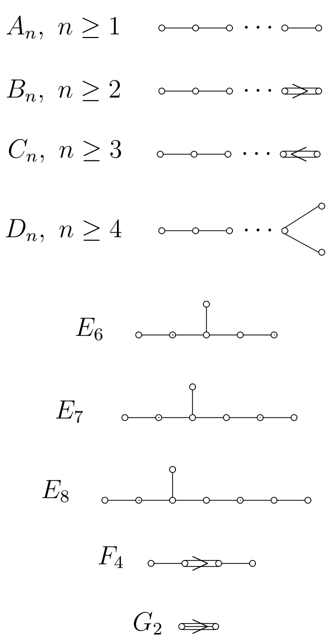


Figure 1: Simple Dynkin diagrams.

We enunciate the following fundamental

Theorem 2. *1.* $S(\mathbf{FinGroup}) \subset S(\mathbf{LieGroup}) \subset S(\mathbf{Group})$.

2. $S(\mathbf{LieGroup}) \cong S(\mathbf{LieAlg}) \cong S(\mathbf{Root}) \cong S(\mathbf{Dynkin}) \to \mathbf{WeylGroup}$.

Part 1 of Theorem 2 is obvious. Although we shall not give a complete proof of part 2 the reader will find in the body of this paper many statements that shed light into its meaning. The map $S(\mathbf{Dynkin}) \longrightarrow \mathbf{WeylGroup}$ is surjective but fails to be injective. Diagrams B_n and C_n of the list above have both $\mathbb{Z}_2^n \rtimes S_n$ as its associated Weyl group.

2 Lie Algebras

We proceed to consider in details the category of Lie algebras. First we recall the notion of a Lie group.

Definition 3. *A group (G, m) is said to be a complex Lie group if*

1. *G is a finite dimensional complex manifold.*

2. *the map $m : G \times G \longrightarrow G$ given by $m(a,b) = ab$ for all $a, b \in G$ is analytic smooth.*

3. *The map $I : G \to G$ given by $I(a) = a^{-1}$ for all $a \in G$, is analytic smooth.*

Definition 4. *A Lie algebra $(\mathfrak{g}, [\ ,\ ])$ over a field k is a vector space $\mathfrak{g}$ together with a binary operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, called the Lie bracket, satisfying*

1. *$[\ ,\ ]$ is a bilinear operation.*

2. *Antisymmetry: $[x,y] = -[y,x]$ for each $x, y \in \mathfrak{g}$.*

3. *Jacobi identity: $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ for each $x, y, z \in \mathfrak{g}$.*

A k-algebra A may be regarded as a Lie algebra $(A, [\ ,\ ])$, with bracket $[x,y] = xy - yx$ for all $x, y \in A$. In particular $\mathrm{End}(V)$ is a Lie algebra for any k-vector space V.

Let M be a smooth manifold and TM the tangent bundle of M. The space

$$\Gamma(M) = \{X : M \longrightarrow TM,\ X(m) \in T_m M,\ m \in M\}$$

of vector fields on M is a Lie algebra with the Lie bracket

$$[X,Y] = X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}, \quad \text{for all} \quad X, Y \in \Gamma(M).$$

Let G be a Lie group. The space $T_e(G)$ tangent to the identity $e \in G$ is a Lie algebra since $T_e(G) \cong \Gamma(G)^G$ is a Lie subalgebra of $\Gamma(G)$. For the classical Lie groups one gets

$$\mathfrak{sl}_n(\mathbb{C}) = T_I(SL_n(\mathbb{C})) = \{A \in M_n(\mathbb{C}) |\ \mathrm{tr(A)} = 0\}.$$

$$\mathfrak{so}_n(\mathbb{C}) = T_I(SO_n(\mathbb{C})) = \{A \in M_n(\mathbb{C}) |\ \langle Ax, y\rangle + \langle x, Ay\rangle = 0 \text{ for } x, y \in \mathbb{C}^n\}.$$

$$\mathfrak{sp}_{2n}(\mathbb{C}) = T_I(Sp_{2n}(\mathbb{C})) = \{A \in M_n(\mathbb{C}) |\ w(Ax,y) + w(x,Ay) = 0 \text{ for } x, y \in \mathbb{C}^{2n}\}.$$

Definition 5. *A morphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear map ρ from $\mathfrak{g}$ to $\mathfrak{h}$ such that $\rho([x,y]) = [\rho(x), \rho(y)]$ for $x, y \in \mathfrak{g}$. A representation ρ of a Lie algebra $\mathfrak{g}$ on a k-vector space V is a morphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of Lie algebras.*

The functor

$$\begin{array}{rccll} T_e : & \mathbf{LieGroup} & \longrightarrow & \mathbf{LieAlg} & \\ & G & \longmapsto & T_e(G) & \\ & \varphi : G \to H & \longmapsto & d_e\varphi : & T_e(G) \to T_e(H) \end{array}$$

induces an equivalence between $S(\mathbf{LieGroup})$ and $S(\mathbf{LieAlg})$.

Definition 6. *For any Lie algebra* $\mathfrak{g}$ *the adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is given by* $\mathrm{ad}(x)(y) = [x, y]$ *for all* $x, y \in \mathfrak{g}$.

Definition 7.
1. *A subspace* I *of a Lie algebra* $\mathfrak{g}$ *is called a Lie subalgebra if* $[x, y] \in I$ *for all* $x, y \in I$.
2. *A subalgebra* I *of* $\mathfrak{g}$ *is said to be abelian if* $[x, y] = 0$ *for all* $x, y \in I$.
3. *A subalgebra* I *of a Lie algebra* $\mathfrak{g}$ *is called an ideal if* $[x, y] \in I$ *for all* $x \in I$ *and* $y \in \mathfrak{g}$.

For any k-algebra the space of derivations of A

$$\mathrm{Der}(A) = \{d : A \longrightarrow A \;\mid\; d(xy) = d(x)y + xd(y) \text{ for all } x, y \in A\}$$

is a Lie subalgebra of $\mathrm{End}(A)$.

Definition 8.
1. *A Lie algebra* $\mathfrak{g}$ *is called simple if it has no ideals other than* $\mathfrak{g}$ *and* $\{0\}$.
2. *A Lie algebra* $\mathfrak{g}$ *is called semisimple if it has no abelian ideals other than* $\{0\}$.
3. *A maximal abelian subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is called a Cartan subalgebra.*

Next theorem is due to Cartan. A proof of it may be found in [5].

Theorem 9. *Let* $\mathfrak{g}$ *be a finite dimensional simple Lie algebra over* $\mathbb{C}$*, then* $\mathfrak{g}$ *is isomorphic to one of the list* $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$, $\mathfrak{so}_{2n}(\mathbb{C})$, E_6, E_7, E_8, F_4 *and* G_2.

Lie Algebras $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$ and $\mathfrak{so}_{2n}(\mathbb{C})$ are called classical and will be explained using our graphical notation in Sections 5, 6, 7 and 8. Lie algebras E_6, E_7, E_8, F_4 and G_2 are called exceptional and the reader may find their definitions in [5].

Definition 10. *The* Killing form *on* $\mathfrak{g}$ *is the bilinear map* $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ *given for all* $x, y \in \mathfrak{g}$ *by* $\langle x, y\rangle = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$*, where* $\circ$ *denotes the product in* $\mathrm{End}(\mathfrak{g})$ *and* $\mathrm{tr} : \mathrm{End}(\mathfrak{g}) \to \mathbb{C}$ *is the trace map.*

Denote by $\mathfrak{h}^*$ the linear dual of vector space $\mathfrak{h}$. The following proposition describes representations of abelian Lie algebras.

Proposition 11. *Let* $\mathfrak{h}$ *be an abelian Lie algebra and* $\rho : \mathfrak{h} \to \mathrm{End}(V)$ *a representation of* $\mathfrak{h}$*. Then* V *admits a decomposition*

$$V = \bigoplus_{\alpha\in\Phi} V_\alpha \tag{1}$$

where for each $\alpha \in \mathfrak{h}^*$, $V_\alpha = \{x \in V : \rho(h)(x) = \alpha(h)x, \quad \text{for all} \quad h \in \mathfrak{h}\}$*, and* $\Phi = \{\alpha \in \mathfrak{h}^* |\ \mathfrak{h}_\alpha \neq 0\}$.

Equation (1) is called Cartan decomposition of the representation ρ of $\mathfrak{h}$. Proposition 11 yields a map from $S(\mathbf{LieAlg})$ into $S(\mathbf{Root})$, which turns out to be a bijection, as follows. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. It is not difficult to see that $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$ for some natural real vector space $\mathfrak{h}_{\mathbb{R}}$. The killing form $\langle\ ,\ \rangle : \mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}} \longrightarrow \mathbb{R}$ restricted to $\mathfrak{h}_{\mathbb{R}}$ is non-degenerated and makes the pair $(\mathfrak{h}_{\mathbb{R}}, \langle\ ,\ \rangle)$ an Euclidean space. The linear dual $\mathfrak{h}^*_{\mathbb{R}}$ has an induced Euclidean structure, which we still denote by $\langle\ ,\ \rangle$ induced by the linear isomorphism $f : \mathfrak{h}_{\mathbb{R}} \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$ given by $f(x)(y) = \langle x, y\rangle$, for all $x, y \in \mathfrak{h}_{\mathbb{R}}$.

The adjoint representation ad : $\mathfrak{h} \longrightarrow \mathrm{End}(\mathfrak{g})$ restricted to $\mathfrak{h}$ give us a Cartan decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ where for each $\alpha \in \mathfrak{h}$ $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}|\ \ [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}$ and $\Phi = \{\alpha \in \mathfrak{h}^*_{\mathbb{R}} :\ \mathfrak{h}_\alpha \neq 0\}$.

Definition 12. *The triple* $(\mathfrak{h}^*_{\mathbb{R}}, \langle\ ,\ \rangle, \Phi)$ *is the root system associated to Lie algebra* $\mathfrak{g}$.

Definition 13. *Given a root system* Φ *the group* W *generated by all reflections* S_α *with* $\alpha \in \Phi$, *where* $S_\alpha(\beta) = \beta - 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha$, *is known as the* Weyl group *associated to* Φ.

One can show that there exists a subset $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of Φ such that Π is a basis of $\mathfrak{h}^*_{\mathbb{R}}$ and each root $\alpha \in \Phi$ can be written as a linear combination of roots in Π with coefficients in $\mathbb{Z}$ which are either all non-negative or all non-positive. The set Π is called a set of fundamental roots. The integers

$$A_{ij} = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle} \tag{2}$$

are called the Cartan integers and the matrix $A = (A_{ij})$ is called the Cartan matrix. Notice that $A_{ii} = 2$, and that for any $\alpha_i, \alpha_j \in \Pi$ with $i \neq j$, $S_{\alpha_i}(\alpha_j)$ is a $\mathbb{Z}$-combination of α_i and α_j. Since the coefficient of α_j is 1, the coefficient associated to α_i in $S_{\alpha_i}(\alpha_j)$ must be a non-positive integer, i.e., $A_{ij} \in \mathbb{Z}^{\leq 0}$. The angle θ_{ij} between α_i, α_j is given by the cosine formula

$$\langle\alpha_i, \alpha_j\rangle = \langle\alpha_i, \alpha_i\rangle^{\frac{1}{2}}\langle\alpha_j, \alpha_j\rangle^{\frac{1}{2}}\cos(\theta_{ij}).$$

Then we have

$$4\cos^2(\theta_{ij}) = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_j, \alpha_j\rangle} \cdot 2\frac{\langle\alpha_j, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle},$$

and therefore $4\cos^2(\theta_{ij}) = A_{ij}A_{ji}$. Let $n_{ij} = A_{ij}A_{ji}$ clearly $n_{ij} \in \mathbb{Z}$ and $n_{ij} \geq 0$. Since $-1 \leq \cos(\theta_{ij}) \leq 1$ the only possible values for n_{ij} are $n_{ij} = 0, 1, 2$ or 3.

Definition 14. *The Dynkin diagram* Δ *associated to a simple Lie algebra* $\mathfrak{g}$ *is the graph* Δ *with vertices* $\{1, \ldots, n\}$ *in bijective correspondence with the set* Π *of fundamental roots of* $\mathfrak{g}$ *such that*

1. *Vertices* i, j *with* $i \neq j$ *are joined by* $n_{ij} = A_{ij}A_i$ *edges, where* A_{ij} *is given by formula (2).*
2. *Between each double edge or triple we attach the symbol* $<$*, or the symbol* $>$ *pointing towards the shorter root with respect to Killing form.*

Theorem 15. *Consider the root system* Φ *associated to a simple Lie algebra* $\mathfrak{g}$*, let* $\alpha \in \Phi$ *be a root. For each nonzero* $x_\alpha \in \mathfrak{g}_\alpha$ *there is* $x_{-\alpha} \in \mathfrak{g}_\alpha$ *and* $h_\alpha \in \mathfrak{h}$ *such that* $\alpha(h_\alpha) = 2$*,* $[x_\alpha, x_{-\alpha}] = h_\alpha$*,* $[h_\alpha, x_\alpha] = 2x_\alpha$ *and* $[h_\alpha, x_{-\alpha}] = -2x_{-\alpha}$*.*

Theorem 15 implies that for any root $\alpha \in \Phi$, $x_\alpha, x_{-\alpha}$ and h_α span a subalgebra $\mathfrak{s}_\alpha$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. See Section 4 for more on $\mathfrak{sl}_2(\mathbb{C})$. This fact explain the distinguished role played by $\mathfrak{sl}_2(\mathbb{C})$ in the representation theory of simple Lie algebras.

Definition 16. *Let* $\Phi \subset \mathfrak{h}$ *be the root system associated with a simple Lie algebra* $\mathfrak{g}$*.*

1. *For any* $\alpha \in \Phi$*, the Cartan element* $h_\alpha \in \mathfrak{h}$ *given by Theorem 15 is called the coroot associated to root* α*.* $\Phi_c = \{h_\alpha : \alpha \in \Phi\}$ *is the coroot system associated to* $\mathfrak{h}$ *and* $\Pi_c = \{h_\alpha : \alpha \in \Pi\}$ *is the set of fundamental coroots.*
2. *The elements* $w_1, \dots, w_n$ *in* $\mathfrak{h}^*$ *given by the relations* $w_i(h_j) = \delta_{ij}$*, for all* $1 \leq i, j \leq n$*, where* h_j *is the coroot associated to fundamental root* α_j*, are called the fundamental weights.*

One can recover a simple Lie algebra $\mathfrak{g}$ from its associated Dynkin diagram Δ as follows: Let n_{ij} be the adjacency matrix of Δ. The relation $n_{ij} = a_{ij}a_{ji}$ determines univocally the Cartan matrix a_{ij}. Consider the free Lie algebra generated by the symbols $h_1, \dots, h_n, x_1, \dots, x_n, y_1, \dots, y_n$. Form the quotient of this free Lie algebra by the relations

$$[h_i, h_j] = 0 \ \ (\text{all } i, j); \qquad [x_i, y_i] = h_i \ \ (\text{all } i); \qquad [x_i, x_j] = 0 \ \ (i \neq j);$$

$$[h_i, x_j] = a_{ij}x_j \ \ (\text{all } i, j); \quad [h_i, y_j] = -a_{ij}y_j \ \ (\text{all } i, j);$$

and for all $i \neq j$,

$$\begin{array}{lll}
[x_i, x_j] = 0, & [y_i, y_j] = 0, & \text{if } \ a_{ij} = 0. \\
[x_i, [x_i, x_j]] = 0, & [y_i, [y_i, y_j]] = 0 & \text{if } \ a_{ij} = -1. \\
[x_i, [x_i, [x_i, x_j]]] = 0, & [y_i, [y_i, [y_i, y_j]]] = 0 & \text{if } \ a_{ij} = -2. \\
[x_i, [x_i, [x_i, [x_i, x_j]]]] = 0, & [y_i, [y_i, [y_i, [y_i, y_j]]]] = 0 & \text{if } \ a_{ij} = -3.
\end{array}$$

Serre shows that the resulting Lie algebra is a finite-dimensional simple Lie algebra isomorphic to $\mathfrak{g}$. See [12] for more details.

2.1 Jacobian Criterion

Let k be a field of characteristic zero and let V be a finite dimensional k-vector space. Set $V = \langle e_1, e_2, \dots, e_n \rangle$ and $V^* = \langle x_1, x_2, \dots, x_n \rangle$ such that $x_i(e_j) = \delta_{ij}$. We have

$$x_k\Big(\sum_{i=1}^{n} a_i e_i\Big) = a_k, \quad 1 \le k \le n.$$

Any $f \in V^*$ is written as $f = b_1x_1 + b_2x_2 + \cdots + b_nx_n$. Denote by $S = S(V^*)$ the symmetric algebra of the dual space V^* which can be identify with the polynomial ring $k[x_1, x_2, \dots, x_n]$. Let G be a finite group which acts on V. G also acts on V^*, and thus it acts on $S = S(V^*)$ as follows

$$\begin{array}{ccc} S(V^*) \times G & \longrightarrow & S(V^*) \\ (p, g) & \longmapsto & p(g) \end{array}$$

where $(pg)(v) = p(gv), \quad \text{for all} \;\; g \in G,\; p \in S(V^*),\; v \in V$. The algebra

$$k[x_1, x_2, \dots, x_n]^G = \{p \in k[x_1, x_2, \dots, x_n] :\; p(g) = p,\; \forall g \in G\}$$

is called the G-invariant subalgebra of $k[x_1, x_2, \dots, x_n]$.

Definition 17. *Let k be a field and F a extension of k. Let S be a subset of F. The set S is algebraically dependent over k if for some positive integer n there is a non-zero polynomial $f \in k[x_1, \dots, x_n]$ such that $f(s_1, \dots, s_n) = 0$ for some different $s_1, \dots, s_n \in S$. Otherwise S is algebraically independent.*

Theorem 18. *Let $\mathbb{C}[x_1, x_2, \dots, x_n]^W$ be the subalgebra of $\mathbb{C}[x_1, x_2, \dots, x_n]$ consisting of W-invariant polynomials, then $\mathbb{C}[x_1, x_2, \dots, x_n]^W$ is generated as an $\mathbb{C}$-algebra by n homogeneous, algebraically independent elements of positive degree together with* 1.

The idea of proof of Theorem 18 goes as follows: let I be the ideal of $\mathbb{C}[x_1, x_2, \dots, x_n]$ generated by all homogeneous W-invariant polynomials of positive degree. Using Hilbert's Basis Theorem we may choose a minimal generating set $f_1, f_2, \dots, f_r$ for I consisting of homogeneous W-invariant polynomials of positive degree. One can show that $r = n$ and furthermore $\mathbb{C}[x_1, x_2, \dots, x_n]^W = \mathbb{C}[f_1, f_2, \dots, f_n]$.

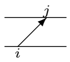
Proposition 19. *Let $f_1, \dots, f_n$ and $g_1, \dots, g_n$ be two sets of homogeneous, algebraically independent generators of $\mathbb{C}[x_1, \dots, x_n]^W$ with degrees d_i and e_i respectively, then (after reordering) $d_i = e_i$ for all $i = 1, \dots, n$.*

The numbers $d_1, \dots, d_n$ written in increasing order are called the *degrees* of W. Theorem 20 below is a simple criterion for the algebraic independence of polynomials $f_1, \dots, f_n$ expressed in terms of the Jacobian determinant. We write $J(f_1, \dots, f_n)$ for the determinant of the $n \times n$ matrix whose (i, j)-entry is $\frac{\partial f_i}{\partial x_j}$.

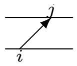
Theorem 20 (Jacobian criterion)**.** *The set of polynomials* $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ *are algebraically independent over a field* k *of characteristic zero if and only if* $J(f_1, \dots, f_n) \neq 0$.

3 Graph and matrices

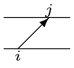
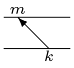
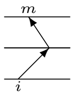
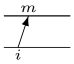
We denote by $\mathbf{Digraph}^1(n,n)$ the vector space generated by bipartite directed graphs with a unique edge starting on the set $[n]$ and ending on the set $[n]$. We describe $\mathbf{Digraph}^1(n,n)$ pictorially as follows

$\mathbf{Digraph}^1(n,n) = \Big\langle$  $, \quad 1 \le i, j \le n \Big\rangle$

where the symbol

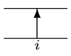
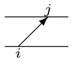


denotes the graph whose unique edge stars at vertex at i and ends at vertex j. We define a product on $\mathbf{Digraph}^1(n,n)$ as follows

 $\cdot$  $=$ $\Bigg\{$  $=$  , if $j = k$

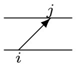
$0, \qquad$ otherwise

The trace on $\mathbf{Digraph}^1(n,n)$ is defined as the linear functional $\mathrm{tr} : \mathbf{Digraph}^1(n,n) \longrightarrow \mathbb{C}$ given by

$\mathrm{tr}\Big($  $\Big) = 1$ and $\mathrm{tr}\Big($  $\Big) = 0$

Algebra $\mathbf{Digraph}^1(n,n)$ is isomorphic to $\mathrm{End}(\mathbb{C}^n)$ through the application

$$\textbf{Digraph}^1(n,n) \cong \mathfrak{gl}_n(\mathbb{C})$$



$$\longrightarrow \quad E_{ij}$$

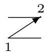
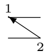
We will use this isomorphism to give combinatorial interpretation of results on $\mathrm{End}(\mathbb{C}^n)$ that are traditionally expressed in the language of matrices.

4 Linear special algebra $\mathfrak{sl}_2(\mathbb{C})$

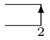
We begin studying the special linear algebra $\mathfrak{sl}_2(\mathbb{C})$. It plays a distinguished role in the theory of Lie algebras. By definition we have

$$\mathfrak{sl}_2(\mathbb{C}) = \{A \in \mathrm{End}(\mathbb{C}^2) : \mathrm{tr}(A) = 0\}.$$

As subspace of $\textbf{Digraph}^1(2,2)$, $\mathfrak{sl}_2(\mathbb{C})$ is the following vector space

$\mathfrak{sl}_2(\mathbb{C}) = \Big\langle$  $-$ , ,  $\Big\rangle \subset \textbf{Digraph}^1(2,2)$

We fix as Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ the 1-dimensional subspace

$\mathfrak{h} = \Big\{\ a_1$  $+\ a_2$  $,\quad a_1 + a_2 = 0\ \Big\}$

The dual space is $\mathfrak{h}^* = \langle a_1, a_2\rangle/\{a_1 + a_2 = 0\}$, where

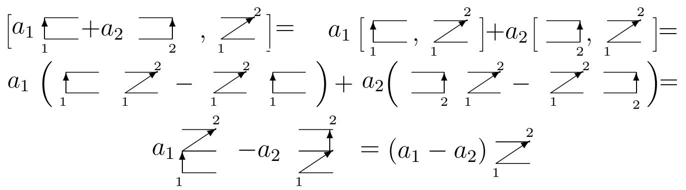
$a_i\Big($  $\Big) = \delta_{ij}$

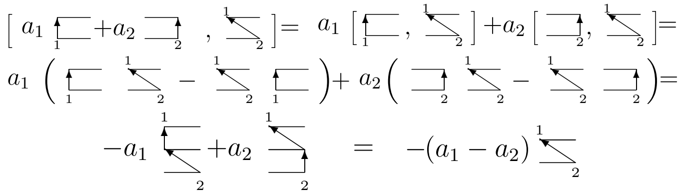
4.1 Root system of $\mathfrak{sl}_2(\mathbb{C})$

Consider the projection map $\langle a_1, a_2\rangle \longrightarrow \langle a_1, a_2\rangle/\{a_1+a_2 = 0\}$. We still denote by a_i the image of a_i under the projection above. Now each $h \in \mathfrak{h}$ is of the form

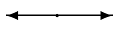
$h = \quad a_1$  $+a_2$  $,\quad a_1, a_2 \in \mathfrak{h}^*$

Let us compute the roots



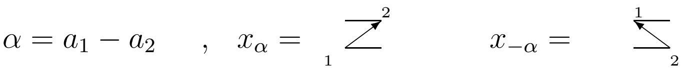


Therefore the root system of $\mathfrak{sl}_2(\mathbb{C})$ is $\Phi = \{a_1 - a_2, a_2 - a_1\}$. Setting $\alpha = a_1 - a_2$ we have that the roots are α and $-\alpha$ and the set of fundamental roots is $\Pi = \{\alpha\}$. In pictures

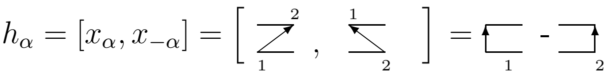


4.2 Coroot system of $\mathfrak{sl}_2(\mathbb{C})$

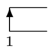
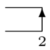
Let x_α and $x_{-\alpha}$ be the covectors associated with the roots α and $-\alpha$ of $\mathfrak{sl}_2(\mathbb{C})$ respectively.



A vector $h_\alpha \in \mathfrak{h}$ is said to be the coroot associated to the root $\alpha \in \mathfrak{h}^*$, if $h_\alpha = c[x_\alpha, x_{-\alpha}]$, $c \in \mathbb{C}$ and $\alpha(h_\alpha) = 2$.



since

$(a_1 - a_2)$ ( $-$ )$= 2$

4.3 Killing form of $\mathfrak{sl}_2(\mathbb{C})$

Let x and y in $\mathfrak{h}$. Set

$$\begin{aligned}\langle x, y\rangle &= \sum_{\alpha\in\Phi} \alpha(x)\alpha(y)\\ &= 2(x_1 - x_2)(y_1 - y_2)\\ &= 2x_1y_1 + 2x_2y_2 - 2x_1y_2 - 2x_2y_1\\ &= 2(x_1y_1 + x_2y_2) - 2(x_1 + x_2)(y_1 + y_2) + 2(x_1y_1 + x_2y_2)\\ &= 4\mathrm{tr}(xy).\end{aligned}$$

4.4 Dynkin diagram of $\mathfrak{sl}_2(\mathbb{C})$

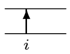
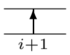
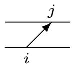
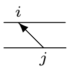
We have only one fundamental root, so the Dynkin diagram is just $\circ$.

5 Special linear algebra $\mathfrak{sl}_n(\mathbb{C})$

Let us recall the special linear algebra $\mathfrak{sl}_n(\mathbb{C})$

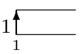
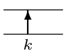
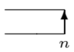
$$\mathfrak{sl}_n(\mathbb{C}) = \{A \in \mathrm{End}(\mathbb{C}^n) : \ \mathrm{tr}(A) = 0\}$$

$\mathfrak{sl}_n(\mathbb{C})$ consider as a subspace of $\mathbf{Digraph}^1(n, n)$ is following subspace

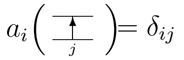
$\mathfrak{sl}_n(\mathbb{C}) =$ $\Big\langle$  $-$  ,  ,  ; $1 \le i < j \le n-1$ $\Big\rangle$

5.1 Root system of $\mathfrak{sl}_n(\mathbb{C})$

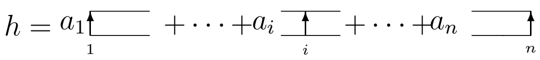
We take as Cartan subalgebra the subspace of $\mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{h} = \Big\{$ a_1  $+\cdots+ a_k$  $+\cdots+ a_n$  , $\sum a_k = 0$ $\Big\}$

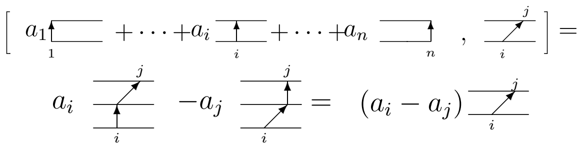
The dual space is $\mathfrak{h}^* = \langle a_1, \ldots, a_n\rangle/(\sum a_i = 0)$, where



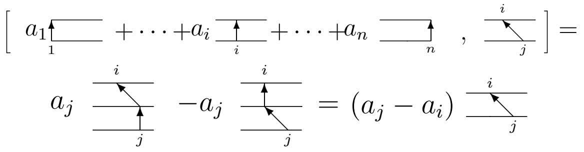
Consider the projection $\langle a_1,\ldots,a_n\rangle \longrightarrow \langle a_1,\ldots,a_n\rangle/(\sum a_k = 0)$. The image of a_i under the projection above is still denote by a_i. Then vector $h \in \mathfrak{h}$ can be written as



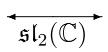
Let us compute the root system

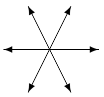


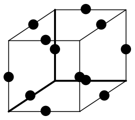
Also



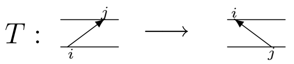
Thus the root system of $\mathfrak{sl}_n(\mathbb{C})$ is $\Phi = \{a_i - a_j,\ a_j - a_i,\ 1 \le i < j \le n-1\} \subset \mathfrak{h}^*$. The set of fundamental roots is $\Pi = \{a_i - a_{i+1},\ i = 1,\ldots,n-1\}$. In pictures for $n = 2, 3$ and 4 the root systems look like


$\mathfrak{sl}_2(\mathbb{C})$


$\mathfrak{sl}_3(\mathbb{C})$


$\mathfrak{sl}_4(\mathbb{C})$

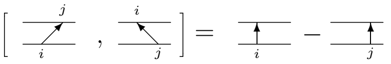
Consider the linear map $T : \textbf{Digraph}^1(n,n) \longrightarrow \textbf{Digraph}^1(n,n)$ sending each directed graph into its opposite graph. Clearly T is an antimorphism, i.e, $T(ab) = T(b)T(a)$, for all $a, b \in \textbf{Digraph}^1(n,n)$. For example,



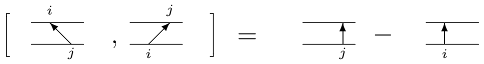
Notice that negative roots can be obtain from the positive ones through an application of T.

5.2 Coroots and weights for $\mathfrak{sl}_n(\mathbb{C})$

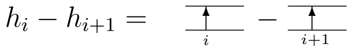
1. Coroot associated to the root $a_i - a_j$



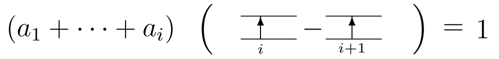
2. Coroot associated to the root $a_j - a_i$

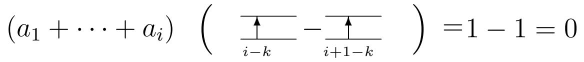


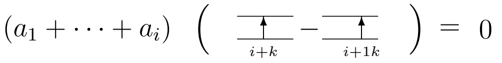
The set of fundamental coroots has the form $\Pi_c = \{h_i - h_{i+1}, 1 \leq i \leq n-1\}$ where



The set of fundamental weights is $w_i = a_1 + a_2 + \cdots + a_i$ since







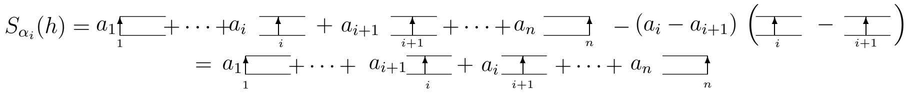
5.3 The Killing form of $\mathfrak{sl}_n(\mathbb{C})$

Let x and y in $\mathfrak{h}$. Set

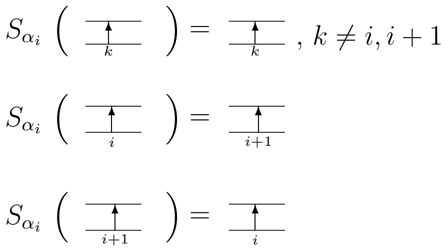
$$\begin{aligned}
\langle x, y\rangle &= \sum_{\alpha\in\Phi}\alpha(x)\alpha(y)\\
&= \sum_{i<j}(x_i - x_j)(y_i - y_j) + \sum_{i<j}(x_j - x_i)(y_j - y_i)\\
&= 2\sum_{i<j}(x_i - x_j)(y_i - y_j)\\
&= 2\big(\sum_{i<j}x_iy_i + \sum_{i<j}x_jy_j - \sum_{i<j}x_iy_j - \sum_{i<j}x_jy_i\big)\\
&= 2\big(\sum(n-i)x_iy_i + \sum(i-1)x_iy_i + \sum x_iy_i\big)\\
&= 2n\mathrm{tr}(xy).
\end{aligned}$$

5.4 Weyl group of $\mathfrak{sl}_n(\mathbb{C})$

Consider the fundamental roots $\alpha_i = a_i - a_{i+1}$, $i = 1, \ldots, n-1$ and S_{α_i} the reflection associated to the fundamental root α_i. Let $h \in \mathfrak{h}$ and h_{α_i} be the coroot associated to the fundamental root α_i. By definition we have $S_{\alpha_i}(h) = h - \alpha_i(h)h_{\alpha_i}$



so we see that reflections S_{α_i} has the form



Therefore the Weyl group A_n associated with $\mathfrak{sl}_{n+1}(\mathbb{C})$ is the symmetric group on n letters

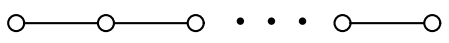
$$A_n = \langle S_{\alpha_i} |\ \ i = 1, \ldots, n-1\rangle = S_n.$$

5.5 Dynkin diagram of $\mathfrak{sl}_n(\mathbb{C})$ and Cartan matrix.

Using equation (2) one can checks that the Cartan matrix associated to the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is

$$A_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ & & & \ddots & & \\ & & & & \ddots & \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

The Dynkin diagram associated to $\mathfrak{sl}_n(\mathbb{C})$ is

$A_{n-1},\ n \geq 2$ 

5.6 Invariant polynomials for $\mathfrak{sl}_n(\mathbb{C})$

Consider the action of S_{n+1} on $\mathbb{R}^{n+1}$ given by

$$\begin{array}{ccc} S_{n+1} \times \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R}^{n+1} \\ (\pi, x) & \longmapsto & (\pi x)_i = x_{\pi^{-1}(i)} \end{array}$$

notice that the permutation (ij) acts as a reflection on $\mathbb{R}^{n+1}$ since

$$(ij)(x_i - x_j) = x_j - x_i = -(x_i - x_j)$$

$$(ij)(x) = x, \ \text{ si } \ x \in (x_i - x_j)^{\perp} \text{ (es decir } x_i = x_j)$$

Since S_{n+1} is generated by transpositions $(i\ i+1),\ i = 1, \dots, n$, then S_{n+1} is an example of what is called a reflection group. Recall that a linear action of a group G on a vector space V is said to be effective if the only fixed point is 0. The action of S_{n+1} on $\mathbb{R}^{n+1}$ fixes points in $\mathbb{R}^{n+1}$ lying on the straight line $\{(x, x, \dots, x)|\ \ x \in \mathbb{R}\}$. Thus the action of A_n on $\mathbb{R}^{n+1}$ fails to be effective. If we instead let A_n act on the hyperplane $V = \{(x_1 \dots x_{n+1}) \in \mathbb{R}^{n+1}|\ x_1 + \dots + x_{n+1} = 0\}$ then the action becomes effective. Consider the power symmetric functions

$$f_i = x_1^{i+1} + \dots + x_{n+1}^{i+1}, \quad 1 \leq i \leq n.$$

Each f_i is S_{n+1}-invariant, and together the power symmetric functions form a set of basic invariants. This fact can be proven as follows: first notice that

$$\mathrm{gr}(f_1)\mathrm{gr}(f_2)\cdots\mathrm{gr}(f_n) = 2 \cdot 3 \cdots n(n+1) = (n+1)! = |S_{n+1}| = |A_n|.$$

Next, it is easy to compute the Jacobian $J(f_1, f_2, \cdots, f_n)$ yielding the non-vanishing polynomial

$$J(f_1, f_2, \cdots, f_n) = (n+1)! \prod_{1 \leq i < j \leq n} (x_j - x_i) \prod_{i=1}^{n} (x_1 + \cdots + 2x_i + \cdots + x_n)$$

Finally, use the Jacobian criterion.

6 Symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$

Recall that the symplectic Lie algebra $\mathfrak{sp}_{2n}$ is defined as

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{X : X^t S + SX = 0\}.$$

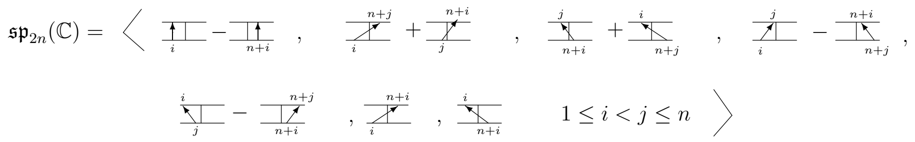
Here $S \in M_{2n}(\mathbb{C})$ is the matrix

$$S = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

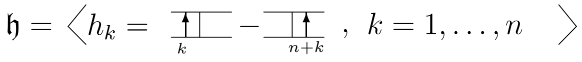
Equivalently,

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix};\ A, B, C \in M_n(\mathbb{C}) \text{ y } B = B^t,\ C = C^t \right\}$$

$\mathfrak{sp}_{2n}(\mathbb{C})$ as a subspace of $\textbf{Digraph}^1(2n, 2n)$ is given by

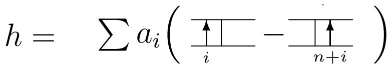


We take as a Cartan subalgebra of $\mathfrak{sp}_{2n}(\mathbb{C})$ the following subspace

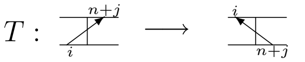


6.1 Root system of $\mathfrak{sp}_{2n}(\mathbb{C})$

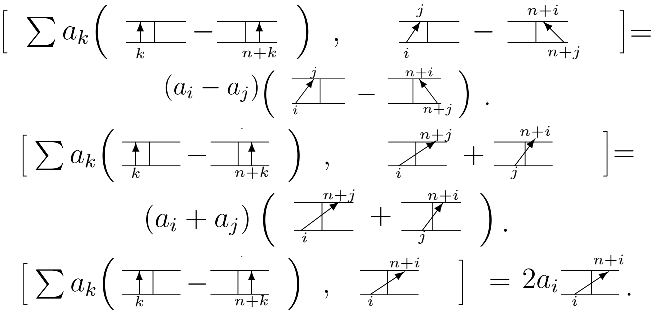
Consider $h \in \mathfrak{h}$



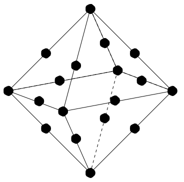
where $\{a_i\}$ denotes de base of $\mathfrak{h}^*$ dual to the given base of $\mathfrak{h}$. Let us define $T : \mathbf{Digraph}^1(2n, 2n) \longrightarrow \mathbf{Digraph}^1(2n, 2n)$ to be the linear map that sends each directed graph into its opposite. Clearly T es un antimorphism, i.e, $T(ab) = T(b)T(a)$ for all $a, b \in \mathbf{Digraph}^1(2n, 2n)$. For example,



We will compute explicitly the positive roots. To obtain the negative roots it is enough to apply the transformation T to each positive root.

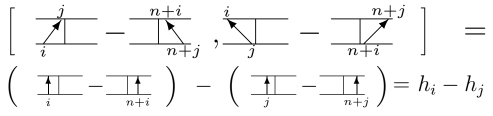


Thus the root system of $\mathfrak{sp}_{2n}(\mathbb{C})$ is $\Phi = \{a_i - a_j, a_j - a_i, a_i + a_j, -a_i - a_j, 2a_i, -2a_i$ $1 \leq i < j \leq n\}$. The set of fundamental roots is $\Pi = \{\alpha_1, \dots, \alpha_n\}$ where $\alpha_i = a_i - a_{i+1}, i = 1, \dots, n-1$ and $\alpha_n = 2a_n$. In pictures, the root system of $\mathfrak{sp}_6(\mathbb{C})$ looks like



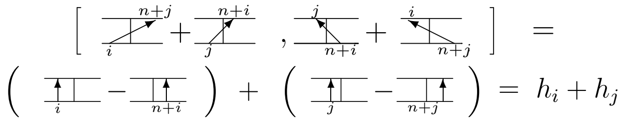
6.2 Coroots and weights of $\mathfrak{sp}_{2n}(\mathbb{C})$

1. Coroot asociated to the root $a_i - a_j$



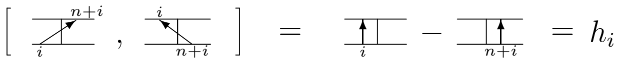
   $h_i - h_j$ is the coroot associated to the root $a_i - a_j$, since $(a_i - a_j)(h_i - h_j) = 2$.

2. Coroot associated to the root $a_i + a_j$

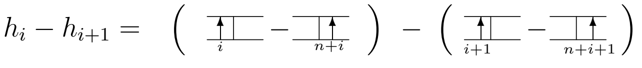


   $h_i + h_j$ is the coroot associated to the la root $a_i + a_j$, since $(a_i + a_j)(h_i + h_j) = 2$.

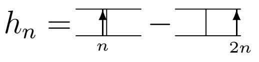
3. Coroot associated to the root $2a_i$



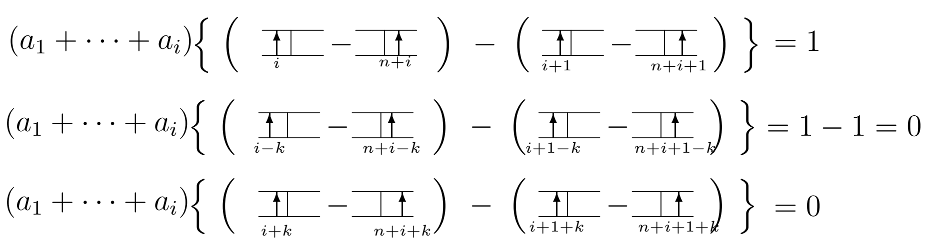
We conclude that $\Phi_c = \{h_i - h_j, h_j - h_i, h_i + h_j, -h_i - h_j, h_i, -h_i,\ 1 \le i < j \le n\}$ is the coroot system of $\mathfrak{sp}_{2n}(\mathbb{C})$. The set of fundamental coroots is given by $\Pi_c = \{h_i - h_{i+1}, h_n;\ \ 1 \le i \le n-1\}$ where



and



The fundamental weights are $w_i = a_1 + a_2 + \cdots + a_i$ since



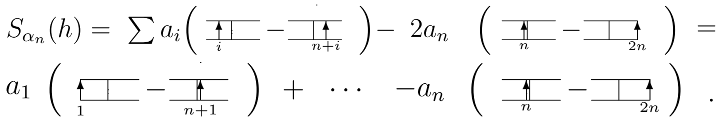
6.3 Killing form of $\mathfrak{sp}_{2n}(\mathbb{C})$

Let x and y in $\mathfrak{h}$. Set

$$\begin{aligned}
\langle x, y\rangle &= \sum_{\alpha\in\Phi} \alpha(x)\alpha(y) \\
&= \sum_{i\neq j}(x_i - x_j)(y_i - y_j) + \sum_{i\neq j}(x_i + x_j)(y_i + y_j) + 2\sum_{i}(2x_i)(2y_i) \\
&= 4(n+1)\sum x_i y_i \\
&= 4(n+1)\mathrm{tr}(xy).
\end{aligned}$$

6.4 Weyl group of $\mathfrak{sp}_{2n}(\mathbb{C})$

Consider the fundamental roots of the form $\alpha_i = a_i - a_{i+1}$. Similarly to the $\mathfrak{sl}_n(\mathbb{C})$, it is easy to check that they generate a copy of S_n. Let us compute the reflection associated to the root $\alpha_n = 2a_n$. Given $h \in \mathfrak{h}$, we have that $S_{\alpha_n}(h) = h - \alpha_n(h)h_{\alpha_n}$, where h_{α_n} is the coroot associated to the root α_n



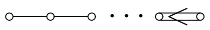
This reflections are the sign changes and they generate a copy of the group $\mathbb{Z}_2^n$. Altogether the Weyl group associated to $\mathfrak{sp}_{2n}(\mathbb{C})$ is

$$C_n = \mathbb{Z}_2^n \rtimes S_n.$$

6.5 Cartan matrix and Dynkin diagram of $\mathfrak{sp}_{2n}(\mathbb{C})$

$$C_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

There are n vertices in this case, one for each fundamental root. The Killing form is $\langle \alpha_i, \alpha_{i+1} \rangle = 1$, if $i = 1, \dots, n-1$ and $\langle \alpha_{n-1}, \alpha_n \rangle = 2$. Moreover $\langle \alpha_{n-1}, \alpha_{n-1} \rangle < \langle \alpha_n, \alpha_n \rangle$, and thus the Dynkin diagram of $\mathfrak{sp}_{2n}(\mathbb{C})$ has the form

$C_n,\ n \geq 3$ 

6.6 Invariant functions under the action of $C_n = \mathbb{Z}_2^n \rtimes S_n$

Let us recall that the group structure on $\mathbb{Z}_2^n \rtimes S_n$ is given by

$$(a, \pi)(b, \sigma) = (a \cdot \pi(b), \pi \circ \sigma)$$

where $(\pi b)_i = b_{\pi^{-1}(i)}$.

Proposition 21. *$\mathbb{Z}_2^n \rtimes S_n$ acts on $\mathbb{R}^n$ as follows*

$$\begin{array}{ccc} \mathbb{Z}_2^n \rtimes S_n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ ((a,\pi)x) & \longmapsto & ((a,\pi)x)_i = a_i x_{\pi^{-1}(i)} \end{array}$$

Consider the polynomials

$$f_i = x_1^{2i} + x_2^{2i} + \cdots + x_n^{2i}, \quad 1 \leq i \leq n$$

Each polynomial f_i is invariant under the action of $(\mathbb{Z}_2)^n \rtimes S_n$ given by

$$(f(a,\pi))(x) = f((a,\pi)x).$$

The set of invariants

$$\begin{array}{ccc} f_1 & = & x_1^2 + x_2^2 + \cdots + x_n^2 \\ f_2 & = & x_1^4 + x_2^4 + \cdots + x_n^4 \\ \vdots & & \\ f_n & = & x_1^{2n} + x_2^{2n} + \cdots + x_n^{2n} \end{array}$$

is a basic set. This follows from the Jacobian criterion since

$$\mathrm{gr}(f_1)\mathrm{gr}(f_2)\cdots\mathrm{gr}(f_n) = 2\cdot 4\cdot 6\cdots 2n = 2^n n! = |\mathbb{Z}_2^n \rtimes S_n|$$

and

$$J = 2^n n!\; x_1\cdots x_n \prod_{1\le i<j\le n}(x_j^2 - x_i^2) \neq 0.$$

7 Orthogonal Lie Algebra $\mathfrak{so}_{2n}(\mathbb{C})$

Recall that the $2n$-orthogonal Lie Algebra is defined as follows

$$\mathfrak{so}_{2n}(\mathbb{C}) = \{X : X^tS + SX = 0\}$$

where $S \in M_{2n}(\mathbb{C})$ is the matrix

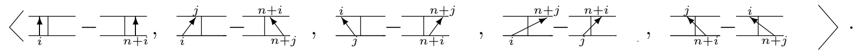
$$S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

Explicitly

$$\mathfrak{so}_{2n}(\mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix};\ A, B, C \in M_n(\mathbb{C}) \text{ y } B = -B^t,\ C = -C^t \right\}.$$

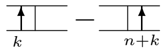
$\mathfrak{so}_{2n}(\mathbb{C})$ as a subspace of **Digraph**$^1(2n, 2n)$ is given by

$\mathfrak{so}_{2n}(\mathbb{C}) =$



where $1 \le i < j \le n$. We fix $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{so}_{2n}(\mathbb{C})$ to be

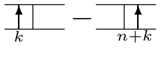
$\mathfrak{h} = \langle h_k =$



$,\ k = 1, \ldots, n \rangle$

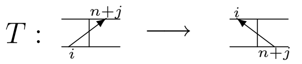
7.1 Root System of $\mathfrak{so}_{2n}(\mathbb{C})$

$h = \sum a_k \Big($

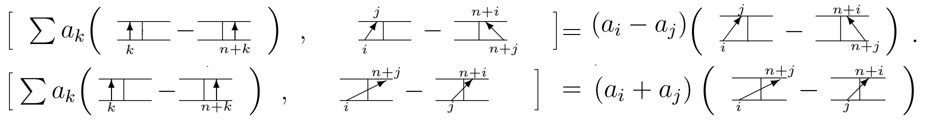


$\Big)$

where $\{a_i\}$ is the base of $\mathfrak{h}^*$ dual to the natural base of $\mathfrak{h}$. As for the case of the symplectic algebra we define a map $T : \mathbf{Digraph}^1(2n,2n) \longrightarrow \mathbf{Digraph}^1(2n,2n)$. T sends a given graph to its opposite if it does not cross the vertical line, and to minus its opposite if it crosses the vertical line. We have again that $T(ab) = T(b)T(a)$. For example,



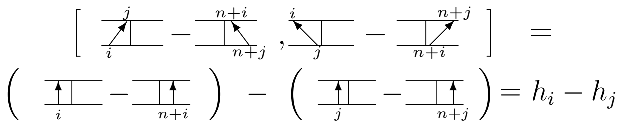
Let us find out the positive roots



To get the negative roots it is enough to apply T to the positive roots. Therefore the root system is $\Phi = \{a_i - a_j, a_j - a_i, a_i + a_j, -a_i - a_j,\ \ 1 \leq i < j \leq n\}$ and the fundamental roots can be taken to be $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ where
$\alpha_i = a_i - a_{i+1},\ \ i = 1, \ldots, n-1$ and $\alpha_n = a_{n-1} + a_n$

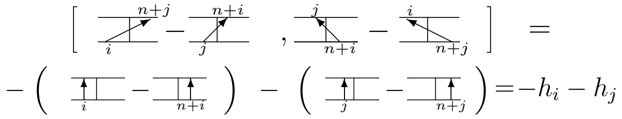
7.2 Coroots and weights of $\mathfrak{so}_{2n}(\mathbb{C})$

1. Coroot associated to the root $a_i - a_j$ in this case $(a_i - a_j)(h_i - h_j) = 2$,



thus $h_i - h_j$ is the coroot associated to the root $a_i - a_j$.

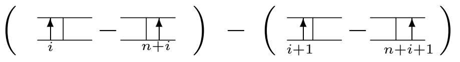
2. Coroot associated to the root $a_i + a_j$



since $(a_i + a_j)(-h_i - h_j) = -2$, $h_i + h_j$ is the coroot associated to the root $a_i + a_j$.

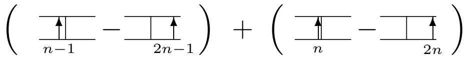
We concluded that $\Phi_c = \{h_i - h_j, h_j - h_i, h_i + h_j, -h_i - h_j, 1 \leq i < j \leq n\}$ is the coroot system of $\mathfrak{so}_{2n}(\mathbb{C})$. The set of fundamental coroots is given by $\Pi_c = \{h_i - h_{i+1}, h_{n-1} + h_n;\ \ 1 \leq i \leq n-1\}$ where

$$h_i - h_{i+1} =$$



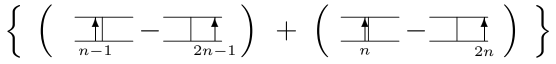
and

$$h_{n-1} + h_n =$$



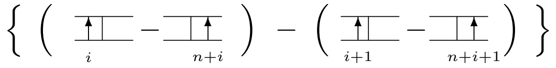
The fundamental weights are given by $w_i = a_1 + \cdots + a_i,\ \ i = 1, \cdots, n-1$ and $w_n = \frac{a_1 + a_2 + \cdots + a_n}{2}$ In a similar as for $\mathfrak{sp}_{2n}(\mathbb{C})$ one can prove that $w_i(h_j - h_{j+1}) = \delta_{ij}$. For w_n we get

$$\frac{(a_1 + \cdots + a_n)}{2}$$



$$= 1$$

$$\frac{(a_1 + \cdots + a_n)}{2}$$



$$= 0$$

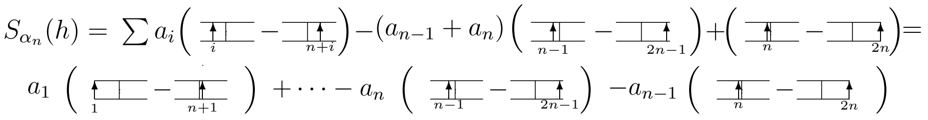
7.3 Killing form of $\mathfrak{so}_{2n}(\mathbb{C})$

Let x and y in $\mathfrak{h}$. Set

$$\begin{aligned}
\langle x, y \rangle &= \sum_{\alpha \in \Phi} \alpha(x)\alpha(y) \\
&= \sum_{i \leq j} (x_i - x_j)(y_i - y_j) + \sum_{i \leq j} (x_i + x_j)(y_i + y_j) + \sum_{j \leq i} (x_i - x_j)(y_i - y_j) \\
&\quad + \sum_{j \leq i} (x_i + x_j)(y_i + y_j) \\
&= \sum_{i \neq j} 2x_i y_j + 2x_i y_j \\
&= 4(n-1) \sum x_i y_i \\
&= 4(n-1)\mathrm{tr}(xy).
\end{aligned}$$

7.4 Weyl group of $\mathfrak{so}_{2n}(\mathbb{C})$

Consider the fundamental roots $\alpha_i = a_i - a_{i+1}$. Just as for $\mathfrak{sl}_n(\mathbb{C})$, the associated reflections associated to these roots generate the group S_n. We compute the reflections associated to the roots $\alpha_n = a_{n+1} + a_n$. Given $h \in \mathfrak{h}$, we have $S_{\alpha_n}(h) = h - \alpha_n(h)h_{\alpha_n}$ where h_{α_n} is the coroot associated to the root α_n



This reflection correspond to a change of sign. Thus we have that the Weyl group associated with $\mathfrak{so}_{2n}(\mathbb{C})$ is

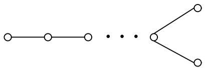
$$D_n = \mathbb{Z}_2^{n-1} \rtimes S_n$$

7.5 Cartan matrix and Dynkin diagram of $\mathfrak{so}_{2n}(\mathbb{C})$.

$$D_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & \dots & -1 & 0 & 2 \end{pmatrix}$$

The Dynkin diagram has n vertices corresponding with the fundamental roots. The Killing has the form $\langle \alpha_i, \alpha_{i+1} \rangle = 1$, if $i = 1, \dots, n-2$, $\langle \alpha_{n-2}, \alpha_{n-1} \rangle = 1$, $\langle \alpha_{n-2}, \alpha_n \rangle = 1$ and $\langle \alpha_{n-1}, \alpha_n \rangle = 0$. Thus the Dynkin diagram $\mathfrak{so}_{2n}(\mathbb{C})$

$D_n,\ n \geq 4$



7.6 Invariant functions under the action of $\mathbb{Z}_2^{n-1} \rtimes S_n$

Consider the polynomials

$$\begin{aligned} f_i &= \sum_{j=1}^{n} x_j^{2i}, \quad 1 \leq i \leq n-1 \\ f_n &= x_1 \cdots x_n \end{aligned}$$

clearly each f_i is invariant under the action of $\mathbb{Z}_2^{n-1} \rtimes S_n$. It is easy to check that

$$\mathrm{gr}(f_1)\mathrm{gr}(f_2)\ldots\mathrm{gr}(f_n) = 2^{n-1}n! = |\mathbb{Z}_2^{n-1} \rtimes S_n|$$

and

$$J = (-2)^{n-1}(n-1)! \prod_{1\leq i<j\leq n} (x_j^2 - x_i^2) \neq 0,$$

so the Jacobian criterion tell us that $f_1, \ldots, f_n$ is a basic set of invariants.

8 Orthogonal algebra $\mathfrak{so}_{2n+1}(\mathbb{C})$

The orthogonal odd algebra

$$\mathfrak{so}_{2n+1}(\mathbb{C}) = \{X : X^tS + SX = 0\}$$

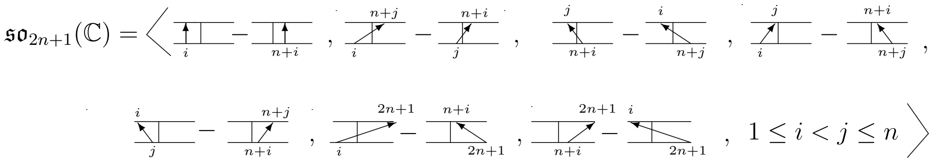
where $S \in M_{2n+1}(\mathbb{C})$ is of the form

$$S = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

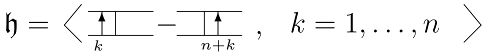
An explicit form

$$\mathfrak{so}_{2n+1}(\mathbb{C}) = \left\{ \begin{pmatrix} A & B & -H^t \\ C & -A^t & -G^t \\ G & H & 0 \end{pmatrix};\ \begin{array}{ll} A, B, C \in M_n(\mathbb{C}), & H, G \in M_{1\times n}(\mathbb{C}), \\ B = -B^t, & C = -C^t \end{array} \right\}.$$

$\mathfrak{so}_{2n+1}(\mathbb{C})$ as a subspace of **Digraph**$^1(2n+1, 2n+1)$ is given by

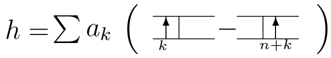


Let us fix $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{so}_{2n+1}(\mathbb{C})$

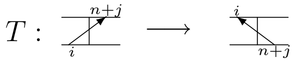


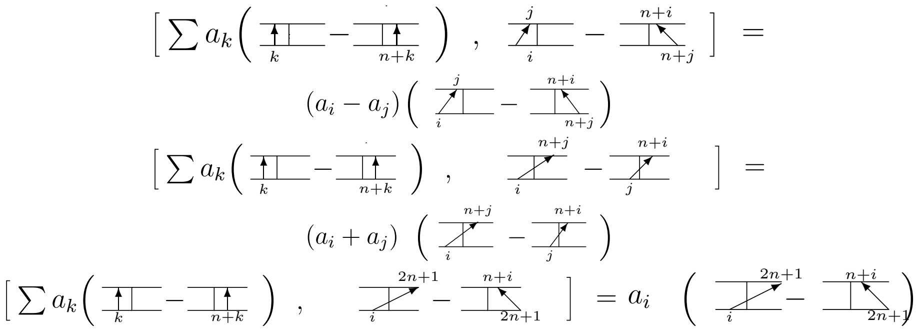
8.1 Root system of $\mathfrak{so}_{2n+1}(\mathbb{C})$

Let $h \in \mathfrak{h}$,

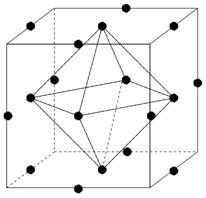


where $\{a_i\}$ is a base of $\mathfrak{h}^*$ dual to the natural base of $\mathfrak{h}$. We compute the positive roots. The negative roots are obtain applying the following antimorphism to the positive roots. $T : \textbf{Digraph}^1(2n+1, 2n+1) \longrightarrow \textbf{Digraph}^1(2n+1, 2n+1)$. For example



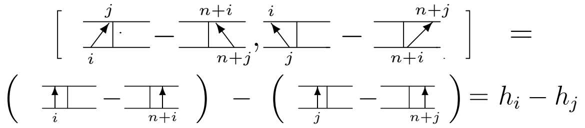


Thus the set of roots is $\Phi = \{a_i - a_j, a_j - a_i, a_i + a_j, a_i, -a_i\}$. The fundamental roots are $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_i = a_i - a_{i+1},\ \ i = 1, \ldots, n-1$ and $\alpha_n = a_n$. In pictures the root system of $\mathfrak{so}_7(\mathbb{C})$ looks like



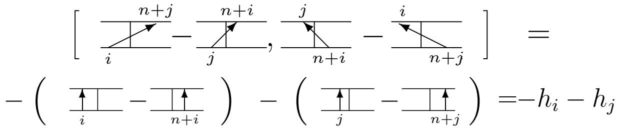
8.2 Coroots and weights of $\mathfrak{so}_{2n+1}(\mathbb{C})$

1. Coroots associated to the root $a_i - a_j$



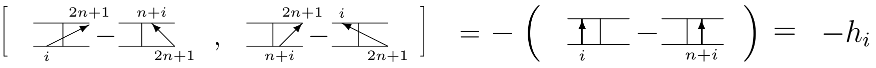
Since $(a_i - a_j)(h_i - h_j) = 2$, we see that $h_i - h_j$ is the coroot associated to the root $a_i - a_j$.

2. Coroot associated to the root $a_i + a_j$



Here $(a_i + a_j)(-h_i - h_j) = -2$, and thus $h_i + h_j$ is the coroot associated to the root $a_i + a_j$.

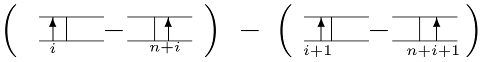
3. Coroot associated to the root a_i



Thus $2h_i$ is the coroot associated to the root a_i.

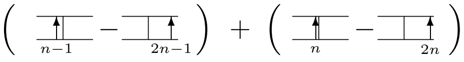
We have that $\Phi_c = \{h_i - h_j, h_j - h_i, h_i + h_j, -h_i - h_j, 2h_i, -2h_i,\ 1 \le i < j \le n\}$ is the coroot system of $\mathfrak{so}_{2n+1}(\mathbb{C})$. The set of fundamental coroot has the form $\Pi_c = \{h_i - h_{i-1}, 2h_n;\ \ 1 \le i \le n-1\}$ where

$$h_i - h_{i+1} =$$



and

$$h_{n-1} + h_n =$$



The fundamental weights are $w_i = a_1 + \cdots + a_i, \quad i = 1, \cdots, n-1$ and $w_n = \dfrac{a_1 + \cdots + a_n}{2}$. In a similar fashion to the $\mathfrak{so}_{2n}(\mathbb{C})$ case we have that $w_i(h_j - h_{j+1}) = \delta_{ij}$ and $w_i(2h_n) = \delta_{in}$.

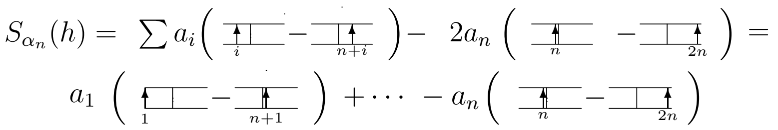
8.3 Killing form of $\mathfrak{so}_{2n+1}(\mathbb{C})$

Let x and y in $\mathfrak{h}$. Set

$$\begin{aligned}
\langle x, y\rangle &= \sum_{\alpha\in\Phi} \alpha(x)\alpha(y) \\
&= \sum_{i\leq j}(x_i - x_j)(y_i - y_j) + \sum_{i\leq j}(x_i + x_j)(y_i + y_j) + \sum_{j\leq i}(x_i - x_j)(y_i - y_j) \\
&\quad + \sum_{j\leq i}(x_i + x_j)(y_i + y_j) + \sum_i x_i y_i + \sum_i (2x_i)(2y_i) \\
&= (4n-2)\sum x_i y_i \\
&= (4n-2)\mathrm{tr}(xy)
\end{aligned}$$

8.4 Weyl group of $\mathfrak{so}_{2n+1}(\mathbb{C})$

Consider the fundamental roots $\alpha_i = a_i - a_{i+1}$. Just like for $\mathfrak{sl}_{n+1}(\mathbb{C})$, the reflections associated to these roots generate the symmetric group S_n. Let us analyze the reflection associated to the root $\alpha_n = a_n$. Let $h \in \mathfrak{h}$, we have $S_{\alpha_n}(h) = h - \alpha_n(h)h_{\alpha_n}$ where h_{α_n} is the coroot associated to the root α_n



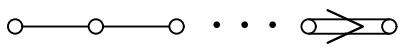
This reflections represent sign changes and generate the group $\mathbb{Z}_2^n$, therefore the Weyl group associated with $\mathfrak{so}_{2n+1}(\mathbb{C})$ is

$$B_n = \mathbb{Z}_2^n \rtimes S_n$$

8.5 Cartan matrix and Dynkin diagram of $\mathfrak{so}_{2n+1}(\mathbb{C})$.

$$B_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

The diagram has n vertices, one for each fundamental root. The killing form is given by $\langle \alpha_i, \alpha_{i+1} \rangle = 1$, if $i = 1, \dots, n-1$ and $\langle \alpha_{n-1}, \alpha_n \rangle = 2$. Furthermore $\langle \alpha_{n-1}, \alpha_{n-1} \rangle > \langle \alpha_n, \alpha_n \rangle$, and thus, the Dynkin diagram of $\mathfrak{so}_{2n+1}(\mathbb{C})$ has form

$B_n, \; n \geq 2$ 

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Rafael Díaz & Eddy Pariguan
Instituto Venezolano de Investigaciones Científicas.
Universidad Central de Venezuela.