A mathematical model for core-annular fluids with surfactants

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Abstract

We investigate the influence of surfactants in core-annular fluids without basic flow. The surfactant is considered insoluble in both film and core fluids. We derive a complete problem and obtain an asymptotic solution which consists of a system of two coupled non-linear partial differential equations. One describing the evolution of the interface and the other, the evolution of the surfactant concentration throughout the interface.

Resumen

Investigamos la influencia de surfactantes en fluidos centro-anulares en ausencia de flujo básico. El surfactante se considera insoluble tanto en el líquido anular como en el central. Derivamos un problema completo y obtenemos una solución asintótica del mismo, que consiste de un sistema acoplado de dos ecuaciones diferenciales parciales no lineales. Una para la evolución de la interfaz y la otra para la evolución de la concentración de surfactantes a lo largo de la interfaz.

Key words. core-annular flow, surfactants, interfacial tension.

Palabras claves. flujo centro-anular, surfactantes, tensión interfacial.

AMS subject classifications. 35Q35, 76E17, 76T99

1 Introduction

If a liquid desplaces another one that initially is contained in a capillar cylinder, a layer of the first fluid remains coating the cylinder walls (Taylor [12], 1961). **Core-annular flows** are parallel flows of immiscible fluids inside a cylinder. A fluid is at the center of the cylinder and the others in successive annuli that surround the core. An important application to the oil industry is lubricated pipelining, where the annular liquid lubricates the core liquid movement. Also, the core-annular flows occur during the liquid-liquid displacements in pourous media, and in lung airways since the internal surfaces of them are coated with a thin layer of liquid.

There are several studies related to core-annular flows for interfaces free of surfactants. Goren ([4], 1962) found that an annular film is unstable in the presence of infinitesimal sinusoidal disturbaces. He studied the linear stability of an annular film that coats the internal surface of a cylinder when the core fluid is inviscid or the surface of a wire when the ambient is inviscid.

A non-linear analysis for the adjustment, under surface tension, of a thin annular film, was developed by Hammond ([7], 1983) based on lubrication theory indicating that an initial sinusoidal disturbance of the interface could lead to the rupture of the film in the form of axisymmetric drops or "lenses" of the annular liquid separated by the core fluid. Later, Gauglitz and Radke ([3], 1988) developed an alternative approximation based on Hammond's analysis, including the exact expression of the curvature in the theory used by Hammond. Thus, the approximate equation reduces to the Young-Laplace's equation (see Adamson [1], 1976) for capillar static in regions where the fluid is almost static, and acts as the usual evolution equation when the film is thin. They found a critical thickness which marks the transition between films that evolve into collars and those that brakeup to form liquid lenses.

On the other hand, the presence of surfactant on a fluid-fluid interface can have a substantial effect on the evolution of the interface (Edwards, Brenner and Wasan [2], 1991). There are two known ways how surfactants affect the interfacial dynamic. One way is reducing the interfacial tension; i.e. the surface tension on an interface without surfactants is bigger than the surface tension on an interface with surfactants. The other way is introducing the Marangoni force caused by the presence of a gradient in the surfactant concentration. This is a force directed from regions of high surfactant concentration to regions of low surfactant concentration through the interface. In general, the Marangoni force acts to oppose any external flow that promotes accumulation of surfactant throughout the interface.

Applications to pulmonary fluid dynamics have been the motivation of most of the works on the effects of surfactants on core-annular flows. A thin liquid film coats the internal walls of the airways forming a liquid-air interface. The interfacial tension tries to minimize the interfacial area, so a colapse of the fine airways could happens due to the formation of a meniscus (from the liquid) while exhalation is in process. The biological surfactants tend to reduce the interfacial tension decrecing the attractive force between the film molecules. This way, surfactants have a stabilizing effect which prevent the colapse and keep the airways open. Halpern and Grotberg ([5], 1992) considered the stability, including the effect of surfactants, in a flexible cylinder. In ([6], 1993), they developed a nonlinear model taking into account the fluid mechanic of the film, the equation of motion of the tube and the transport equation. They concluded that surfactant convection has the effect of reducing the surface tension which prevent the meniscus formation.

Kas-Danouche, Papageorgiou and Siegel ([8], 2004) solved the problem of two fluids with a core-annular configuration inside a cylindrical tube with rigid walls. In the problem, they considered a pressure gradient (basic flow) and the presence of insoluble surfactants at the interface.

In this article, we explore the influence of surfactants on a core-annular configuration in the absence of the pressure gradient. The core liquid is sorrounded by another (annular) liquid. We assume that the surfactant is insoluble in both the film and the core. Physically, this corresponds to surfactants that have a very low solubility in the film and core fluids. So, the surfactant remains at the interface between the two fluids.

Here, as in ([8], 2004), we employ asymptotic analysis to derive very carefully a system of two coupled nonlinear partial differential equations that govern the evolution of the interface and the surfactant concentration.

2 Physical Problem and Governing Equations

We consider an annular liquid film, which we call fluid 2, that surrounds a cylindrical core fluid, which we call fluid 1, infinitely long inside a cylindrical horizontal tube of radius R. The internal surface of the tube is coated by the liquid film. The fluid 1 has an undisturbed radius b and viscosity μ_1. The viscosity of fluid 2 is μ_2. The gravitational effects are neglected (Hammond [7], 1983); i.e., the gravity does not appreciably change the shape of the interface.

Also, we consider that at the interface between the two fluids there are surfactants. The surfactant concentration, given in units of surfactant mass per unit of interfacial area, is denoted by Γ^*. The relationship between the interfacial tension σ and the surfactant concentration is given by the surface equation of state for the interfacial tension (Edwards, Brenner, and Wasan [2], 1991, Milliken, Stone, and Leal [10], 1993, and Stone and Leal [11], 1990)

$$\sigma \equiv \sigma(\Gamma) = \sigma_o + \Re T \Gamma_\infty \ln(1-\Gamma), \tag{1}$$

where σ_o is the interfacial tension of the clean (without surfactant) interface, $\Re$ is the ideal gas constant and T is the temperature. The dimensionless surfactant concentration is given by $\Gamma = \frac{\Gamma^*}{\Gamma_\infty}$, where Γ_∞ is the maximum packing concentration that the interface can support. Expanding $\ln(1-\Gamma)$ in Taylor series about $\Gamma = 0$ we obtain the linear relation between the interfacial tension σ and the surfactant concentration Γ expressed as follows

$$\sigma(\Gamma) = \sigma_o(1 - \beta\Gamma), \tag{2}$$

where $\beta = \dfrac{\Re T \Gamma_\infty}{\sigma_o}$ and σ_o, $\Re$, T, Γ_∞, and Γ are defined as before. Expression (2) is expected to hold in the dilute Γ limit. Even though this appears to be a restrictive assumption, our asymptotic solution is developed for small surfactant variations about a uniform state, in which instance (2) is the appropriate starting point.

We use cylindrical coordinates $\vec{x} = (r, \theta, z)$ with associated velocity components $\vec{u}_1 = (u_1, v_1, w_1)$ for the core and $\vec{u}_2 = (u_2, v_2, w_2)$ for the film. Let $h(z,t)$ be the typical thickness of the film, we define $S(z,t)$, the interface, as

$$S(z,t) = R - h(z,t), \tag{3}$$

where R is the tube radius.

For the evolution of the interface we start from the **Navier-Stokes** equations for axisymmetric flows with kinematic viscosity and pressure ν_1, p_1 for the core and, kinematic viscosity and pressure ν_2, p_2 for the film. We consider both regions with the same density ρ.

In order to complete the mathematical model, we require a no slip condition at the tube wall $\vec{u}_2 = 0$, continuity of velocity at the interface $\vec{u}_1 = \vec{u}_2$, kinematic condition which take the form

$$u_i = \frac{\partial S}{\partial t} + w_i \frac{\partial S}{\partial z}, \tag{4}$$

where $i = 1, 2$. Also, we require normal stress balance

$$\left[\vec{n} \cdot \underline{\underline{\mathrm{T}}} \cdot \vec{n}\right]_1^2 = \sigma \bigtriangledown_s \cdot \vec{n} \tag{5}$$

and tangential stress balance

$$\left[\vec{t} \cdot \underline{\underline{\mathrm{T}}} \cdot \vec{n}\right]_1^2 = - \bigtriangledown_s \sigma \cdot \vec{t}, \tag{6}$$

where $[\cdot]_1^2 = (\cdot)_2 - (\cdot)_1$, $\underline{\underline{\mathrm{T}}}$ is the stress tensor, $\bigtriangledown_s$ is the surface gradient operator, $\vec{n}$ the normal unit vector, and $\vec{t}$ the tangential unit vector.

Now, for the evolution of surfactant concentration, we start from the convective-diffusion equation for surfactant transport (Wong, Rumschitzki and Maldarelli [14], 1996)

$$\frac{\partial \Gamma}{\partial t} - \frac{\partial \vec{x}}{\partial t} \cdot \bigtriangledown_s \Gamma + \bigtriangledown_s (\Gamma u_s) - D_s \bigtriangledown_s^2 \Gamma + \Gamma \kappa \vec{u} \cdot \vec{n} = 0, \tag{7}$$

where κ is the surface curvature and D_s is the diffusivity constant. Using the surface gradient operator $\bigtriangledown_s$, surface divergence operator $\bigtriangledown_s \cdot$, the surface Laplacian oparator $\bigtriangledown_s^2$ (Wheeler and McFadden [13], 1994 and Kas-Danouche [9], 2002), and parameterizing the interface in terms of θ and z; i.e., we write $\vec{x} \equiv \vec{x}(\theta, z)$, we obtain:

The **normal stress balance**

$$\left\{-p+\frac{2\mu_i}{1+(S')^2}\left[(S')^2\frac{\partial w}{\partial z}-S'\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)-\frac{\partial u}{\partial r}\right]\right\}_1^2= \\ =\frac{\sigma(\Gamma)}{S\sqrt{1+(S')^2}}\left\{1-\frac{SS''}{1+(S')^2}\right\}, \tag{8}$$

where $i=1,2$.

The **tangential stress balance**

$$\left\{\frac{\mu_i}{1+(S')^2}\left[2S'\left(\frac{\partial u}{\partial r}-\frac{\partial w}{\partial z}\right)+(1-(S')^2)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\right]\right\}_1^2= \\ =\frac{-1}{\sqrt{1+(S')^2}}\frac{\partial\sigma}{\partial z}, \tag{9}$$

where $i=1,2$.

The **convective-diffusion equation for surfactant transport**

$$\begin{aligned}\frac{\partial\Gamma}{\partial t} &- \frac{\dot{S}S'}{1+(S')^2}\frac{\partial\Gamma}{\partial z}+\frac{1}{S\sqrt{1+(S')^2}}\left\{\frac{\partial}{\partial z}\left[\frac{S\Gamma}{\sqrt{1+(S')^2}}(w+S'u)\right]\right\}\\ &- D_s\frac{1}{S\sqrt{1+(S')^2}}\frac{\partial}{\partial z}\left(\frac{S}{\sqrt{1+(S')^2}}\frac{\partial\Gamma}{\partial z}\right)\\ &+ \frac{\Gamma}{S(1+(S')^2)}\left[1-\frac{SS''}{1+(S')^2}\right](-S'w+u)=0.\end{aligned} \tag{10}$$

3 Non-dimensionalization Process

We non-dimensionalize using the tube radius R as the length unit, the interfacial tension σ_0 in the absence of surfactant, the film viscosity μ_2 and the surfactant uniform concentration Γ_0. Γ_0 is considered as the surfactant concentration at the interface in the absence of any movement of the fluids.

This way, we re-scale the pressure with σ_o/R, velocities with σ_o/μ_2, time with $\mu_2 R/\sigma_o$, and the concentration of surfactant with Γ_o. In what follows we write the dimensionless equations of our model. For the Navier-Stokes equations, the non-dimensionalization process introduces the **Reynolds number** (Re) which is defined as $Re=\sigma_0\rho R/\mu_2^2$. The non-dimensionalization of the transport equation produces the **Peclet number** (Pe) which is given by $Pe=\sigma_0 R/(\mu_2 D_s)$. We are using the same notation for the dimensional and non-dimensional variables.

The **non-dimensional Navier-Stokes and continuity** equations are

$$Re\,[(w_i)_t + w_i(w_i)_z + u_i(w_i)_r] = -(p_i)_z + \frac{\mu_i}{\mu_2}\,\bigtriangledown^2 w_i \tag{11}$$

$$Re[(u_i)_t + w_i(u_i)_z + u_i(u_i)_r] = -(p_i)_r + \frac{\mu_i}{\mu_2}\left(\bigtriangledown^2 u_i - \frac{u_i}{r^2}\right) \tag{12}$$

$$(w_i)_z + \frac{1}{r}(ru_i)_r = 0, \tag{13}$$

with $i = 1$ for the core and $i = 2$ for the film, where

$$\bigtriangledown^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) \quad \text{and} \quad Re = \sigma_0 \rho R/\mu_2^2. \tag{14}$$

The **non-dimensional normal stress balance** equation is

$$\left\{-p + \frac{2\lambda_i}{1+(S')^2}\left[(S')^2\frac{\partial w}{\partial z} - S'\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) - \frac{\partial u}{\partial r}\right]\right\}_1^2 = \\ = \frac{\sigma(\Gamma)}{S\sqrt{1+(S')^2}}\left\{1 - \frac{SS''}{1+(S')^2}\right\}, \tag{15}$$

where $\lambda_i = \frac{\mu_i}{\mu_2}, i = 1, 2.$

The **non-dimensional tangential stress balance** equation is

$$\left\{\frac{\lambda_i}{1+(S')^2}\left[2S'\left(\frac{\partial u}{\partial r} - \frac{\partial w}{\partial z}\right) + (1-(S')^2)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right]\right\}_1^2 = \\ = \frac{-1}{\sqrt{1+(S')^2}}\frac{\partial \sigma}{\partial z}, \tag{16}$$

where $\lambda_i = \frac{\mu_i}{\mu_2}, i = 1, 2.$

Finally, the **non-dimensional convective-diffusion equation for the surfactant transport** is

$$\begin{aligned}\frac{\partial \Gamma}{\partial t} &- \frac{\dot{S}S'}{1+(S')^2}\frac{\partial \Gamma}{\partial z} + \frac{1}{S\sqrt{1+(S')^2}}\left\{\frac{\partial}{\partial z}\left[\frac{S\Gamma}{\sqrt{1+(S')^2}}(w + S'u)\right]\right\} \\ &- \frac{1}{P_e}\frac{1}{S\sqrt{1+(S')^2}}\frac{\partial}{\partial z}\left(\frac{S}{\sqrt{1+(S')^2}}\frac{\partial \Gamma}{\partial z}\right) \\ &+ \frac{\Gamma}{S(1+(S')^2)}\left[1 - \frac{SS''}{1+(S')^2}\right](-S'w + u) = 0\end{aligned} \tag{17}$$

where $P_e = \frac{\sigma_0 R}{\mu_2 D_s}$.

4 Derivation of the Model

Here, we take advantage of the film thickness relative to the core thickness and derive a coupled system of two approximated evolution equations. One equation describes the evolution of the interface between the two fluids, and the other one describes the evolution of surfactant concentration at the interface. In the **non-dimensional unperturbed state** the tube radius is 1 and the distance from the tube wall to the interface is $\varepsilon = a/R << 1$, where $h(z,t) = aH(z,t)$ and $a = R - b$.

At the film, we define the radial component of a point as $r = 1 - \varepsilon y$, where y is 0 at the tube wall and $H(z,t)$ on the interface. The axial component is defined as $z = \hat{z}$. Thus,

$$\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial \hat{z}} \quad \text{and} \quad \frac{\partial}{\partial r} \longrightarrow -\frac{1}{\varepsilon}\frac{\partial}{\partial y}. \tag{18}$$

From the continuity condition we know that, in order to balance both terms, the order of u_2 must equals ε times the order of w_2. From the first momentum equation (11) for $i = 2$, we find the order of w_2. But before that we look for the order of pressure. Let $\bar{p}_1$, $\bar{p}_2$, $\tilde{p}_1$, and $\tilde{p}_2$ be the basic state pressure in region 1 which is the core fluid, the basic state pressure in region 2 which is the film, the perturbed pressure in region 1 and the perturbed pressure in region 2, respectively. Then,

$$p_1 = \bar{p}_1 + \tilde{p}_1 \tag{19}$$

$$p_2 = \bar{p}_2 + \tilde{p}_2. \tag{20}$$

First, let us consider that the core is a non-viscous fluid, then, $\lambda_1 = \frac{\mu_1}{\mu_2} = 0$. The pressure of the core fluid is constant and, by a convinient choice of reference pressure, $p_1 = 0$ ($\bar{p}_1 = \tilde{p}_1 = 0$). Let us consider the asymptotic expansion of the Navier-Stokes equation (11) in the film and $Re \sim O(1)$. Balancing the pressure with the leading order term ($\partial^2 w_2/\partial r^2$), we obtain

$$\tilde{p}_2 \sim \frac{1}{\varepsilon^2}(\text{order of } w_2). \tag{21}$$

The basic state for the surfactant concentration Γ is taken as a uniform covering of the interface, which in dimensionless terms is given by $\Gamma = 1$. Therefore, the perturbed surfactant concentration can be expressed by $\Gamma = 1 + \tilde{\Gamma}$, where $\tilde{\Gamma}$ is the perturbation of the surfactant concentration. The dimensionless interfacial tension is $\sigma(\Gamma) = 1 - \beta\Gamma$ and so, the basic state of the interfacial tension is $\sigma(\Gamma) = 1 - \beta$. All the left hand side terms of the normal stress balance equation (15), are of order w_2, except the pressure p_2. Due to (21), $\bar{p}_2$ and $\tilde{p}_2$ are the leading order terms of the left hand side of (15). In the right hand side of (15) we have $1-\beta+O(\varepsilon)$. Thus

$$\bar{p}_2 = \beta - 1 \sim O(1) \tag{22}$$
$$\tilde{p}_2 \sim O(\varepsilon). \tag{23}$$

4.1 Derivation of the Interface Evolution Equation

Now, let us assume that $\lambda_1 > 0$, but not too big. The asymptotic expansion of the Navier-Stokes equation (11) in the core and $Re \sim O(1)$ give us

$$\tilde{p_1} \sim \text{ order of } u_1. \tag{24}$$

The continuity of the axial velocity at the interface produces order of $w_1 =$ order of w_2. Therefore, order of $\tilde{p}_1 \ll$ order of $\tilde{p}_2$.

Similarly, all the left hand side terms of the normal stress balance equation (15), are of the order of w_2, except the pressures which are the leading order terms in the left hand side. In the right hand side of (15) we have $1-\beta+O(\varepsilon)$. Then, $\bar{p}_1 - \bar{p}_2 = 1-\beta$ and $\tilde{p}_2 \sim O(\varepsilon)$.

Therefore, $w_2 \sim O(\varepsilon^3)$ and $u_2 \sim O(\varepsilon^4)$. The continuity of the axial velocity at the interface implies that $w_1 \sim O(\varepsilon^3)$, the axial velocity in the core. Thus, in the left hand side of Navier-Stokes equation (11), in order to balance the second and third terms, we have $u_1 \sim O(\varepsilon^3)$. By (24) we have that $\tilde{p}_1 \sim O(\varepsilon^3)$, and from the tangential stress balance equation (16) we need that $\lambda_1 \ll \frac{1}{\varepsilon}$ for the core to decouple from the film.

In what follows, we look for the time scale. Let us consider (4), the kinematic condition, u and $w\frac{\partial S}{\partial z}$ are of order ε^4; therefore, $\frac{\partial S}{\partial t} \sim O(\varepsilon^4)$. Since $\frac{\partial S}{\partial t} = -\varepsilon H_t(z,t)$, we have that $H_t \sim O(\varepsilon^3)$. This motivates the introduction of the scaled variable for time $\tau = \varepsilon^3 t$ such that

$$\frac{\partial}{\partial t} \longrightarrow \varepsilon^3 \frac{\partial}{\partial \tau}. \tag{25}$$

Continuing in this way and keeping only the leading order terms in (11) for $i=2$, we obtain $(\tilde{u}_2)_{yy} = (\tilde{p}_2)_z$; but, from the second momentum equation (12) for $i=2$, we have $p_y \sim O(\varepsilon^3)$. So, $\tilde{p}_2 \equiv \tilde{p}_2(z)$ is not a function of y in the resulting equation retaining the leading order terms. Therefore, we can integrate twice with respect to y and apply $\tilde{w}_2(0) = 0$ (no slip at the tube wall) to obtain

$$\tilde{u}_2(y) = \frac{1}{2}(\tilde{p}_2)_z y^2 + A(z)y. \tag{26}$$

However, keeping the leading order terms $\frac{1}{\varepsilon}(w_2)_y$ and $\beta\Gamma_z$ in the tangential stress balance (16), we obtain

$$\frac{1}{\varepsilon}(w_2)_y = \beta\Gamma_z, \tag{27}$$

but, $(w_2)_y \sim O(\varepsilon^3)$. Choosing $\beta = \varepsilon^2\beta_o$, we have $(\tilde{w}_2)_y = \beta_0\Gamma_z$. Now, using (26) we find

$$A(z) = \beta_o\Gamma_z - (\tilde{p}_2)_z H. \tag{28}$$

Thus,

$$(\tilde{w}_2)(y) = \frac{1}{2}(\tilde{p}_2)_z y^2 + (\beta_o\Gamma_z - (\tilde{p}_2)_z H)y. \tag{29}$$

On the other hand, using the condition of normal stress balance (15) we find the pressure. Firstly, we take the terms of $O(1)$ and obtain $\bar{p}_1 - \bar{p}_2 = 1 - \beta$. In the next step, we take the terms of $O(\varepsilon)$ obtaining $\tilde{p}_2 = -H_{zz} - H$. Therefore, $\tilde{w}_2$ becomes

$$(\tilde{u}_2)(y) = -\frac{1}{2}\left(H_{zzz} + H_z\right) y^2 + \left[\beta_o\Gamma_z + \left(H_{zzz} + H_z\right) H\right] y. \tag{30}$$

Now, using the continuity condition (13) for $i = 2$ and the scales for w_2, u_2 and r, we obtain considering only leading order terms

$$\frac{\partial \tilde{u}_2}{\partial z} = \frac{\partial \tilde{v}_2}{\partial y}. \tag{31}$$

So, differentiating $\tilde{w}_2$ with respect to z and integrating it with respect to y, we find an expression for $\tilde{u}_2$

$$\begin{aligned} \tilde{u}_2 &= -\frac{1}{6}(H_{zzzz} + H_{zz})y^3 + \frac{1}{2}\beta_o\Gamma_{zz}y^2 + \frac{1}{2}(H_{zzzz} + H_{zz})Hy^2 \\ &+ \frac{1}{2}(H_{zzz} + H_z)H_z y^2. \end{aligned} \tag{32}$$

In order to obtain the evolution equation for H, we consider the kinematic equation (4) and the fact that

$$\begin{aligned} \frac{\partial S}{\partial t} = -\varepsilon H_t &= -\varepsilon^4 H_\tau \\ \frac{\partial S}{\partial z} = -\varepsilon H_z. & \end{aligned}$$

Then,

$$\tilde{u}_2 = -H_\tau - \tilde{w}_2 H_z. \tag{33}$$

Substituting $\tilde{w}_2$ and $\tilde{u}_2$ in (33), and evaluating it at the interface ($y = H$), we obtain the **evolution equation for the interface**

$$H_\tau = -\frac{1}{3}\left[(H_{zzz} + H_z)H^3\right]_z - \frac{1}{2}\beta_o\left(\Gamma_z H^2\right)_z. \tag{34}$$

4.2 Derivation of the Evolution Equation of Surfactant Concentration

Now, in order to derive the equation for Γ, we use the scales in the equation of surfactant concentration (17) and take the leading order terms, to find

$$\Gamma_\tau + \frac{\partial}{\partial z}(\tilde{w}_2\Gamma) + \frac{1}{\tilde{Pe}}\frac{\partial^2\Gamma}{\partial z^2} = 0. \tag{35}$$

Finally, substituting $\tilde{w}_2$ evaluated at $y = H$, we obtain the **evolution equation of surfactant concentration** through the interface

$$\Gamma_\tau = -\left[\left(\frac{1}{2}(H_{zzz} + H_z)H^2 + \beta_o \Gamma_z H\right)\Gamma\right]_z - \frac{1}{\tilde{Pe}}\Gamma_{zz}. \tag{36}$$

5 Re-scaling of the Model

We want to re-scale z from $[0, L]$ to $[0, 2\pi]$. For that, we consider the change of variables $z = \frac{2\pi}{L}\tilde{z}$ and $t = \left(\frac{2\pi}{L}\right)^2\tilde{t}$, where $\tilde{z} \in [0, L]$, $\tilde{t} \geq 0$, and the variables with '$\tilde{}$' represent the non-scaled variables. Then, we have $z \in [0, 2\pi]$, $t \geq 0$,

$$\frac{\partial}{\partial \tilde{z}} = \frac{2\pi}{L}\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial \tilde{t}} = \left(\frac{2\pi}{L}\right)^2\frac{\partial}{\partial t}.$$

Writing (34) and (36) in terms of the new scaled variables, we obtain the following coupled system of two non linear partial differential equations

$$H_t = -\frac{1}{3}\left[H^3\left(\lambda^2 H_{zzz} + H_z\right)\right]_z - \frac{1}{2}\beta_0\left(\Gamma_z H^2\right)_z \tag{37}$$

and

$$\Gamma_t = -\left\{\left[\frac{1}{2}\left(\lambda^2 H_{zzz} + H_z\right)H^2 + \beta_0\Gamma_z H\right]\Gamma\right\}_z - \frac{1}{Pe}\Gamma_{zz}\left(\Gamma_z H^2\right)_z, \tag{38}$$

where $\lambda = \frac{2\pi}{L}$. This system is more complicated than the one obtained for the problem with basic flow studied in ([8], 2004).

Now, we are exploring numerical schemes in order to solve (37) and (38). Thus, it is convinient to isolate the terms in (37) and (38) that contain the derivatives of H and Γ of higher order. Therefore, we re-write them as

$$H_t = -\frac{1}{3}\lambda^2 H^3 H_{zzzz} - \frac{1}{2}\beta_0 H^2\Gamma_{zz} - f(H, H_z, H_{zz}, H_{zzz}, \Gamma_z)$$

and

$$\Gamma_t = -\frac{1}{2}\lambda^2 H^2 \Gamma H_{zzzz} - \left(\beta_0 \Gamma H + \frac{1}{Pe}\right)\Gamma_{zz} - g(H, H_z, H_{zz}, H_{zzz}, \Gamma, \Gamma_z)$$

where,

$$\begin{aligned} f(H, H_z, H_{zz}, H_{zzz}, \Gamma_z) &= \frac{1}{3}H^3 H_{zz} + \lambda^2 H^2 H_z H_{zzz} + H^2 H_z^2 \\ &+ \beta_0 H H_z \Gamma_z, \\ g(H, H_z, H_{zz}, H_{zzz}, \Gamma, \Gamma_z) &= \frac{1}{2}H^2 H_{zz}\Gamma + \lambda^2 H H_z H_{zzz}\Gamma + H H_z^2 \Gamma \\ &+ \frac{1}{2}\lambda^2 H^2 H_{zzz}\Gamma_z + \frac{1}{2}H^2 H_z \Gamma_z + \beta_0 H \Gamma_z^2 \\ &+ \beta_0 H_z \Gamma \Gamma_z \end{aligned}$$

and $\lambda = \frac{2\pi}{L}$.

For the case when $\Gamma = 0$; i.e., there is not surfactant in our problem, we have

$$H_t = -\frac{1}{3}\left(H^3(\lambda^2 H_{zzz} + H_z)\right)_z, \tag{39}$$

which is the equation, originally, derived by Hammond in ([7], 1983).

Hammond solved this equation numerically replacing the spacial derivatives with finite differences (**The Line Method**). However, we have developed some numerical methods that can be applied to (39). One method uses **Fast Fourier Transform (FFT)** for the spacial derivatives and **finite differences** for H_t. This is what is known as a **Pseudospectral Method**. Even though this method is spectrally precise, has a restriction in the time step $\Delta t \sim O(\Delta x^4)$. But, this explicit method is simple to code and helpful to compare with results of implicit methods which are more complicated. An efficient implicit method for (39) that does not have such restriction with the time step will be the subject of future work which will help us to construct a numerical scheme able to solve the system (37) and (38).

6 Conclusion

We have derived a system of two coupled non linear partial differential equations which model the evolution of core-annular fluids without pressure gradient. One equation describes the evolution of the interface and the other the evolution of the surfactant concentration at the interface.

We note that, when $\Gamma = 0$, the system of equations (37) and (38) derived in this research is transformed in equation (39) which, originally, was derived by Hammond ([7], 1983). So, we can conclude that the presence of surfactants in our problem influence the evolution of the interface in the sense that a new term appears in the interface evolution equation as we compare (37) with (39).

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