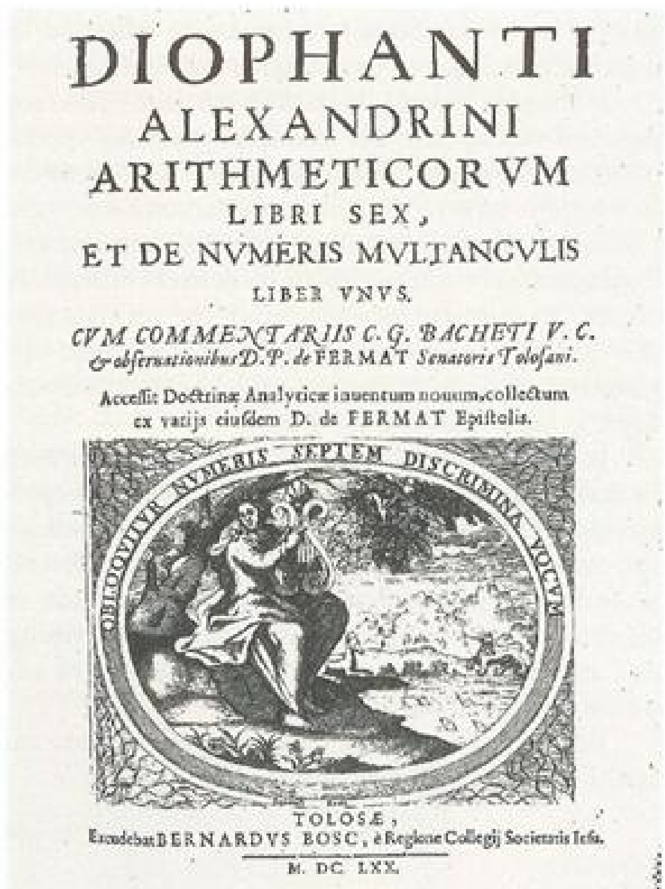

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La ecuación $\dfrac{x^2+y^2}{xy+1}=n^2$, pag. 143

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Asociación Matemática Venezolana

La Asociación Matemática Venezolana fue legalmente fundada en 1990 como una organización civil cuya finalidad es trabajar por el desarrollo de la matemática en Venezuela. Para más información ver su portal de internet:
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Asociación Matemática Venezolana

Boletín de la Asociación Matemática Venezolana

Vol. XVIII • No. 2 • Año 2011

La sección de Artículos, de este número, comienza con un trabajo de Marquina, Quintero y Viloria sobre, "Expansión de las soluciones para ecuaciones integrales". A continuación tenemos los trabajos de Murugusundaramoorthy et al; Olaleru y Mogbademu, y finalizamos con el artículo de Antonio Oller sobre, "Counting domino trains". Considerando lo popular que es el juego de dominó, en algunos países de Latino América y el Caribe, sin duda que despertará la curiosidad de nuestros colegas aficionados a ese pasatiempo.

En la 29° International Mathematical Olympic Games of Mathematics (IMO) realizada en Canberra, Australia, en 1988, fue propuesto un interesante problema: Sean a y b enteros positivos tales que $ab+1$ divide a^2+b^2. Demostrar que $\frac{a^2+b^2}{ab+1}$ es el cuadrado de un entero. En su artículo Luis Gómez Sánchez analiza la ecuación diofántica $F(x,y)=n^2$ y su conexión con ese problema.

En Información Nacional, Rafael Sánchez nos informa de la actividad olímpica efectuada en el segundo semestre del 2011. Por otro lado, le anunciamos a nuestros lectores la organización de la XXV Jornada Venezolana de Matemática, evento a realizarse en Cumaná, Venezuela, del 26 al 29 de marzo de 2012. Para mayor información recomendamos consultar la página web de las jornadas 2012:

http://postmat.sucre.udo.edu.ve/jornadas

Finalmente, expresamos nuestro agradecimiento a nuestros colegas que colaboraron en la edición de este volumen.

Oswaldo Araujo G.
Editor

ARTÍCULOS

Expansión de las soluciones para ecuaciones integrales cuadráticas.

Eribel Marquina, Javier Quintero, Nelson Viloria.

Resumen. En este artículo encontramos la expansión de la solución de

$$x(t) + \int_a^t d_s K(t,s) g(s) B x(s) = u(t), \qquad t \in [a,b],$$

basada en la teoría de representación de operadores multilineales aplicada a operadores bilineales.

Abstract. In this article we find the expansion of the solution of

$$x(t) + \int_a^t d_s K(t,s) g(s) B x(s) = u(t), \qquad t \in [a,b],$$

based on the theory of representation of multilinear operators applied to bilinear operators.

Introducción

Sean X, Y espacios de Banach y consideremos la ecuación integral no lineal de Volterra-Stieltjes del tipo

$$x(t) + \int_a^t d_s K(t,s) f(s, x(s)) = u(t), \qquad t \in [a,b], \qquad (K),$$

donde x es una función reglada incógnita, u es una función reglada conocida, K es una función simplemente reglada como función de t y uniformemente de semivariación de Fréchet como función de s, anulándose en la diagonal; y la no

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Keywords: Ecuaciones Integrales de Volterra-Stieltjes, Ecuaciones Integrales Cuadráticas, Integral de Dushnik, Semivariación Acotada, Funciones Regladas, Teoremas de Representación de Riez, Expansiones de Volterra.

linealidad de (K) está dada por $f : [a,b] \times X \longrightarrow X$, con $f(t,x) = g(t)Bx$, donde g es una función reglada y $Bx = L_2 x$ un operador polinomial de grado dos sobre X.

Daremos la expansión de la solución de (K), basándonos en la Teoría de representación de operadores multilineales, aplicada a operadores bilineales.

1 Funciones regladas

Consideremos X, Y, W y Z espacios de Banach, y $[a,b] \subset \mathbb{R}$ un intervalo cerrado.

Una función $x : [a,b] \longrightarrow X$ es una **función reglada** si sólo tiene discontinuidades de primera especie, es decir, si

i) para todo $t \in [a,b)$ existe $x(t^+) = \lim_{h \downarrow 0} x(t+h)$ y

ii) para todo $t \in (a,b]$ existe $x(t^-) = \lim_{h \downarrow 0} x(t-h)$.

Al espacio de las funciones regladas de $[a,b]$ en X lo denotamos por $G([a,b];X)$, tal espacio es un espacio de Banach con la norma del supremo.

Teorema 1.1. *(Hönig[3], Theorem 1.3.1) Una función $x : [a,b] \longrightarrow X$ es reglada si, y sólo si, existe una sucesión de funciones escalonadas*

$$(\varphi_n)_{n \geq 1} : [a,b] \longrightarrow X, \text{ tal que } \lim_{n \to \infty} \varphi_n(t) = x(t) \ \forall \ t \in [a,b].$$

Una función $x : [a,b] \longrightarrow X$ es **reglada por la izquierda** si $x(a) = 0$ y $x(t) = x(t^-)$ para todo $t \in (a,b]$. A este subespacio cerrado de $G([a,b];X)$ lo denotamos por $G^-([a,b];X)$.

Una función $x : [a,b] \longrightarrow L(W,X)$ es **simplemente reglada** si, para todo $w \in W$, la función

$$\begin{aligned} xw : [a,b] &\longrightarrow X \\ t &\mapsto x(t)w, \qquad \text{es reglada} \end{aligned}$$

y escribimos $x \in G^\sigma([a,b];L(W,X))$, que es un espacio de Banach con la norma dada por

$$\|x\| = \sup_{t \in [a,b]} \|x(t)\|_{L(W,X)} \qquad \forall \, x \in G^\sigma([a,b];L(W,X)).$$

Además,

$$G([a,b];L(W,X)) \subset G^\sigma([a,b];L(W,X)). \qquad \text{(Arbex [1])}$$

2 Funciones de variación y semivariación acotada

Una partición de un rectángulo $[a_1, b_1] \times [a_2, b_2]$ es el conjunto $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ con $\mathcal{P}_r \in \mathbb{P}[a_r, b_r]$, donde $a_r = t_0 < \ldots < t_{n(r)} = b_r$.

$\mathbb{P}\left([a_1, b_1] \times [a_2, b_2]\right)$ denota el conjunto de todas las particiones del rectángulo. Además, $n(\mathcal{P}) = n(\mathcal{P}_1) \times n(\mathcal{P}_2)$, $|\mathcal{P}| = |\mathcal{P}_1| \times |\mathcal{P}_2|$.

Sean $z : [a_1, b_1] \times [a_2, b_2] \longrightarrow Z$ y $\mathcal{P} \in \mathbb{P}\left([a_1, b_1] \times [a_2, b_2]\right)$. Consideremos $i(1), i(2) \in \mathbb{N}$, con $1 \leq i(r) \leq n(\mathcal{P}_r)$. Definimos

- $\Delta_{i(1)} z : [a_2, b_2] \longrightarrow Z$ por

$$\Delta_{i(1)} z(s) = z(t_{i(1)}, s) - z(t_{i(1)-1}, s) \qquad \forall\, s \in [a_2, b_2]$$

- $\Delta_{i(2)} z : [a_1, b_1] \longrightarrow Z$ por

$$\Delta_{i(2)} z(s) = z(s, t_{i(2)}) - z(s, t_{i(2)-1}) \qquad \forall\, s \in [a_1, b_1]$$

En particular, para $[a, b] \subset \mathbb{R}, z : [a, b] \longrightarrow Z$ está dada por

$$\Delta_i z = z(t_i) - z(t_{i-1}).$$

Luego, $\Delta_{i(1)} \Delta_{i(2)} z$ denota el cálculo $\Delta_{i(1)} \left(\Delta_{i(2)} z\right)(s)$.

Dados $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ y $(Y, \|\cdot\|)$ espacios de Banach. Consideremos, en $X_1 \times X_2$, la topología producto inducida por las normas sobre X_1, X_2, es decir,

$$\|x\|_{X_1 \times X_2} = \sup\{\|x_1\|_1, \|x_2\|_2\}.$$

Una aplicación $q : X_1 \times X_2 \longrightarrow Y$ es bilineal si es lineal en cada variable por separado. Diremos que $q \in L\left(X_1 \times X_2, Y\right)$ si q es bilineal y continua (es decir, $\exists\, M > 0$ tal que $\|q(x_1, x_2)\| \leq M \|x_1\| \|x_2\|$).

Sea $\alpha : [a, b] \longrightarrow L(X, Y)$. Se define la **semivariación de** α en $[a, b]$ por

$$\begin{aligned} SV[\alpha] &= \sup_{\mathcal{P} \in \mathbb{P}[a,b]} SV[\alpha; \mathcal{P}] \\ &= \sup_{\mathcal{P} \in \mathbb{P}[a,b]} \sup_{\substack{x_i \in X \\ \|x_i\| \leq 1}} \left\{ \left\| \sum_{i=1}^{n(\mathcal{P})} [\alpha(t_i) - \alpha(t_{i-1})] x_i \right\| \right\}. \end{aligned}$$

Si $SV[\alpha] < \infty$, entonces α es una función de **semivariación acotada** y se escribe $\alpha \in SV([a,b]; L(X,Y))$.

Sea $K : [a_1, b_1] \times [a_2, b_2] \longrightarrow Z$. Se define **variación de Vitali de** K en $[a_1, b_1] \times [a_2, b_2]$ por

$$V[K] = \sup_{\mathcal{P} \in \mathbb{P}([a_1,b_1]\times[a_2,b_2])} V[K; \mathcal{P}],$$

donde

$$V[K; \mathcal{P}] = \sum_{i(1),i(2)}^{n(\mathcal{P})} \|\Delta_{i(1)} \Delta_{i(2)} K\|$$

Si $V[K] < \infty$, entonces K es una función de **variación acotada de Vitali en** $[a_1, b_1] \times [a_2, b_2]$ y se escribe $K \in BV\left([a_1, b_1] \times [a_2, b_2]; Z\right)$.

Sea $K : [a_1, b_1] \times [a_2, b_2] \longrightarrow L(X, Y)$, se define la **semivariación de Vitali de** K en $[a_1, b_1] \times [a_2, b_2]$ como

$$SV[K] = \sup_{\mathcal{P} \in \mathbb{P}([a_1,b_1]\times[a_2,b_2])} SV[K; \mathcal{P}],$$

donde

$$SV[K; \mathcal{P}] = \sup_{\|x_{i(1)i(2)}\| \leq 1} \left\{ \left\| \sum_{i(1),i(2)}^{n(\mathcal{P})} \Delta_{i(1)} \Delta_{i(2)} K x_{i(1)i(2)} \right\| : x_{i(1)i(2)} \in X \right\}.$$

Si $SV[K] < \infty$, entonces K se dice de **semivariación acotada de Vitali** y se escribe

$$K \in SV\left([a_1, b_1] \times [a_2, b_2]; L(X, Y)\right).$$

Teorema 2.1. *(Viloria[8]; Teorema 2.1.1)*
$BV\left([a_1, b_1] \times [a_2, b_2], L(X, Y)\right) \subset SV\left([a_1, b_1] \times [a_2, b_2], L(X, Y)\right)$ y si $K \in BV\left([a_1, b_1] \times [a_2, b_2], L(X, Y)\right)$, entonces $SV[K] \leq V[K]$.

Sea $K : [a_1, b_1] \times [a_2, b_2] \longrightarrow L(X_1, X_2; Y)$. Se define la **variación de Fréchet** de K en $[a_1, b_1] \times [a_2, b_2]$ por

$$SF[K] = \sup_{\mathcal{P}\in\mathbb{P}([a_1,b_1]\times[a_2,b_2]} SF[K;\mathcal{P}]$$

donde

$$SV[K;\mathcal{P}] = \sup_{\|x_{i(r)}\|\leq 1}\left\{\left\|\sum_{i(1),i(2)}^{n(\mathcal{P})}\Delta_{i(1)}\Delta_{i(2)}K\left(x_{i(1)},x_{i(2)}\right)\right\| : x_{i(r)}\in X_r\right\}$$

Si $SF[K]<\infty$, entonces K se dice de **semivariación acotada de Fréchet** y se escribe $K\in SF\left([a_1,b_1]\times[a_2,b_2];L(X_1,X_2;Y)\right)$.

Teorema 2.2. *(Viloria [8]; Teorema 2.1.2)*
$BV\left([a_1,b_1]\times[a_2,b_2],L\left(X_1,X_2;Y\right)\right)\subset SF\left([a_1,b_1]\times[a_2,b_2];L\left(X_1,X_2;Y\right)\right)$
Además, si $K\in BV\left([a_1,b_1]\times[a_2,b_2],L\left(X_1,X_2;Y\right)\right)$, *entonces*

$$SF[K]<V[K].$$

3 Integral interior de Dushnik

La integral interior de Dushnik fue concebida por Pollard, en 1920, y redescubierta por Dushnik, en 1931. Posteriormente fue utilizada por Kaltenborn, en 1934, para representar los elementos del espacio $L((G[a,b],\mathbb{R}),\mathbb{R})$, resultado generalizado por Hönig, en 1975, al espacio $L(G([a,b],X);Z)$ y extendido por Viloria [7], en 1997, para representar los elementos de los espacios $L\left(\prod_{r=1}^{m}G^-([a,b],X);Z\right)$ y $L\left(\prod_{r=1}^{m}G^-([a_r,b_r];X_r),G^-([a,b];Y)\right)$, y también para los operadores causales en $L\left(\prod_{r=1}^{m}G^-([a_r,b_r];X),G^-([a,b];Y)\right)$. En este trabajo es empleada para representar, de manera particular, los elementos de $L\left(G^-([a_1,b_1];X)\times G^-([a_2,b_2];X);G^-([a,b];Y)\right)$.

Sean $e,(e_{\mathcal{P}})_{\mathcal{P}\in\mathbb{P}}$ en un espacio topológico E, escribiremos $\lim_{\mathcal{P}\in\mathbb{P}} e_{\mathcal{P}}=e$ si, para todo entorno V de e, existe $\mathcal{P}_V\in\mathbb{P}$ tal que

$$\mathcal{P}\geq\mathcal{P}_V\Leftrightarrow e_{\mathcal{P}}\in V.$$

Definición 3.1. *(Integral doble de Dushnik)*
Sean $K:[a_1,b_1]\times[a_2,b_2]\longrightarrow L(X_1,X_2;Y), x_1:[a_1,b_1]\longrightarrow X_1$ *y* $x_2:[a_2,b_2]\longrightarrow X_2$. *Si existe* $\lim_{P\in\mathbb{P}}\sigma_P$, *donde* $\mathbb{P}=\mathbb{P}([a_1,b_1]\times[a_2,b_2])$ *y*

$\sigma_P = \sum_{i(1)}^{n(P_1)} \sum_{i(2)}^{n(P_2)} \Delta_{i(1)}\Delta_{i(2)} K(x_1(\xi_{i(1)}), x_2(\xi_{i(2)})$ *con* $\xi_{i(r)} \in (t_{i(r)-1}, t_{i(r)})$, *entonces es llamado* ***integral interior de Dushnik*** *de la función* $x = (x_1, x_2)$ *con respecto al núcleo* K *y se denota por*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} d_{s_1 s_2} K(s_1, s_2)(x_1(s_1), x_2(s_2)).$$

Un resultado, que nos ofrece una condición suficiente para la existencia de la integral, es el siguiente

Teorema 3.1. *(Viloria [8]; Lema 2.2.1)*
Sean $K \in SF([a_1, b_1] \times [a_2, b_2]; L(X_1, X_2; Y))$ *y* $x_r \in G^-([a_r, b_r]; X_r), r = 1, 2.$
Entonces

(i) Existe $\Lambda_K : G^-([a_1, b_1]; X_1) \times G^-([a_2, b_2]; X_2) \longrightarrow Y$*, definida por*

$$\Lambda_K(x_1; x_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} d_{s_1 s_2} K(s_1; s_2)(x_1(s_1), x_2(s_2)),$$

(ii) Λ_K *es bilineal,*

(iii) $\|\Lambda_K x\| \leq SF[K]\|x_1\|\|x_2\|$,

(iv) Si $x_r \in \Omega_0([a_r, b_r]; X_r)$ *para algún* $r = 1, 2$*, entonces* $\Lambda_K x = 0$.

4 Teoremas de representación integral para operadores bilineales

Definición 4.1. *Sea* $K : [a_1, b_1] \times [a_2, b_2] \longrightarrow L([a_1, b_1] \times [a_2, b_2]; Z)$ *tal que*

$$K(a_1, s_2) = K(s_1, a_2) = 0 \qquad \forall\, s_1 \in [a_1, b_1] \text{ y } \forall\, s_2 \in [a_2, b_2].$$

Entonces diremos que $K \in SF_{a^2}([a_1, b_1] \times [a_2, b_2], L(X_1 \times X_2; X))$.

Teorema 4.1. *(Viloria [8]; Teorema 2.3.1)*
La aplicación $K \mapsto \Lambda_K$, *definida por*

$$\Lambda_K(x_1, x_2) = \int_{a_2}^{b_2} d_{s_2} \int_{a_1}^{b_1} d_{s_1} K(s_1, s_2) x_1(s_1) x_2(s_2),$$

es una isometría entre los espacios de Banach

$$SF_{a^2}\left([a_1, b_1] \times [a_2, b_2], L\left(X_1, X_2; Z\right)\right)$$

y

$$L\left(G^-([a_1, b_1]; X_1), G^-([a_2, b_2]; X_2); Z\right).$$

Además,

$$K(s_1, s_2)(\overline{x_1}, \overline{x_2}) = \Lambda_K\left(\mathcal{X}_{(a_1, s_1]}\overline{x_1}, \mathcal{X}_{(a_2, s_2]}\overline{x_2}\right)$$

con $\|\Lambda_K\| = SF[K]$.

Definición 4.2. *Sea* $K : [a, b] \times [a_1, b_1] \times [a_2, b_2] \longrightarrow L\left(X_1, X_2; Y\right)$. *Definimos*
$K^t : [a_1, b_1] \times [a_2, b_2] \longrightarrow L\left(X_1, X_2; Y\right)$ *y* $K_{s^2} : [a, b] \longrightarrow L\left(X_1, X_2; Y\right)$ *por*

$$K^t(s_1, s_2) = K(t_1, s_1, s_2) = K_{s^2}(t).$$

Además, consideremos las siguientes propiedades:
(G^σ) : *K es simplemente reglada como función de* t, *es decir,*

$$K_{s^2} \in G^\sigma\left([a, b]; L\left(X_1, X_2; Y\right)\right).$$

(SF^u) : *K es uniformemente de semivariación de Fréchet acotada como función de* (s_1, s_2), *esto es,*

$$SF^u[K] = \sup_{t \in [a,b]} [K^t] < \infty.$$

$(SF^u_{a^2})$: *K satisface* (SF^u) *y* $K^t \in SF_{a^2}\left([a_1, b_1] \times [a_2, b_2], L\left(X_1, X_2; Y\right)\right)$ *para todo* $t \in [a, b]$.
Si K verifica (G^σ) *y* (SF^u), *escribimos*

$$K \in G^\sigma \cdot SF^u\left([a, b] \times [a_1, b_1] \times [a_2, b_2], L\left(X_1, X_2; Y\right)\right).$$

Análogamente definimos $K \in G^\sigma \cdot SF^u_{a^2}$.

Teorema 4.2. *(Viloria [8]; Teorema 2.3.2)*
La aplicación $K \mapsto \Lambda_K$ dada por

$$\Lambda_K(x_1, x_2)(t) = \int_{a_2}^{b_2} d_{s_2} \int_{a_1}^{b_1} d_{s_1} K(t, s_1, s_2) x_1(s_1) x_2(s_2)$$

es una isometría entre los espacios de Banach

$$G^\sigma \cdot SF^u_{a^2}\left([a,b] \times [a_1, b_1] \times [a_2, b_2]; L\left(X_1, X_2; Y\right)\right)$$

y

$$L\left(G^-([a_1, b_1]; X_1), G^-([a_2, b_2]; X_2); G([a,b]; Y)\right),$$

con $K(t, s_1, s_2)(\overline{x_1}, \overline{x_2}) = \Lambda_K(\mathcal{X}_{(a_1,s_1]}\overline{x_1}, \mathcal{X}_{(a_2,s_2]}\overline{x_2})(t)$ *y* $\|\Lambda_K\| = SF^u[K]$.

Definición 4.3. *Sea* $K \in G^\sigma \cdot SF^u\left([a,b]^3; L_2\left(X; Y\right)\right)$, *donde* $[a,b]^3 = [a,b] \times [a,b] \times [a,b]$ *y* $L_2(X;Y) = L\left(X \times X; Y\right)$.
Si, para todo $x \in X$, la función $K_\Delta : [a,b] \longrightarrow Y$ definida por

$$K_\Delta(t) = K(t,t,t)(x,x) \quad \forall\, t \in [a,b]$$

es reglada, se dice que K es simplemente reglada en la diagonal y escribimos

$$K \in G^\sigma_\Delta SF^u\left([a,b]^3, L_2(X;Y)\right).$$

Si, además, $K_\Delta(t) = 0 \quad \forall\, t \in [a,b]$, se dice que K se anula en la diagonal y escribimos

$$K \in G^\sigma_0 \cdot SF^u\left([a,b]^3, L_2(X;Y)\right).$$

Definición 4.4. *$P \in L_2(G([a,b],X); G([a,b],Y))$ es un operador* ***causal*** *si para todo $x \in G([a,b];X)$ y para todo $T \in [a,b]$,*

$$x|_{[a,T]} = 0 \Rightarrow P(x,x)|_{[a,T]} = 0.$$

Definición 4.5. *Sea $K \in G^\sigma \cdot SF^u([a,b]^3; L_2(X,Y))$. Para $x = (x_1, x_2)$ con $x_r \in G^-([a,b];X), r = 1,2$, definimos*

$$(kx)(t) = \int_a^t d_{s_2} \int_a^t d_{s_1} K(t, s_1, s_2) x_1(s_1) x_2(s_2) \quad \forall\, t \in [a,b]$$

A continuación mostramos que los operadores bilineales causales también pueden ser representados.

Teorema 4.3. *(Viloria [8]; Teorema 2.3.3)*
La aplicación $K \mapsto k$ *es una isometría entre el espacio de Banach* $G_0^\sigma \cdot SF^u([a,b]^3, L_2(X;Y))$ *y el subespacio de los operadores causales de* $L_2(G^-([a,b],X); G([a,b],Y))$, *donde* $\|k\| = SF^u[K]$ *y*

$$K(t,s_1,s_2)(\overline{x_1},\overline{x_2}) = k\left(\mathcal{X}_{(a,t]}\overline{x_1}, \mathcal{X}_{(a,t]}\overline{x_2}\right)(t).$$

Ahora presentaremos el concepto de polinomial de Volterra-Stieltjes de grado dos.

Definición 4.6. *Si* $h_2 \in G^\sigma \cdot SF_a^u([a,b]^3, L_2(X;Y))$, *el operador*

$$P_2 : G^-([a,b];X) \longrightarrow G^-([a,b];X),$$

definido por

$$P_2x(t) = h_0(t) + \int_a^b d_{s_1} h_1(t,s_1)x(s_1) + \int_a^b d_{s_2} \int_a^b d_{s_1} K(t,s_1,s_2)x_1(s_1)x_2(s_2)$$

es llamado Polinomio de Volterra-Stieltjes de grado dos.

Daremos a continuación la definición de Expansión de Volterra-Stieltjes de un operador en $G^-([a,b];X)$.

Definición 4.7. *Un operador* $T : G^-([a,b];X) \longrightarrow G^-([a,b];X)$ *posee una expansión de Volterra-Stieltjes de grado dos, en una vecindad de* $x_0 \in G^-([a,b];X)$, *si existen núcleos* h_1, h_2 *tales que*

$$\begin{aligned} T(x_0+x) - T(x_0) = & \int_a^t d_{s_1} h_1(t_1 s_1)x(s_1) \\ & + \int_a^t d_{s_2} \int_a^t d_{s_1} h_2(t,s_1,s_2)x(s_1)x(s_2) + R_2(x_0;x), \end{aligned}$$

donde $\lim\limits_{\lambda \to 0} \dfrac{\|R_2(x_0;\lambda x)\|}{\lambda^2} = 0.$

Teorema 4.4. *(Hille - Phillips [2]; Theorem 26.3.5)*
Sean $V \subset X$ *no vacío, abierto y* $T : V \longrightarrow X$ *un operador 2 veces diferenciable Gâteaux. Entonces*

a) ∂T_{x_0} *es lineal simétrica,* $\partial^2 T_{x_0}$ *es bilineal y simétrico con* $\partial T_{x_0}[x]$ *y* $\partial^2 T_{x_0}[x]$ *polinomios homogéneos de grado 1 y 2, respectivamente.*

b) $T(x_0 + x) - T(x_0) = \partial T_{x_0}[x] + \frac{1}{2}\partial^2 T_{x_0}[x] + R_2(x_0, x)$, *con*

$$\lim_{\lambda \to 0} \frac{\|R_2(x_0, \lambda x)\|}{\lambda^2} = 0$$

Corolario 4.1. *Sea* $V \subset G^-([a,b];X)$ *no vacío, abierto y* $T : V \longrightarrow G^-([a,b];X)$ *2 veces diferenciable Gâteaux en* x_0*, con* $\partial^2 T_{x_0}$ *causal. Entonces* T *tiene una expansión de Volterra-Stieltjes de grado dos.*

Definición 4.8. *Una aplicación* $T : V \longrightarrow Y$ *es analítica en* V*, si puede representarse como*

$$T(x) = \sum_{n=1}^{\infty} a_n[x]^n,$$

donde a_n *es un operador polinomial homogéneo de grado* n *y la serie converge absolutamente en* V*, y uniformemente sobre conjuntos cerrados y acotados de* V*.*

Teorema 4.5. *(Pisanelli [6])*
Sea $V \subset X$ *no vacío, abierto, acotado y* $T : V \longrightarrow X$ *un operador localmente acotado tal que*

i) T *es diferenciable Gâteaux,*

ii) ∂T_{x_0} *es invertible en una vecindad de* x_0.

Entonces existen vecindades de x_0 *y* T_{x_0} *respectivamente donde* T *posee una única inversa dada por:*

$$T^{-1}u = \sum_{m=1}^{\infty} Q_m[u],$$

donde

$$\begin{aligned} Q_1[u] &= [\partial T_{x_0}]^{-1}[u] \\ Q_m[u] &= -[\partial T_{x_0}]^{-1} \sum_{n=1}^{m} \sum_{j_1+\ldots+j_n=m} \frac{1}{n!}\partial^m T_{x_0}(Q_{j_1}, \ldots, Q_{j_n})[u]. \end{aligned}$$

Como resultado del teorema anterior, obtenemos cómo podemos definir la inversa local de un operador polinomial de grado dos, sobre espacios de Banach de forma explícita.

Corolario 4.2. *Sea $V \subset X$ no vacío, abierto, acotado y $T : V \longrightarrow X$ un operador polinomial de grado dos, con ∂T_{X_0} invertible en una vecindad de x_0. Entonces, existen vecindades de x_0 y de T_{x_0}, respectivamente, donde T tiene una única inversa dada por:*

$$T^{-1}u = Q_1[u] + Q_2[u],$$

con

$$\begin{aligned} Q_1[u] &= [\partial T_{x_0}]^{-1}[u] \\ Q_2[u] &= -[\partial T_{x_0}]^{-1}(\partial^2 T_{x_0})(Q_1, Q_1)[u] - \frac{1}{2}[\partial T_{x_0}]^{-1}\partial^2 T_{x_0}(Q_1, Q_1)[u]. \end{aligned}$$

Definición 4.9. *Consideremos el espacio vectorial*

$$\mathbb{F} = \left\{ x : [a,b] \longrightarrow X \ : \ x \text{ es una función} \right\}$$

y $f : [a,b] \times X \longrightarrow X$ una función cualquiera. El operador no lineal $F : \mathbb{F} \longrightarrow \mathbb{F}$ dado por

$$Fx(t) = f(t, x(t)) \ \ \forall\, t \in [a,b], \ \forall\, x \in \mathbb{F},$$

es llamado el operador composición asociado a la función f.

Definición 4.10. *El conjunto de las funciones $f : [a,b] \times X \longrightarrow X$ tales que*

- *f es reglada por la izquierda en la primera variable,*
- *f es Lipschitz en la segunda variable,*

forman un espacio de Banach, denotado $G^- \cdot Lips([a,b] \times X; X)$ con la norma

$$\|f\| = \sup \left\{ \|f_0\|; [f] \right\},$$

donde $f_0 : [a,b] \longrightarrow X$ definida por $f_0(t) = f(t, \theta)$ y

$$[f] = \inf \left\{ M \ : \ \|f(t, x_2) - f(t, x_1)\| \leq M\|x_2 - x_1\|, \quad x_1, x_2 \in X \right\}.$$

A continuación expondremos el resultado de Viloria [7], que indica las condiciones necesarias y suficientes en f, para que F actúe en el espacio $G^-([a,b]; X)$.

Teorema 4.6. *Sea $f : [a,b] \times X \longrightarrow X$ Lipschitz en la segunda variable. Entonces, el operador F de composición asociado a f, es tal que $F : G^-([a,b];X) \longrightarrow G^-([a,b];X)$ si, y sólo si, $f \in G^- \cdot Lips([a,b] \times X;X)$. Además, F es acotado.*

5 Condiciones de existencia y unicidad de soluciones

Consideremos la ecuación

$$x(t) + \int_a^t d_s K(t,s) f(s,x(s)) = u(t), \qquad t \in [a,b] \tag{K}$$

Teorema 5.1. *(Hönig)*
Sean $K \in G_0^\sigma \cdot SV^u\,([a,b] \times [a,b]; L(X,X))$ y $f \in G^- \cdot Lips([a,b] \times X, X)$ con $C(K,\mathcal{P})[f] < 1$, para algún $\mathcal{P} \in \mathbb{P}([a,b])$. Entonces, para cada $u \in G([a,b];X)$, existe una única $x \in G([a,b];X)$, solución de (K), que depende continuamente de u, K y f; donde

$$C(K,\mathcal{P}) = \sup_{0 \le i \le n-1} \left\{ \|K(t_{i+}, t_i)\|; \sup_{t_i \le t \le t_{i+1}} SV_{(t_i,t]}[K^t] \right\}.$$

El siguiente resultado expresa en el caso lineal, es decir cuando $f(t,x(t)) = x(t)$, que el resolvente de la solución puede ser expresado por una Serie de Neumann.

Teorema 5.2. *(Arbex [1], Corolario 3.22 parte b))*
Sea $K \in G_0^\sigma \cdot SV^u\,([a,b] \times [a,b]; L(X,X))$ y $C(K,\mathcal{P}) < 1$, para algún $\mathcal{P} \in \mathbb{P}([a,b])$. Entonces, para cada $u \in G([a,b];X)$, existe una única $x \in G([a,b];X)$, solución de (K), la cual esta dada por

$$x(t) = u(t) - \int_a^t d_s R(t,s) u(s),$$

donde el operador resolvente se escribe

$$R(s,t) = I(t,s) + \sum_{n=1}^{\infty} (-1)^{n-1} K_n(t,s),$$

con

$$\begin{cases} K_1(t,s) &= K(t,s) \\ K_{n+1}(t,s) &= \displaystyle\int_a^t d_s K(t,\sigma) K_n(\sigma,s) \quad \forall\, n \ge 1 \end{cases}$$

núcleos iterados.

6 Forma Explícita de la Solución para ecuaciones cuadráticas

Consideremos la ecuación integral no lineal de Volterra-Stieltjes

$$x(t) + \int_a^t d_s K(t,s) f(s, x(s)) = u(t), \qquad t \in [a,b] \qquad (K)$$

donde $x \in G([a,b];X)$ es incógnita, $u \in G([a,b];X)$ es conocida y $K \in G_0^\sigma \cdot SF^u\left([a,b]\times[a,b]\times[a,b]; L_2(X,X)\right)$.

Ahora supongamos $V \subset X$ no vacío, abierto, acotado y $f : [a,b] \times V \longrightarrow X$, definida por

$$f(t,x) = g(t)Bx,$$

donde $g \in G^-([a,b];\mathbb{C})$ y B un operador polinomial homogéneo de grado 2 sobre X, definido por $Bx = L_2(x,x)$, con L_2 bilineal simétrico.

Veamos que

$$f \in G^- \cdot Lips([a,b] \times V, X).$$

En efecto,

- f es reglada por la izquierda en la primera variable,
- f es Lipschitz en la segunda variable.
- Sea $t \in (a,b]$

$$\begin{aligned} \lim_{h\to 0} f(t-h,x) &= \lim_{h\to 0} g(t-h)Bx \\ &= g(t^-)Bx \qquad (\text{ya que } \ g \in G^-([a,b],X)) \ , \\ &= f(t^-,x) \end{aligned}$$

luego,

$$f(t^-,x) = \lim_{h\to 0} f(t-h,x) \qquad \forall\, x \in X \ ,$$

esto demuestra que f es reglada por la izquierda en la primera variable.

- Sean $x_1, x_2 \in V$ tales que $[x_1, x_2] \subset V$. Queremos ver que existe $L \in [0,1)$ tal que:

$$\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\| \qquad \forall\, t \in [a,b]$$

Así,

$$\|f(t,x_1) - f(t,x_2)\| \leq \sup_{x_0 \in [x_1,x_2]} \|\partial_x f(t,x_0)\| \|x_1 - x_2\| \qquad \forall\, t \in [a,b],$$

y como $[a,b]$ es compacto y $g \in G^-([a,b];\mathbb{C})$, sabemos que g es acotada. Además, V es acotado, por lo tanto existe $M > 0$ tal que

$$\sup_{x_0 \in [x_1,x_2]} \|\partial_x f(t,x_0)\| \leq M \qquad \forall\, t \in [a,b].$$

De donde, se sigue que

$$\|f(t,x_1) - f(t,x_2)\| \leq M\|x_1 - x_2\| \qquad \forall\, t \in [a,b].$$

En consecuencia, f es Lipschitz respecto a X.

□

Consideremos ahora, el operador $F : G^-([a,b],X) \longrightarrow G^-([a,b],X)$ de composición asociado a f,

$$Fx(t) = f(t,x(t)) \qquad \forall\, x \in G^-([a,b];X),$$

el cual es acotado (por el Teorema 4.4).

Y, definiendo el operador lineal k por:

$$(kF)x(t) = \int_a^t d_s K(t,s) Fx(s), \tag{1}$$

la ecuación (K) se puede expresar como

$$\begin{aligned} x(t) + kFx(t) &= u(t) \qquad \forall\, t \in [a,b] \\ (I + kF)x &= u \qquad (x \in G^-([a,b];X)), \end{aligned}$$

donde $I : G^-([a,b];X) \longrightarrow G^-([a,b];X)$ es el operador identidad.

Si consideramos el operador $T : G^-([a,b];X) \longrightarrow G^-([a,b];X)$,

$$T = I + kF$$

que actúa sobre funciones $x \in G^{-}([a,b];X)$, entonces el operador T se puede invertir localmente hallando $[\partial T_{x_0}]^{-1}$ y $\partial^2 T_{x_0}$, como lo muestra el siguiente teorema.

Teorema 6.1. *Sean* $K \in G_0^{\sigma} \cdot SV^{u}([a,b]\times[a,b]\times[a,b];L_2(X,X))$ *y* $f \in G^{-} \cdot Lips([a,b]\times X;X)$. *Si* $C(K;\mathcal{P})[f] < 1$, *para algún* $\mathcal{P} \in \mathbb{P}([a,b])$, *y* $\|\partial F\| \leq \inf\{1, 1/SF[K]\}$, *entonces*

(i) *T es 2 veces diferenciable Gâteaux y*

(ii) *la diferencial de T posee inversa.*

Demostración:

(i) De su misma definición, tenemos que Fx es diferenciable Gâteaux, lo que nos induce la diferenciabilidad de T. Así que, dado $x_0 \in W \subset G^{-}([a,b];X)$, abierto y acotado,

$$\begin{aligned}
\partial T_{x_0}x(t) &= \lim_{\lambda\to 0}\frac{T(x_0+\lambda x)(t)-T(x_0)(t)}{\lambda}\\
&= \lim_{\lambda\to 0}\frac{(I+kF)(x_0+\lambda x)(t)-(I+kF)(x_0)(t)}{\lambda}\\
&= Ix(t)+k\lim_{\lambda\to 0}\frac{F(x_0+\lambda x)-F(x_0)(t)}{\lambda}\\
&= Ix(t)+k\partial F_{x_0}[x](t).
\end{aligned}$$

Siendo,

$$\begin{aligned}
\partial F_{x_0}[x](t) &= \lim_{\lambda\to 0}\frac{F(x_0+\lambda x)(t)-F(x_0)(t)}{\lambda}\\
&= \lim_{\lambda\to 0}\frac{f(t,x_0+\lambda x)-f(t,x_0)}{\lambda}
\end{aligned}$$

donde, empleando las reglas del cálculo diferencial en espacios de Banach,

$$\begin{aligned}
\partial f_x(t,x_0) &= g(t)\partial B_{x_0}[x]\\
&= g(t)[L_2(x_0,x)+L_2(x,x_0)]\\
&= 2g(t)L_2(x_0,x).
\end{aligned}$$

Luego, $\partial T_{x_0} = I + k\partial F_{x_0}$ con $\partial T_{x_0} \in L(G^-([a,b];X); G^-([a,b];X))$.
Ahora, calculemos

$$\begin{aligned}
\partial^2 T_{x_0}[x](t) &= \partial^2 T_{x_0}(x_1(t), x_2(t)) \\
\partial^2 T_{x_0}(x_1(t), x_2(t)) &= \lim_{\lambda \to 0} \frac{1}{\lambda}\left\{ \partial T_{x_0+\lambda x_2(t)}(x_1)(t) + \partial T_{x_0}(x_1)(t) \right\} \\
&= \lim_{\lambda \to 0} \frac{1}{\lambda}\Big\{ I(x_1)(t) + k\partial F_{x_0+\lambda x_2(t)}[x_1](t) \\
&\qquad - I(x_1)(t) - k\partial F_{x_0}[x_1](t) \Big\} \\
\partial^2 T_{x_0}[x](t) &= k \lim_{\lambda \to 0} \frac{1}{\lambda}\left\{ \partial F_{x_0+\lambda x_2(t)}[x_1](t) + \partial F_{x_0}[x_1](t) \right\} \\
&= k\partial^2 F_{x_0}[x_1, x_2](t) = k\partial^2 F_{x_0}[x](t),
\end{aligned}$$

siendo

$$\begin{aligned}
\partial^2 F_{x_0}[x](t) &= \partial_x(\partial_x f(t, x_0)) \\
&= \partial_x(2g(t)L_2(x_0, x)) = 2g(t)[L_2(x_0, x) + L_2(x, x_0)] \\
&= 4g(t)L(x_0, x).
\end{aligned}$$

Esto es, $\partial^2 T_{x_0}$ existe y, además,

$$\partial^2 T_{x_0} = k\partial^2 F_{x_0} \in L\left(G^-([a,b];X) \times G^-([a,b];X); G^-([a,b];X)\right).$$

Así, T es un operador 2 veces diferenciable Gâteaux.

Por otro lado, en virtud del Teorema 4.4,

$$T(x_0 + x) - T(x_0) = \partial T_{x_0}[x] + \frac{1}{2}\partial^2 T_{x_0}[x]$$

y resulta T un operador polinomial de grado dos.
Por otro lado, como por hipótesis

$$\|\partial F\| \leq \inf\left\{1, \frac{1}{SF[K]}\right\},$$

se tiene que

$$\|k\partial F\| \leq \|k\|\|\partial F\| \leq SF[K]\frac{1}{SF[K]} = 1$$

de donde ∂T_{x_0} es invertible en una vecindad de x_0. Se sigue entonces, del corolario anterior, que existen vecindades de x_0 y $T(x_0)$, respectivamente, donde T posee una única inversa definida por

$$T^{-1}u = Q_1[u] + Q_2[u] \tag{2}$$

donde

$$\begin{aligned} Q_1[u] &= [\partial T_{x_0}]^{-1}[u] \\ Q_2[u] &= -[\partial T_{x_0}]^{-1}\partial^2 T_{x_0}(Q_1, Q_1)[u] - \frac{1}{2}[\partial T_{x_0}]^{-1}\partial^2 T_{x_0}(Q_1, Q_1)[u]. \end{aligned}$$

Por consiguiente, para hallar $(\partial T_{x_0})^{-1}$, basta resolver la ecuación lineal

$$Hx = u, \tag{3}$$

donde $H = \partial T_{x_0} = I + k\partial F_{x_0}$.

La cual, según demostró Arbex, tiene resolvente determinada a partir de una serie de Neumann.

Pasemos a calcular H^{-1}.

La ecuación (3) puede escribirse como

$$x(t) + \int_a^t d_s K(t,s)\partial F_{x_0} x(s) = u(t) \tag{4}$$

donde, por ser

$$\partial F_{x_0} \in L(G^-([a,b];X); G^-([a,b];X)),$$

posee una representación integral, esto es existe

$$\overline{h_1} \in G_0^\sigma \cdot SV^u\left([a,b]\times[a,b]; L(X,X)\right)$$

tal que

$$\partial F_{x_0} x(t) = \int_a^t d_s \overline{h_1}(t,s)x(s) \qquad t \in [a,b],$$

con

$$\begin{aligned}\overline{h_1}(t,s)x &= \partial F_{x_0}(x\mathcal{X}_{(a,t]})(s)\\ &= 2g(s)L_2(x_0(s),x\mathcal{X}_{(a,t]}(s))\end{aligned}$$

y

$$\|\partial F_{x_0}\| = SV[\overline{h_1}] \quad = \quad \sup_{t\in[a,b]} SV[\overline{h_1}^t] < \infty.$$

Entonces, la ecuación (4) posee una única solución dada por

$$x(t) = u(t) - \int_a^t d_s R(t,s)u(s), \tag{5}$$

donde la resolvente está representada por el operador

$$R(t,s) = I + \sum_{n=1}^{\infty}(-1)^{n-1}K_n(t,s)$$

siendo $K_1 = \overline{h_1}$ y $K_{n+1}(t,s) = \displaystyle\int_a^t d_\sigma \overline{h_1}(t,\sigma)K_n(\sigma,s)$.

Así, hemos hallado $[\partial T_{x_0}]^{-1}u$, mediante la expresión (5).

Ahora, dado que

$$\partial^2 F_{x_0} \in L\left(G^-([a,b];X), G^-([a,b];X); G^-([a,b];X)\right)$$

existe un único $\overline{h_2} \in G^\sigma \cdot SF^u_{a^2}\left([a,b]\times[a,b]\times[a,b]; L(X,X;X)\right)$ tal que

$$\partial^2 F_{x_0}(x_1,x_2)(t) = \int_a^t d_{s_2}\int_a^t d_{s_1}\overline{h_2}(t,s_1,s_2)x_1(s_1)x_2(s_2),$$

donde

$$\begin{aligned}
\overline{h_2}(t,s_1,s_2)(x_1,x_2) &= \partial^2 F_{x_0}(x_1\mathcal{X}_{(a,t]}(s_1), x_2\mathcal{X}_{(a,t]}(s_2))\\
&= 4g(t)L_2\Big(x_0(t), x_1\mathcal{X}_{(a,t]}(s_1)+x_2\mathcal{X}_{(a,t]}(s_2)\Big)\\
&= 4g(t)\Big[L_2(x_0(t),x_1\mathcal{X}_{(a,t]}(s_1))\\
&\qquad +L_2(x_0(t),x_2\mathcal{X}_{(a,t]}(s_2))\Big]\\
&= 2\Big[2g(t)L_2(x_0(t),x_1\mathcal{X}_{(a,t]}(s_1))\\
&\qquad +2g(t)L_2(x_0(t),x_2\mathcal{X}_{(a,t]}(s_2))\Big]\\
&= 2\Big[\overline{h_1}(t,s_1)x_1+\overline{h_1}(t,s_2)x_2\Big].
\end{aligned}$$

□

Finalmente, podemos concluir que la única solución para un sistema no lineal dado por una ecuación integral de Volterra-Stieltjes de la forma

$$x(t)+\int_a^t d_sK(t,s)g(s)Bx = u(t),$$

se expresa mediante la igualdad

$$x(t) = Q_1[u](t)+Q_2[u](t)$$

donde

$$Q_1[u](t) = u(t)-\int_a^t d_sR(t,s)u(s) \quad \text{con} \quad R = I+\sum_{n=1}^{\infty}(-1)^{n-1}K_n$$

para $K_1 = \partial F_{x_0}$ y $K_{n+1}(t,s) = \displaystyle\int_a^t d_\sigma \partial F_{x_0}(x\mathcal{X}_{(s,t]}(\sigma))K_n(\sigma,s)$.

Y,

$$Q_2[u](t) = -\frac{3}{2}(I+k\partial F_{x_0})^{-1}(k\partial^2 F_{x_0})(Q_1,Q_1)[u](t)$$

con

$$\partial^2 F_{x_0}(Q_1,Q_1)u(t) = \int_a^t d_{s_2} \int_a^t d_{s_1} \overline{h_2}(t,s_1,s_2)(Q_1u(s_1),Q_1u(s_2)),$$

siendo

$$\overline{h_2} \in G_0^{\sigma} \cdot SF^u\left([a,b]\times[a,b]\times[a,b]; L(X,X;X)\right),$$

tal que

$$\begin{aligned}\overline{h_2}(t,s_1,s_2)(x_1,x_2) &= \partial^2 F_{x_0}(x_1\mathcal{X}_{(a,t]}(s_1), x_2\mathcal{X}_{(a,t]}(s_2)) \\ &= 2[\overline{h_1}(t,s_1)x_1 + \overline{h_1}(t,s_2)x_2]\end{aligned}$$

con

$$\begin{aligned}\overline{h_1}(t,s_1)x_1 &= 2g(s_1)L_2(x_0(s_1), x_1\mathcal{X}_{(a,t]}(s_1)) = \partial F_{x_0}x_1(s_1) \\ \overline{h_2}(t,s_2)x_2 &= 2g(s_2)L_2(x_0(s_2), x_2\mathcal{X}_{(a,t]}(s_2)) = \partial F_{x_0}x_2(s_2).\end{aligned}$$

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Analytic functions associated with Caputos fractional differentiation defined by Hilbert space operator

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Abstract. In this paper, we introduce a new class of functions which are analytic and univalent with negative coefficients defined by using certain fractional operators described in the Caputo sense. Characterization property, the results on modified Hadamard product and integral transforms are discussed. Further, distortion theorem and radii of starlikeness and convexity are also determined here.

Resumen. En este trabajo, presentamos una nueva clase de funciones que son analíticas y univalente con coeficientes negativos, definidos usando ciertos operadores fraccionarios en el sentido de Caputo. Discutimos la propiedad de caracterización, los resultados sobre el producto de Hadamard modificado y transformaciones integrales. Además, determinamos el teorema de distorsión y los radios de "starlikeness" y convexidad.

1 Introduction

Fractional calculus operators have recently found interesting application in the theory of analytic functions. The classical definition of fractional calculus and their other generalizations have fruitfully been applied in obtaining, the characterization properties, coefficient estimates and distortion inequalities for various subclasses of analytic functions. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

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which are analytic in the open disc $U = \{z : z \in \mathbb{C};\ |z| < 1\}$. Also denote by $\mathcal{T}$, a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0; z \in U \tag{1.2}$$

introduced and studied by Silverman [9]. For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, we define the convolution product (or Hadamard) of f and g by

$$(f * g)(z) = (g * f)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \tag{1.3}$$

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on$\mathcal{H}$. For a complex-valued function f analytic in a domain $\mathbb{E}$ of the complex z-plane containing the spectrum $\sigma(\mathbb{P})$ of the bounded linear operator $\mathbb{P}$, let $f(\mathbb{P})$ denote the operator on $\mathcal{H}$ defined by Dunford [3],

$$f(\mathbb{P}) = \frac{1}{2\pi i} \int_{\mathcal{C}} (z\mathbb{I} - \mathbb{P})^{-1} f(z) dz, \tag{1.4}$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$ and $\mathcal{C}$ is a positively-oriented simple rectifiable closed contour containing the spectrum $\sigma(\mathbb{P})$ in the interior domain. The operator $f(\mathbb{P})$ can also be defined by the following series:

$$f(\mathbb{P}) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \mathbb{P}^n$$

which converges in the norm topology (cf. [3]).

Now we look at the Caputos [2]definition which shall be used throughout the paper. Caputos definition of the fractional-order derivative is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \tag{1.5}$$

where $n - 1 < Re(\alpha) \leq n, n \in N$, and the parameter α is allowed to be real or even complex, a is the initial value of the function f .

We recall the following definitions [6] .

Definition 1. *[6] Let the function $f(z)$ be analytic in a simply - connected region of the $z-$ plane containing the origin. The fractional integral of f of* *order μ is defined by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi, \quad \mu > 0, \tag{1.6}$$

where the multiplicity of $(z-\xi)^{1-\mu}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 2. *[6] The fractional derivatives of order μ, is defined for a function $f(z)$, by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi, \quad 0 \leq \mu < 1, \tag{1.7}$$

where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\xi)^{-\mu}$ is removed as in Definition 1.

Definition 3. *Under the hypothesis of Definition 2, the fractional derivative of order $n+\mu$ is defined by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z), \quad (0 \leq \mu < 1 \ ; \ n \in N_0). \tag{1.8}$$

With the aid of the above definitions, and their known extensions involving fractional derivative and fractional integrals, Srivastava and Owa [13] introduced the operator Ω^{δ} $(\delta \in \mathbb{R}; \delta \neq 2, 3, 4, \dots) : \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^{\delta} f(z) = \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z) = z + \sum_{n=2}^{\infty} \Phi(n, \delta) a_n z^n \tag{1.9}$$

where

$$\Phi(n, \delta) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}. \tag{1.10}$$

For $f \in \mathcal{A}$ and various choices of δ, , we get different operators

$$\Omega^0 f(z) := f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.11}$$

$$\Omega^1 f(z) := z f'(z) = z + \sum_{k=2}^{\infty} k a_k z^k \tag{1.12}$$

$$\Omega^j f(z) := \Omega(\Omega^{j-1} f(z)) = z + \sum_{k=2}^{\infty} k^j a_k z^k, (j = 1, 2, 3, ...) \tag{1.13}$$

which is known as Salagean operator[7] .Also note that

$$\Omega^{-1}f(z) = \frac{2}{z}\int_0^z f(t)dt := z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)a_k z^k$$

and

$$\Omega^{-j}f(z) := \Omega^{-1}(\Omega^{-j+1}f(z)) := z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^j a_k z^k, (j = 1, 2, 3, ...) \quad (1.14)$$

called Libera integral operator.We note that the Libera integral operator is generalized as Bernardi integral operator given by Bernardi[1],

$$\frac{1+\nu}{z^\nu}\int_0^z t^{\nu-1}f(t)dt := z + \sum_{k=2}^{\infty}\left(\frac{1+\nu}{k+1}\right)a_k z^k, (\nu = 1, 2, 3, ...).$$

Making use of these results Recently Salah and Darusin [8], introduced the following operator

$$\mathcal{J}_\mu^\eta = \frac{\Gamma(2+\eta-\mu)}{\Gamma(\eta-\mu)}z^{\mu-\eta}\int_0^z \frac{\Omega^\eta f(t)}{(z-t)^{\mu+1-\eta}}dt \quad (1.15)$$

where η(real number) and $(\eta - 1 < \mu < \eta < 2)$. By simple calculations for functions $f(z) \in \mathcal{A}$, we get

$$\mathcal{J}_\mu^\eta f(z) = z + \sum_{k=2}^{\infty}\frac{(\Gamma(k+1))^2\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(k+\eta-\mu+1)\Gamma(k-\eta+1)}a_k z^k \quad (z \in U), \quad (1.16)$$

and for the sake of brevity we let

$$C_k(\eta,\mu) = \frac{(\Gamma(k+1))^2\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(k+\eta-\mu+1)\Gamma(k-\eta+1)} \quad (1.17)$$

and

$$C_2(\eta,\mu) = \frac{4\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(3+\eta-\mu)\Gamma(1-\eta)}$$

unless otherwise stated.

Further, note that $\mathcal{J}_0^0 f(z) = f(z)$ and $\mathcal{J}_1^1 f(z) = zf'(z)$. In this paper, by making use of the operator $\mathcal{J}_\mu^\eta$ we introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \le \alpha < \frac{1}{2}$, we let $\mathcal{J}_\mu^\eta(\lambda,\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left\|\frac{\mathcal{J}_{\mu,\eta}^\lambda(\mathbb{P}) - 1}{\mathcal{J}_{\mu,\eta}^\lambda(\mathbb{P}) - (2\alpha - 1)}\right\| < 1 \quad (1.18)$$

where

$$\mathcal{J}_{\mu,\eta}^{\lambda} f(\mathbb{P}) = \frac{\mathbb{P}(\mathcal{J}_{\mu}^{\eta} f(\mathbb{P}))'}{\mathcal{J}_{\mu}^{\eta} f(\mathbb{P})} + \frac{\lambda \mathbb{P}^2 (\mathcal{J}_{\mu}^{\eta} f(\mathbb{P}))''}{\mathcal{J}_{\mu}^{\eta} f(\mathbb{P})}, \tag{1.19}$$

$0 \leq \lambda \leq 1$, $\mathcal{J}_{\mu}^{\eta} f(z)$ is given by (1.16) . We further let $\mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha) = \mathcal{J}_{\mu}^{\eta}(\lambda,\alpha) \cap \mathcal{T}$.

In the following section we obtain coefficient estimates for $f \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$.

2 Coefficient Bounds

Theorem 1. *Let the function f be defined by (1.2). Then $f \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$ if and only if*

$$\sum_{k=2}^{\infty} (k[1+\lambda(k-1)] - \alpha) C_k(\eta,\mu) a_k \leq (1-\alpha). \tag{2.1}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{(k[1+\lambda(k-1)] - \alpha) C_k(\eta,\mu)} z^k, \quad k \geq 2. \tag{2.2}$$

Proof. Suppose f satisfies (2.1). Then for $\|z\|$,

$$\begin{aligned}
&\left\|\mathcal{J}_{\mu,\eta}^{\lambda}(\mathbb{P}) - 1\right\| < \left\|\mathcal{J}_{\mu,\eta}^{\lambda}(\mathbb{P}) + 1 - 2\alpha\right\| \\
&= \left\| \frac{z - \sum_{k=2}^{\infty} k C_k(\eta,\mu) a_k z^k - \lambda \sum_{k=2}^{\infty} k(k-1) C_k(\eta,\mu) a_k z^k}{z - \sum_{k=2}^{\infty} C_k(\eta,\mu) a_k z^k} - 1 \right\| \\
&< \left\| 2(1-\alpha) - \frac{z - \sum_{k=2}^{\infty} k C_k(\eta,\mu) a_k z^k - \lambda \sum_{k=2}^{\infty} k(k-1) C_k(\eta,\mu) a_k z^k}{z - \sum_{k=2}^{\infty} C_k(\eta,\mu) a_k z^k} \right\| \\
&\leq \sum_{k=2}^{\infty} (k[1+\lambda(k-1)] - 1)\ C_k(\eta,\mu) a_k \\
&\leq 2(1-\alpha) - \sum_{k=2}^{\infty} (k[1+\lambda(k-1)] + (1-2\alpha)) C_k(\eta,\mu) a_k \\
&= \sum_{k=2}^{\infty} (k[1+\lambda(k-1)] - \alpha) C_k(\eta,\mu) a_k - (1-\alpha) \\
&\leq 0, \quad \text{by (2.1)}.
\end{aligned}$$

Hence, by maximum modulus theorem and (1.18), $f \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$. To prove the converse, assume that

$$\left\| \frac{\mathcal{J}_{\mu,\eta}^{\lambda}(\mathbb{P}) - 1}{\mathcal{J}_{\mu,\eta}^{\lambda}(\mathbb{P}) + 1 - 2\alpha} \right\| = \left\| \frac{-\sum_{k=2}^{\infty} (k[1+\lambda(k-1)] - 1) C_k(\eta,\mu) a_k z^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} (k[1+\lambda(k-1)] + (1-2\alpha)) C_k(\eta,\mu) a_k z^{k-1}} \right\|$$

$$\leq 1, \quad z \in U.$$

Equivalently,

$$\operatorname{Re}\left\{\frac{-\sum\limits_{k=2}^{\infty}(k[1+\lambda(k-1)]-1)C_k(\eta,\mu)a_k z^{k-1}}{2(1-\alpha)-\sum\limits_{k=2}^{\infty}(k[1+\lambda(k-1)]+(1-2\alpha))C_k(\eta,\mu)a_k z^{k-1}}\right\}<1. \quad (2.3)$$

Since $\mathrm{Re}(z) \leq \|z\|$ for all z. Choose values of z on the real axis so that $\mathcal{J}^{\lambda}_{\mu,\eta}(\mathbb{P})$ is real. Upon clearing the denominator in (2.3) and letting $\|z\| = \mathbb{P} = r\mathbb{I}(0 < r < 1)$ and letting $r \to 1^{-}$, we obtain the desired assertion (2.1). □

Corollary 1. *If $f(z)$ of the form (1.2) is in $\mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$, then*

$$a_k \leq \frac{(1-\alpha)}{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}, \quad k \geq 2, \quad (2.4)$$

with equality only for functions of the form (2.2).

In the following theorem we state the distortion bounds and extreme point results for functions $f \in \mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$ without proof.

Theorem 2. *If $f \in \mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$, then*

$$r - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r^2 \leq \|f(\mathbb{P})\| \leq r + \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r^2 \quad (2.5)$$

$$1 - \frac{2(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r \leq \|f'(\mathbb{P})\| \leq 1 + \frac{2(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r. \quad (2.6)$$

The bounds in (2.5) and (2.6) are sharp, since the equalities are attained by the function

$$f(z) = z - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}z^2 \quad z = \pm r. \quad (2.7)$$

Theorem 3. **(Extreme Points)** *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{(1-\alpha)}{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}z^k,$$

$k \geq 2$, for $0 \leq \alpha < \frac{1}{2}$, and $0 \leq \lambda \leq 1$. Then $f(z)$ is in the class $\mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$ if and only if it can be expressed in the form $f(z) = \sum\limits_{k=1}^{\infty}\omega_k f_k(z)$, where $\omega_k \geq 0$ and $\sum\limits_{k=1}^{\infty}\omega_k = 1$.

3 Radius of Starlikeness and Convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{T}\mathcal{J}_\mu^\eta(\lambda,\alpha)$ are given in this section.

Theorem 4. *Let the function $f(z)$ defined by (1.2) belong to the class $\mathcal{T}\mathcal{J}_\mu^\eta(\lambda,\alpha)$. Then $f(z)$ is close-to-convex of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \left[\frac{(1-\delta)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{3.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.2).

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$\|f'(\mathbb{P}) - 1\| < 1 - \delta. \tag{3.2}$$

For the left hand side of (3.2) we have

$$\|f'(\mathbb{P}) - 1\| \le \sum_{k=2}^{\infty} k a_k \|\mathbb{P}\|^{k-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta} a_k \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{T}\mathcal{J}_\mu^\eta(\lambda,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} \frac{(k[1+\lambda(k-1)]-\alpha)a_k\ C_k(\eta,\mu)}{(1-\alpha)} \le 1.$$

We can say (3.2) is true if

$$\frac{k}{1-\delta}\|\mathbb{P}\|^{k-1} \le \frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} = r^{k-1} = \left[\frac{(1-\delta)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{k(1-\alpha)}\right]$$

which completes the proof. □

Theorem 5. *Let $f \in \mathcal{T}\mathcal{J}_\mu^\eta(\lambda,\alpha)$. Then*

1. *f is starlike of order $\delta(0 \le \delta < 1)$ in the disc $\|z\| < r_2$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, where*

$$r_2 = \inf_{k \ge 2}\left\{\frac{(1-\delta)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

2. *f is convex of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_3$, that is $Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, where*

$$r_3 = \inf_{k \ge 2}\left\{\frac{(1-\delta)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)k(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.2).

Proof. Given $f \in \mathcal{T}$ and f is starlike of order δ, we have

$$\left\|\frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| < 1-\delta, \quad (\mathbb{P} = r_2\mathbb{I}\ (0 < r_2 < 1)) \tag{3.3}$$

For the left hand side of (3.3) we have

$$\left\|\frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k\ \|\mathbb{P}\|^{k-1}}{1-\sum_{k=2}^{\infty}a_k\ \|\mathbb{P}\|^{k-1}}.$$

The last expression is less than $1-\delta$, if

$$\sum_{k=2}^{\infty}\frac{k-\delta}{1-\delta}a_k\ \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$, if and only if

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\alpha)a_k\ C_k(\eta,\mu)}{(1-\alpha)} < 1.$$

We can say (3.3) is true if

$$\frac{k-\delta}{1-\delta}\|\mathbb{P}\|^{k-1} < \frac{(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} < \frac{(1-\delta)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\delta)}$$

which yields the starlikeness of the family.
(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar the proof of (1). □

4 Integral transform of the class $\mathcal{TJ}_{\mu}^{\eta}(\lambda, \alpha)$

For $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda, \alpha)$ we define the integral transform

$$V_\mu(f)(z) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt,$$

where μ is real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c$, $c > -1$, for which V_μ is known as the Bernardi operator, and

$$\mu(t) = \frac{(c+1)^\delta}{\mu(\delta)} t^c \left(log\frac{1}{t}\right)^{\delta-1}, \; c > -1, \; \delta \geq 0$$

which gives the Komatu operator. For more details see [4].

First we show that the class $\mathcal{TJ}_{\mu}^{\eta}(\lambda, \alpha)$ is closed under $V_\mu(f)$.

Theorem 6. *Let $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda, \alpha)$. Then $V_\mu(f) \in \mathcal{TJ}_{\mu}^{\eta}(\lambda, \alpha)$.*

Proof. By definition, we have

$$\begin{aligned} V_\mu(f) &= \frac{(c+1)^\delta}{\mu(\delta)} \int_0^1 (-1)^{\delta-1} t^c (logt)^{\delta-1} \left(z - \sum_{k=2}^{\infty} a_k z^k t^{k-1}\right) dt \\ &= \frac{(-1)^{\delta-1}(c+1)^\delta}{\mu(\delta)} \lim_{r\to 0^+} \left[\int_r^1 t^c (logt)^{\delta-1} \left(z - \sum_{k=2}^{\infty} a_k z^k t^{k-1}\right) dt\right], \end{aligned}$$

and a simple calculation gives

$$V_\mu(f)(z) = z - \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right)^\delta a_k z^k.$$

We need to prove that

$$\sum_{k=2}^{\infty} \frac{(k[1+\lambda(k-1)] - \alpha)\; C_k(\eta, \mu)}{(1-\alpha)} \left(\frac{c+1}{c+k}\right)^\delta a_k < 1. \tag{4.1}$$

On the other hand by Theorem 1, $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda, \alpha)$ if and only if

$$\sum_{k=2}^{\infty} \frac{(k[1+\lambda(k-1)] - \alpha)a_k\; C_k(\eta, \mu)}{(1-\alpha)} < 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (4.1) holds and the proof is complete. □

Next we provide a starlike condition for functions in $\mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$ and $V_{\mu}(f)$ on lines similar to Theorem 5 .

Theorem 7. *Let $f\in\mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$. Then*
(i) $V_{\mu}(f)$ is starlike of order $0\leq\gamma<1$ in $|z|<R_1$ where

$$R_1=\inf_k\left[\left(\frac{c+k}{c+1}\right)^{\delta}\frac{(1-\gamma)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\gamma)}\right]^{\frac{1}{k-1}}$$

(ii). $V_{\mu}(f)$ is convex of order $0\leq\gamma<1$ in $|z|<R_2$ where

$$R_2=\inf_k\left[\left(\frac{c+k}{c+1}\right)^{\delta}\frac{(1-\gamma)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\gamma)}\right]^{\frac{1}{k-1}}.$$

5 Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $\mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$.

Lemma 1. *[5] If the functions f and g are analytic in U with $g\prec f$, then for $\kappa>0$, and $0<r<1$,*

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^{\kappa}d\theta\leq\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\kappa}d\theta. \tag{5.1}$$

In [9], Silverman found that the function $f_2(z)=z-\frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to resolve his integral means inequality, conjectured in [10] and settled in [11], that

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\kappa}d\theta\leq\int_0^{2\pi}\left|f_2(re^{i\theta})\right|^{\kappa}d\theta,$$

for all $f\in\mathcal{T}$, $\kappa>0$ and $0<r<1$. In [11], he also proved his conjecture for the subclasses of starlike functions of order α and convex functions of order α.

Applying Lemma 1, Theorem 1 and Theorem 3, we prove the following result.

Theorem 8. *Suppose $f(z)\in\mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$ and $f_2(z)$ is defined by $f_2(z)=z-\frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(b,\mu)}z^2$, Then for $z=re^{i\theta}$, $0<r<1$, we have*

$$\int_0^{2\pi}\|f(z)\|^{\kappa}d\theta\leq\int_0^{2\pi}\|f_2(z)\|^{\kappa}d\theta. \tag{5.2}$$

Proof. For $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, (5.2) is equivalent to proving that

$$\int_0^{2\pi} \left\| 1 - \sum_{k=2}^{\infty} a_k z^{k-1} \right\|^{\kappa} d\theta \le \int_0^{2\pi} \left\| 1 - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(b,\mu)} z \right\|^{\kappa} d\theta.$$

By Lemma 1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{k-1} \prec 1 - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]\|C_2(b,\mu)\|} \|\mathbb{P}\|.$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{k-1} = 1 - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]\|C_2(b,\mu)\|} w(z), \tag{5.3}$$

and using (2.1), we obtain

$$\begin{aligned} \|w(z)\| &= \left\| \sum_{k=2}^{\infty} \frac{(1-\alpha)}{(k[1+k\lambda-\lambda]-\alpha)C_k(b,\mu)} a_k z^{k-1} \right\| \\ &\le \|\mathbb{P}\| \sum_{k=2}^{\infty} \frac{(1-\alpha)}{(k[1+k\lambda-\lambda]-\alpha)\|C_k(b,\mu)\|} |a_k| \\ &\le \|\mathbb{P}\|. \end{aligned}$$

This completes the proof Theorem 8. □

6 Modified Hadamard Products

Let the functions $f_j(z)(j = 1, 2)$ be defined by (1.2). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) \;=\; z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2}\, z^k.$$

Using the techniques of Schild and Silverman [12], we prove the following results.

Theorem 9. *For functions $f_j(z)(j = 1, 2)$ defined by (1.2), let $f_1 \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda, \alpha)$ and $f_2 \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda, \gamma)$. Then $(f_1 * f_2) \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda, \xi)$ where*

$$\xi = 1 - \frac{(3+2\lambda)(1-\alpha)(1-\gamma)}{(2+2\lambda-\gamma)(2+2\lambda-\alpha)C_2(\eta,\mu) - (1-\alpha)(1-\gamma)}.$$

Proof. In view of Theorem 1, it suffice to prove that

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)}a_{k,1}a_{k,2}\leq 1,\quad (0\leq\xi<1)$$

where ξ is defined by (6.1). On the other hand, under the hypothesis, it follows from (2.1) and the Cauchy's-Schwarz inequality that

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}C_k(\eta,\mu)}{\sqrt{(1-\alpha)(1-\gamma)}}\sqrt{a_{k,1}a_{k,2}}\ \leq 1. \tag{6.1}$$

We need to find the largest ξ such that

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)}a_{k,1}a_{k,2}\\ \leq\ &\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}C_k(\eta,\mu)}{\sqrt{(1-\alpha)(1-\gamma)}}\sqrt{a_{k,1}a_{k,2}}\end{aligned}$$

or, equivalently that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}}{\sqrt{(1-\alpha)(1-\gamma)}}\frac{1-\xi}{(k[1+\lambda(k-1)]-\xi)},$$

$(k\geq 2)$. By view of (6.1) it is sufficient to find largest ξ such that

$$\begin{aligned}&\frac{\sqrt{(1-\alpha)(1-\gamma)}}{C_k(\eta,\mu)(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}}\\ \leq\ &\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}}{\sqrt{(1-\alpha)((1-\gamma))}}\times\frac{1-\xi}{(k[1+\lambda(k-1)]-\xi)}\end{aligned}$$

which yields

$$\xi\leq 1-\frac{(k[1+\lambda(k-1)]+1)(1-\alpha)(1-\gamma)}{(k[1+\lambda(k-1)]-\gamma)(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)-(1-\alpha)(1-\gamma)} \tag{6.2}$$

for $k\geq 2$ it is an increasing function of k $(k\geq 2)$ for $0\leq\alpha<1; 0<\beta\leq 1; 0\leq\lambda\leq 1$ and letting $k=2$ in (6.2), we have

$$\xi=1-\frac{(3+2\lambda)(1-\alpha)(1-\gamma)}{(2+2\lambda-\gamma)(2+2\lambda-\alpha)C_2(\eta,\mu)-(1-\alpha)(1-\gamma)}.$$

□

Theorem 10. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$. Also let $g(z)=z-\sum\limits_{k=2}^{\infty}|b_k|z^k$ for $|b_k|\le 1$. Then $(f*g)\in\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$.*

Proof. Since

$$\begin{aligned}
& \sum_{k=2}^{\infty}(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)|a_kb_k| \\
\le\ & \sum_{k=2}^{\infty}(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)a_k \\
\le\ & (1-\alpha)
\end{aligned}$$

it follows that $(f*g)\in\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$, by the view of Theorem 1. □

Theorem 11. *Let the functions $f_j(z)(j=1,2)$ defined by (1.2) be in the class $\in\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$. Then the function $h(z)$ defined by $h(z)=z-\sum\limits_{k=2}^{\infty}(a_{k,1}^2+a_{k,2}^2)z^k$ is in the class $\in\mathcal{TJ}_{\mu}^{\eta}(\lambda,\xi)$, where*

$$\xi=1-\frac{2(1-\alpha)^2[2(1+\lambda)-1]}{[2(1+\lambda)-\alpha]^2C_2(\eta,\mu)-2(1-\alpha)^2}$$

.

Proof. By virtue of Theorem 1, it is sufficient to prove that

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)}(a_{k,1}^2+a_{k,2}^2)\le 1 \tag{6.3}$$

where $f_j\in\mathcal{TJ}_{\mu}^{b}(\lambda,\xi,)$ we find from (2.1) and Theorem 1, that

$$\sum_{k=2}^{\infty}\left[\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}\right]^2a_{k,j}^2\le \left[\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}a_{k,j}\right]^2 \tag{6.4}$$

which yields

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}\right]^2(a_{k,1}^2+a_{k,2}^2)\le 1. \tag{6.5}$$

On comparing (6.4) and (6.5), it is easily seen that the inequality (6.3) will be satisfied if

$$\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)} \leq \frac{1}{2}\left[\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}\right]^2, \quad \text{for } k \geq 2.$$

That is an increasing function of k $(k \geq 2)$. Taking $k = 2$ in (6.6), we have,

$$\xi = 1 - \frac{2(1-\alpha)^2[2(1+\lambda)-1]}{[2(1+\lambda)-\alpha]^2 C_2(\eta,\mu) - 2(1-\alpha)^2}, \tag{6.6}$$

which completes the proof. □

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Modified Noor iterative procedure for uniformly continuous mappings in Banach spaces.

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Abstract. In this paper, a strong convergence theorem is obtained for three uniformly continuous mappings in real Banach spaces. Our results extend, improve and generalize the recent results of Chang et al.(2009) among others.

Resumen. En este trabajo, se obtiene un teorema de la convergencia fuerte para tres funciones uniformemente continuas en espacios de Banach reales. Nuestro resultado amplia, mejora y generaliza los resultados recientes de Chang et al. (2009), entre otros.

1 Introduction

Let E be an arbitrary real Banach Space and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* :< x, f >= \|x\|^2 = \|f\|^2\}, \forall x \in E$$

where E^* denotes the dual space of E and $< ., . >$ denotes the generalized duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j. Let $y \in E$ and $j(y) \in J(y)$; note that $< ., j(y) >$ is a Lipschitzian map.
Let K be a nonempty closed convex subset of E and $T : K \to K$ be a map.
The mapping T is said to be uniformly L- Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for any $x, y \in K$ and $\forall n \geq 1$.
The mapping T is said to be asymptotically pseudocontractive if there exists a

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sequence $(k_n) \subset [1, \infty)$ with $lim_{n\to\infty} k_n = 1$ and for any $x, y \in K$ there exists $j(x-y) \in J(x-y)$ such that

$$< T^n x - T^n y, j(x-y) >\leq k_n \|x-y\|^2, \forall n \geq 1.$$

The concept of asymptotically pseudocontractive mappings was introduced by Schu[14].
Recently, Chang et al.[4] pointed out some gaps in the proofs of result in [12] and then proved a strong convergence theorem for a pair of L-Lipschitzian mappings instead of a single map used in [12]. In fact,they proved the following theorem :

Theorem 1.1 ([4]). Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$,$(i = 1, 2)$ be two uniformly L_i-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{x_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:
(i)$\sum_{n=1}^{\infty} \alpha_n = \infty$ (ii)$\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ (iii)$\sum_{n=1}^{\infty} \beta_n < \infty$
(iv) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n.$$

If there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) >\leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in K$, (i=1,2), then $\{x_n\}$ converges strongly to ρ.
The result above extends and improves the corresponding results of [12] from one uniformly Lipschitzian asymptotically pseudocontractive mapping to two uniformly Lipschitzian asymptotically pseudocontractive mappings. In fact, if the iteration parameter $\{\beta_n\}$ in Theorem 1.1 above is equal to zero for all n and $T_1 = T_2 = T$ then we have the main result of Ofoedu [12].
Within the past 5years or so, considerable research efforts have been devoted to developing iterative methods for approximating the common fixed points (assuming existence) for families of two or more maps for several classes of non-linear mappings (see,[4] and [13]).
Rafiq [13], introduced a new type of iteration- the modified three-step iteration process, to approximate the common fixed point of three strongly pseudocontractive mappings in a real Banach space. It is defined as follows:

Let $T_1, T_2, T_3 : K \to K$ be three mappings. For any given $x_0 \in K$, the modified Noor iteration $\{x_n\}_{n=0}^{\infty} \subset K$ is defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T_2 z_n$$

$$z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \quad \geq 0 \tag{1.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences satisfying (i) $a_n, b_n \to 0$ as $n \to \infty$ and (ii)$\sum_{n=0}^{\infty} a_n = \infty$. It is clear that the iteration scheme (1.1) includes iterations defined in the theorems of Ofoedu[12] as special cases.

In fact, he proved the following theorem:

Theorem 1.2 ([13]). Let X be a real Banach space and K be a nonempty closed convex subset of X. Let T_1, T_2, T_3 be strongly pseudocontrative self maps of K with $T_1(K)$ bounded and T_1, T_3 be uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.1), where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are the three real sequences in [0,1] satisfying the conditions,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0, \sum_{n=0}^{\infty} a_n = \infty.$$

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

The purpose of this paper is to extend and improve the recent results of Chang et al.[4] which in turn is a correction, improvement and generalization of results in Ofoedu[12]. We remove the conditions $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$ from Theorem 1.1 and, replace them with a weaker condition $\alpha_n, \beta_n \to 0$ as $n \to \infty$. We equally extend their pair of maps to three maps. Furthermore, we use a more general iteration procedure. Also, the L- Lipschitzian assumption imposed on T_i in Theorem 1.1 is replaced by more general uniformly continuous mappings. Our method is different from [4].

In order to obtain the main results, the following lemmas are needed.

Lemma 1.1[2].Let E be real Banach Space and $J : E \to 2^{E^*}$ be the normalized duality mapping.Then,for any $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y)$$

Lemma 1.2[8]. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n \Phi(a_{n+1}), \quad \forall n \geq N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2.Main results

Theorem 2.1. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T_1, T_2, T_3 : K \to K$ be uniformly continuous mappings such that $T_1(K)$ is bounded and let suppose that $T_1(K), T_2(K)$ and $T_3(K)$ have only one common fixed point. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$ and $\{x_n\}$ be a sequence defined by (1.1) where $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$
and

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K$, (i=1,2,3), then $\{x_n\}$ converges strongly to ρ the unique common fixed point of T_1, T_2, T_3.
Proof. By assumption, we have $F(T_1) \cap F(T_2) \cap F(T_3) = \rho$, say. Let $D_1 = \|x_0 - \rho\| + \sup_{n\geq 0} \|T_1^n y_n - \rho\|$. We prove by induction that $\|x_n - \rho\| \leq D_1$ for all n. It is clear that, $\|x_0 - \rho\| \leq D_1$. Assume that $\|x_n - \rho\| \leq D_1$ holds. We will prove that $\|x_{n+1} - \rho\| \leq D_1$. Indeed, from (1.1), we obtain

$$\begin{aligned} \|x_{n+1} - \rho\| &\leq \|(1 - \alpha_n)(x_n - \rho) + \alpha_n(T_1^n y_n - \rho)\| \\ &\leq (1 - \alpha_n)\|x_n - \rho\| + \alpha_n \|T_1^n y_n - \rho\| \\ &\leq (1 - \alpha_n)D_1 + \alpha_n D_1 = D_1. \end{aligned}$$

Hence the sequence $\{x_n\}$ is bounded.
Using the uniformly continuity of T_3, we have $\{T_3^n x_n\}$ is bounded. Denote $D_2 = \max\{D_1, \sup\{\|T_3^n x_n - \rho\|\}\}$, then

$$\begin{aligned} \|z_n - \rho\| &\leq (1 - \gamma_n)\|x_n - \rho\| + \gamma_n \|T_3^n x_n - \rho\| \\ &\leq (1 - \gamma_n)D_1 + \gamma_n D_2 \\ &\leq (1 - \gamma_n)D_2 + \gamma_n D_2 = D_2 \end{aligned}$$

By the virtue of the uniform continuity of T_2, we get that $\{T_2^n z_n\}$ is bounded. Set $D = \sup_{n\geq 0}\|T_2^n z_n - \rho\| + D_2$. From equation (1.1) we have, in view of Lemma 1.1, that

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &= \langle x_{n+1}-\rho, j(x_{n+1}-\rho)\rangle \\
&\leq (1-\alpha_n)\|x_n-\rho\|\|x_{n+1}-\rho\| + \alpha_n\langle T_1^n y_n - \rho, j(x_{n+1}-\rho)\rangle \\
&= (1-\alpha_n)\|x_n-\rho\|\|x_{n+1}-\rho\| \\
&\quad +\alpha_n\langle T_1^n x_{n+1}-\rho, j(x_{n+1}-\rho)\rangle \\
&\quad +\alpha_n\langle T_1^n y_n - T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_n)\|x_n-\rho\|\|x_{n+1}-\rho\| + \alpha_n\sigma_n\|x_{n+1}-\rho\| \\
&\quad +\alpha_n(k_n\|x_{n+1}-\rho\|^2 - \Phi(\|x_{n+1}-\rho\|))
\end{aligned} \tag{2.1}$$

where $\sigma_n = \|T_1^n y_n - T_1^n x_{n+1}\|$. Observe that

$$\begin{aligned}
\|x_{n+1}-y_n\| &= \beta_n\|x_n - T_2^n z_n\| + \alpha_n\|x_n - T_1^n y_n\| \\
&\leq \beta_n(\|x_n-\rho\| + \|T_2^n z_n - \rho\|) + \alpha_n(\|x_n-\rho\| + \|T_1^n y_n - \rho\|) \\
&\leq \beta_n(D_1+D) + \alpha_n(D_1+D)
\end{aligned}$$

This implies that $\lim_{n\to\infty}\|x_{n+1}-y_n\| = 0$, since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$. Since T_1 is uniformly continuous, we have

$$\sigma_n = \|T_1^n x_{n+1} - T_1^n y_n\| \to 0, \quad (n\to\infty) \tag{2.2}$$

In view of the fact that $(a-1)^2 \geq 0$, if $a = \|x_{n+1}-\rho\|$ then

$$\|x_{n+1}-\rho\| \leq \frac{1}{2}(1 + \|x_{n+1}-\rho\|^2). \tag{2.3}$$

Substituting (2.3) into (2.1), we obtain

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &\leq \tfrac{1}{2}((1-\alpha_n)^2\|x_n-\rho\|^2 + \|x_{n+1}-\rho\|^2) + k_n\alpha_n\|x_{n+1}-\rho\|^2 \\
&\quad -\alpha_n\Phi(\|x_{n+1}-\rho\|) + \alpha_n\sigma_n\cdot\tfrac{1}{2}(1+\|x_{n+1}-\rho\|^2)
\end{aligned}$$

$$(1-2k_n\alpha_n-\alpha_n\sigma_n)\|x_{n+1}-\rho\|^2 \leq (1-\alpha_n)^2\|x_n-\rho\|^2 - 2\alpha_n\Phi(\|x_{n+1}-\rho\|) + \alpha_n\sigma_n. \tag{2.4}$$

Since $\lim_{n\to\infty} k_n\alpha_n = \lim_{n\to\infty}\alpha_n\sigma_n = 0$, there exists a natural number N_0 such that

$$\frac{1}{2} < 1 - 2k_n\alpha_n - \alpha_n\sigma_n < 1$$

for all $n > N_0$. Then, (2.4) implies that

$$\begin{aligned}\|x_{n+1}-\rho\|^2 &\leq \tfrac{(1-\alpha_n)^2}{1-2k_n\alpha_n-\alpha_n\sigma_n}\|x_n-\rho\|^2 - \tfrac{2\alpha_n}{1-2k_n\alpha_n-\alpha_n\sigma_n}\Phi(\|x_{n+1}-\rho\|)\\ &\quad + \tfrac{\alpha_n\sigma_n}{1-2k_n\alpha_n-\alpha_n\sigma_n}\\ &\leq \|x_n-\rho\|^2 + \alpha_n\tfrac{(\alpha_n+\sigma_n-2(1-k_n))}{1-2k_n\alpha_n-\alpha_n\sigma_n}\|x_n-\rho\|^2\\ &\quad - \tfrac{2\alpha_n}{1-2k_n\alpha_n-\alpha_n\sigma_n}\Phi(\|x_{n+1}-\rho\|) + \tfrac{\alpha_n\sigma_n}{1-2k_n\alpha_n-\alpha_n\sigma_n}\end{aligned} \tag{2.5}$$

Since $\|x_n-\rho\| \leq D_1$, it follows from (2.5) that $\forall\ n \geq N_0$,

$$\begin{aligned}\|x_{n+1}-\rho\|^2 &\leq \|x_n-\rho\|^2 + 2\alpha_n(\alpha_n+\sigma_n-2(1-k_n))D_1^2\\ &\quad -2\alpha_n\Phi(\|x_{n+1}-\rho\|) + 2\alpha_n\sigma_n\\ &= \|x_n-\rho\|^2 - 2\alpha_n\Phi(\|x_{n+1}-\rho\|)\\ &\quad +2\alpha_n((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n)\\ &\leq \|x_n-\rho\|^2 - 2\alpha_n\Phi(\|x_{n+1}-\rho\|)\\ &\quad +2\alpha_n((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n), \quad \forall n \geq N_0\end{aligned} \tag{2.6}$$

Taking $b_n = 2\alpha_n$ and observing that

$$\lim_{n\to\infty} \frac{2\alpha_n((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n)}{2\alpha_n}$$

$$= \lim_{n\to\infty}((\alpha_n+\sigma_n-2(1-k_n))D_1^2+\sigma_n) = 0$$

then (2.6) becomes

$$a_{n+1}^2 \leq a_n^2 - b_n\Phi(a_{n+1}) + o(b_n), \quad \forall n \geq N_0$$

This, with Lemma 1.2, showed that $a_n \to 0$ as $n \to \infty$, that is ,

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

This completes the proof.

Theorem 2.2. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T : K \to K$ be uniformly continuous mappings such that $T(K)$ is bounded and let suppose that $F(T)$, the set of fixed points of T, has only one common fixed point. Let $k_n \subset [1,\infty)$ be a sequence with $k_n \to 1$ and let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T^n z_n$$

$$z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0$$

where $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in K$, then $\{x_n\}$ cconverges strongly to the fixed point of T.

Corollary 2.3. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T_1, T_2 : K \to K$ be uniformly continuous mappings such that $T_1(K)$ is bounded and let suppose that $T_1(K)$ and $T_2(K)$ have only one common fixed point. Let $k_n \subset [1,\infty)$ be a sequence with $k_n \to 1$ and let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in K$, (i=1,2), then $\{x_n\}$ converges strongly to the unique common fixed point of T_1, T_2.

Corollary 2.4. Let X be a real Banach space , K a nonempty closed and convex subset of X and $T : K \to K$ be uniformly continuous mappings such that $T(K)$ is bounded and let suppose that $F(T)$, the set of fixed points of T, has only one common fixed point. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$ and let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n$$

$$y_n = (1 - \beta_n)x_n + \beta_n T^n x_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in [0,1] satisfying

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \varphi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K$, then $\{x_n\}$ converges strongly to the fixed point of T.

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Counting domino trains

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Abstract. In this paper we present a way to count the number of trains that we can construct with a given set of domino pieces. As an application we obtain a new method to compute the total number of eulerian paths in an undirected graph as well as their starting and ending vertices.

Resumen. En este artículo presentamos una manera de contar el número de trenes que se puede construir con un determinado conjunto de fichas de dominó. Como una aplicación se obtiene un nuevo método para calcular el número total de caminos eulerianos en un grafo no dirigido, así como sus vértices inicial y final.

1 Introduction

Let G be a directed graph with set of vertices V_G and edge set E_G. A path of length m on G is a sequence of vertices $v_0, v_1, \ldots, v_m$ such that $(v_{i-1}, v_i) \in E_G$ for all $1 \leq i \leq m$. A path is a cycle if $v_0 = v_m$ and $(v_{i-1}, v_i) \neq (v_{j-1}, v_j)$ for all $1 \leq i < j \leq m$. An eulerian path or cycle is a path or cycle of length $|E_G|$. These concepts can be extended to the case when G is an undirected graph in a natural way (see [4] for these and other elementary concepts about Graph Theory).

Two eulerian cycles are called equivalent if one is a cyclic permutation of the other. Let $\text{Eul}(G)$ denote the number of equivalence classes of eulerian cycles. If G is a directed graph there is a well-known theorem, the so called BEST Theorem (see [2, 3]), which computes $\text{Eul}(G)$ but if G is an undirected graph the situation is much more difficult. Nevertheless, in the case of complete graphs interesting results exist. In [1] for instance, an asymptotic value for $\text{Eul}(K_n)$ (as well as the exact number for $n \leq 21$) is given, with K_n being the complete graph with n vertices.

Now let us denote by $\text{Eul}_i^j(G)$ the number of eulerian paths in G starting in vertex v_i and ending in vertex v_j (with no equivalence relation taken into account). In this paper we introduce a new approach and present a new method to compute $\text{Eul}_i^j(G)$ for any undirected graph G. In particular we will count the number of trains that we can construct with a given set of domino pieces, where a train is a chain constructed following the rules of domino and using all the pieces from the given set. Since any graph gives rise to a set of domino pieces where trains correspond to eulerian paths, our method applies to any graph.

The paper is organized as follows. In the first section we present some elementary definitions and fix the notation. The second section is devoted to prove the main result in the paper, namely to present a method to compute the number of trains constructible from a given set of domino pieces. Finally, in the third section we translate this result to a graph theory setting.

2 Definitions and notation

Let us denote by $\{e_{ij} \mid 1 \leq i,j \leq n\}$ the set of matrix units in $\mathcal{M}_n(\mathbb{R})$; i.e., e_{ij} is the square matrix of size n having a 1 in position (i,j) and 0 elsewhere. Now, we define $\overline{e}_{ij} = e_{ij} + e_{ji}$ for all $i \neq j$ and $\overline{e}_{ii} = e_{ii}$. Clearly the set $\mathcal{B} = \{\overline{e}_{ij} \mid 1 \leq i \leq j \leq n\}$ is a basis for the vector space $\mathcal{S}_n(\mathbb{R})$ of real symmetric matrices of size n. If we define a new product over $\mathcal{M}_n(\mathbb{R})$ by $A \bullet B = AB + BA$ our basis multiply in the following way:

$$\overline{e}_{ij} \bullet \overline{e}_{kl} = \begin{cases} \overline{e}_{jl} & \text{if } i = k \text{ and } j \neq l \\ 2\overline{e}_{ii} + 2\overline{e}_{jj} & \text{if } i = k \text{ and } j = l \text{ and } i \neq j \\ 2\overline{e}_{ii} & \text{if } i = j = k = l \\ 0 & \text{if } \{i,j\} \cap \{k,l\} = \emptyset \end{cases} \tag{1}$$

It is well-known that $\bullet$ is commutative but non-associative. Nevertheless, we may define recursively $A_1 \bullet \cdots \bullet A_r = (A_1 \bullet \cdots \bullet A_{r-1}) \bullet A_r$. Now, given $m \in \mathbb{N}$ we consider the polynomial $P_m(X_1, \ldots, X_m) = \sum_{\sigma \in S_m} X_{\sigma(1)} \bullet X_{\sigma(2)} \bullet \cdots \bullet X_{\sigma(m)}$. With the previous convention and choosing $\overline{e}_t = \overline{e}_{i_t j_t}$ for $t = 1, \ldots, m$ it makes sense to compute $P_m(\overline{e}_1, \ldots, \overline{e}_m) = \sum_{\sigma \in S_m} \overline{e}_{\sigma(1)} \bullet \cdots \bullet \overline{e}_{\sigma(m)}$. Moreover, there must exist $\{\alpha_{ij}\} \subseteq \mathbb{R}$ such that $P_m(\overline{e}_1, \ldots, \overline{e}_m) = \sum \alpha_{ij} \overline{e}_{ij}$.

In what follows we will denote by (i,j) the domino piece marked with numbers i and j. We recall that in the game of domino two pieces can be placed together if they share at least one of their numbers. Now let us suppose that we are given certain set of dominoes that we shall denote by $\mathcal{D} = \{(i_1, j_1), \ldots, (i_m, j_m)\}$, we define a train as a sequence $(i_{k_1}, j_{k_1}) \cdots (i_{k_m}, j_{k_m})$ admissible by the rules of domino; i.e., such that $j_{k_r} = i_{k_{r+1}}$ for all $1 \leq r \leq m-1$.

3 Counting domino trains

Given a set of domino pieces $\mathcal{D} = \{(i_1, j_1), \ldots, (i_m, j_m)\}$ we are interested in counting the number of trains that we can construct using all the pieces of $\mathcal{D}$. If an element of $\mathcal{D}$ appears more than once, we will assume that we can distinguish them.

For any domino piece (i, j) we will identify $(i, j) \leftrightarrow \overline{e}_{ij}$. Clearly, two pieces (i, j) and (k, l) can be placed together following the rules of domino if and only if $\overline{e}_{ij} \bullet \overline{e}_{kl} \neq 0$. Consequently we have the following:

Lemma 1. *A sequence* $(i_1, j_1)(i_2, j_2) \cdots (i_n, j_n)$ *is a train if and only if* $\overline{e}_{i_1 j_1} \bullet \overline{e}_{i_2 j_2} \bullet \cdots \bullet \overline{e}_{i_n j_n} \neq 0$.

Proof. By induction on n. Cases $n = 1, 2$ are obvious due to (1). Now let us suppose that $(i_1, j_1)(i_2, j_2) \cdots (i_n, j_n)$ is a train with $n \geq 3$ and $\overline{e}_{i_1 j_1} \bullet \overline{e}_{i_2 j_2} \bullet \cdots \bullet \overline{e}_{i_n j_n} = 0$. Then, there must exist $2 \leq k < n$ such that $\overline{e}_{i_1 j_1} \bullet \cdots \bullet \overline{e}_{i_k j_k} = 0$ and by our induction hypothesis $(i_1, j_1) \cdots (i_k, j_k)$ is not a train which is a contradiction. Conversely if $\overline{e}_{i_1 j_1} \bullet \overline{e}_{i_2 j_2} \bullet \cdots \bullet \overline{e}_{i_n j_n} \neq 0$, then $\overline{e}_{i_1 j_1} \bullet \overline{e}_{i_2 j_2} \neq 0$ and $\overline{e}_{i_2 j_2} \bullet \cdots \bullet \overline{e}_{i_n j_n} \neq 0$ so, by induction, both $(i_1, j_1)(i_2, j_2)$ and $(i_2, j_2) \cdots (i_n, j_n)$ are trains and consequently so is $(i_1, j_1)(i_2, j_2) \cdots (i_n, j_n)$. □

Now, given the set $\mathcal{D}$ and since the set $\mathcal{B}$ is a basis for $\mathcal{S}_n(\mathbb{R})$, we have that $P_m(\overline{e}_{i_1 j_1}, \ldots, \overline{e}_{i_m j_m}) = \sum \alpha_{ij} \overline{e}_{ij}$. In fact, the following lemma holds.

Lemma 2. *In the previous situation,* $\alpha_{i_k j_k} \neq 0$ *if and only if we can construct a train starting with* i_k *and ending with* j_k *(or viceversa) using the pieces of the set* $\mathcal{D}$.

Proof. It is an easy consequence of (1) and Lemma 1. □

Observe that, given a train $(i_1, j_1)(i_2, j_2)(i_3, j_3)$ we could have placed its pieces in different ways according to the order. Namely we could have firstly placed the piece (i_1, j_1) then (i_2, j_2) on its right and finally the piece (i_3, j_3) on the right of the latter. We could also have started by piece (i_2, j_2), then (i_1, j_1) on its left and finally (i_3, j_3) on the right of the former. If we denote by $c(n)$ the number of different ways we can construct a train of length n we have.

Lemma 3. $c(n) = 2^{n-1}$

Proof. Clearly we can finish a given train either by placing the first piece (the one in the left) or the last one (the one in the right). This implies that $c(n) = 2c(n-1)$ and since $c(1) = 1$ the result follows. □

With these technical lemmas in mind, we are in condition to prove the main result of this paper.

Proposition 1. *Given a set of domino pieces* $\mathcal{D} = \{(i_1, j_1), \dots, (i_m, j_m)\}$, *the number* $\frac{\alpha_{i_k j_k}}{2^{m-1}}$ *is exactly the number of domino trains that we can construct starting with* i_k *and ending with* j_k *(or viceversa) using the pieces from the set* $\mathcal{D}$.

Proof. Again, cases $m = 1, 2$ are obvious and can be verified by direct computation using (1). Now if $m \geq 3$ it is enough to observe that

$$P_m(\overline{e}_{i_1,j_1}, \dots, \overline{e}_{i_m,j_m}) = \sum_{k=1}^{m} P_{m-1}(\overline{e}_{i_1 j_1}, \dots, \widehat{\overline{e}_{i_k j_k}}, \dots, \overline{e}_{i_m j_m}) \bullet \overline{e}_{i_k j_k}$$

and proceed by induction on m. □

Example 1. Given the set $\mathcal{D} = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$ we have that $P_6(\overline{e}_{11}, \overline{e}_{12}, \overline{e}_{13}, \overline{e}_{22}, \overline{e}_{23}, \overline{e}_{33}) = 2^5 \left(\sum_{i=1}^{3} 4\overline{e}_{ii} \right)$. So, we can construct 4 trains starting and ending with 1, and the same number with 2 and 3.

Example 2. Given the set $\mathcal{D} = \{(1,2), (1,3), (2,3), (2,3), (2,4), (3,4)\}$ we have that $P_6(\overline{e}_{12}, \overline{e}_{13}, \overline{e}_{23}, \overline{e}_{22}, \overline{e}_{24}, \overline{e}_{34}) = 2^5 (12\overline{e}_{11} + 24\overline{e}_{22} + 24\overline{e}_{33} + 12\overline{e}_{44})$. So, we can construct 12 trains starting and ending with 1 (the same number with 4), and 24 trains with 2 (the same number with 3).

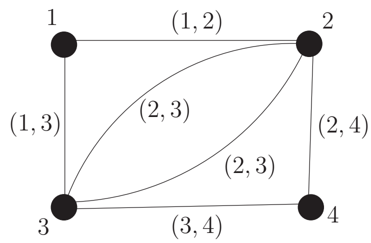
4 Counting eulerian paths in graphs

Given a set of dominoes $\mathcal{D} = \{(i_1, j_1), \dots, (i_m, j_m)\}$ we can construct a graph in the following straightforward way: we draw a vertex for each of the numbers appearing ir our set $\mathcal{D}$ and we join two of such vertices if and only if the corresponding piece lies in our set. Clearly we can proceed backwards and obtain a set of dominoes from every graph. Moreover, domino trains correspond exactly with eulerian paths or cycles. Thus, our previous construction allows us to count them.

Also observe that given a graph G if we denote by $\mathcal{D}_G$ the set of dominoes coming from the previous construction, then $\sum_{(i,j)\in\mathcal{D}_G} \overline{e}_{ij} = A_G$ where A_G is the adjacency matrix of G. Conversely, given a graph G with adjacency matrix A_G we can write $A_G = \sum_{k=1}^{m} \overline{e}_{i_k j_k}$ with $m = |E_G|$ and we know that $P_m(\overline{e}_{i_1 j_1}, \dots, \overline{e}_{i_m j_m}) = \sum \alpha_{ij} \overline{e}_{ij}$. The following result translates Proposition 1 to this setting.

Proposition 2. *In the previous situation and with the notation of Section 1,* $Eul_i^j(G) = \frac{\alpha_{ij}}{2^{m-1}}$.

Example 3. Given the graph below:

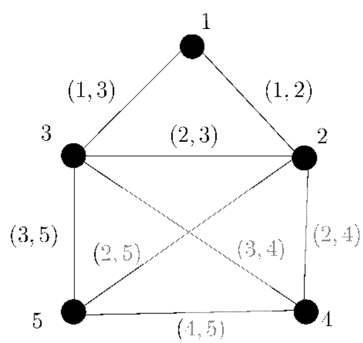


it is clear that the associated set of domino pieces is

$$\mathcal{D}_G = \{(1,2),(1,3),(2,3),(2,3)(2,4),(3,4)\},$$

and according to Example 2 above, the total number of eulerian cycles of G is 72, 12 starting and ending in vertex 1, 24 in vertex 2, 24 in vertex 3 and 12 in vertex 4.

Example 4. Consider the graph G given in the figure below.



In this case $P_8(\bar{e}_{12},\bar{e}_{13},\bar{e}_{23},\bar{e}_{24},\bar{e}_{25},\bar{e}_{34},\bar{e}_{35},\bar{e}_{45}) = 2^7 \cdot 44\bar{e}_{45}$ and thus G has 44 eulerian paths starting in vertex 4 and ending in vertex 5 (and vice versa).

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DIVULGACIÓN MATEMÁTICA

Another perspective on a famous problem, IMO 1988: The equation $\frac{x^2+y^2}{xy+1} = n^2$

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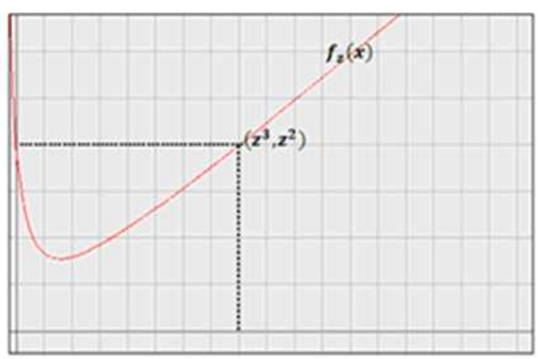
Abstract. In this work we apply a simple property of the function F below to study an interesting IMO problem proposed in 1988 of which we give a solution. We analyze with some detail the diophantine equation $F(x, y) = n^2$ in connection with this problem.

Resumen. En este trabajo se aplica una simple propiedad de la función F, ver abajo, para estudiar un interesante problema propuesto en la OMI de 1988, del cual damos una solución. Se analiza con cierto detalle la ecuación diofántica $F(x, y) = n^2$ en relación con este problema.

The symmetrical function $F(x, y) = \frac{x^2+y^2}{xy+1}$ of $\mathbb{R}_+ \times \mathbb{R}_+$ in $\mathbb{R}_+$ has the remarkable property, trivial to verify: $F(x, x^3) = F(x, 0) = x^2$ for all x.



$f_z(x) = \frac{z^2+x^2}{zx+1}$, $x \geq 0, z \geq 0$. *Always* $f_z(0) = f_z(z^3) = z^2$

Here we use basically this property to determine an infinity of integer solutions of the equation $F(x, y) = n^2$ for all $n \geq 2$. We give first a solution, apparently new, to a famous problem [see (9) below] proposed by Stephan Beck, Federal Germany, in the 29° International Olympic Games of Mathematics held at Canberra, Australia, in 1988. The statement of this problem implies that if n is not a perfect square, the equation $F(x,y) = n$ does not have integer solutions.

Let us define the function f_n from $\mathbb{R}_+$ to $\mathbb{R}_+$ by $f_n(x) = F(n,x)$, i.e.,

$$f_n(x) = \frac{n^2+x^2}{nx+1};\ x \geq 0$$

For $n \in \mathbb{N}$ we have the following properties which are elementary results:

1. $f_n(m) = f_m(n)$ and $f_n(0) = f_n(n^3) = n^2$.
2. f_n is $1-1$ over $x > n^3$.
3. f_n has a unique minimum at $n_0 = \frac{-1+\sqrt{n^4+1}}{n} < n$.
4. f_n decreases over $[0, n_0]$ and increases over $x > n_0$

$$f_n(n_0) = \frac{2(\sqrt{1+n^4}-1)}{n^2} = m_0 < 2 \text{ for all } n;\ 1 < m_0 < 2;\ n \neq 1$$

5. For all $x \neq n_0$ in $[0, n^3]$ there exists a unique

$$y \neq x \text{ such that } f_n(x) = f_n(y);\ \text{in fact } y = \frac{n^3-x}{nx+1} \in [0, n^3]$$

Let h_n be the function defined by $h_n(x) = \frac{n^3-x}{nx+1};\ 0 \leq x \leq n^3$.

Thus $h_n(x) = y$. Note the function h_n is involutive, i.e., $h_n(h_n(x)) = x$.

6. If $x, f_n(x)$ are nonnegative integers, with $0 \leq x < n^3$ then $h_n(x)$ is a nonnegative integer.

$$\text{Moreover, } n_0 < x < n^3 \Longleftrightarrow 0 < h_n(x) < n_0$$

Proof:

$$\frac{n^2+x^2}{nx+1} = \frac{n^2+[h_n(x)]^2}{nh_n(x)+1} = k \Rightarrow \frac{x+h_n(x)}{n} = k \text{ therefore}$$

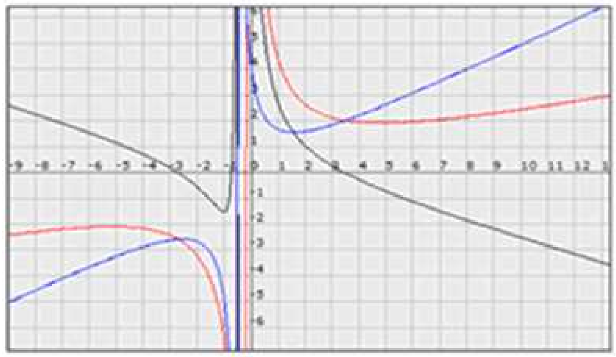
$h_n(x) = kn - x$ is an integer; it must be positive by definition of $h_n(x)$.

7. If $0 \leq a < b$ then $f_a(x) > f_b(x)$ for all $x > \alpha$ where α is the unique positive root of $x^3 - abx - (a+b) = 0$.

Proof: Consider the difference function

$$g(x) = f_b(x) - f_a(x) = \frac{-(b-a)[x^3 - abx - (a+b)]}{(ax+1)(bx+1)};\ x \geq 0.$$

It is easily seen, using the derivative, that $g(x)$ is decreasing over $x \geq 0$ going from $g(0) = b^2 - a^2$ to $-\infty$ so the equation $g(x) = 0$ has a unique positive root α; consequently $f_a(x) > f_b(x)$ if $x > \alpha$.



$f_2(x) = \frac{4+x^2}{2x+1}$, $f_5(x) = \frac{25+x^2}{5x+1}$, $g(x) = f_5(x) - f_2(x) = \frac{-3(x^2-10x-7)}{(2x+1)(5x+1)}$

We are dealing with $x \geq 0$ but the corresponding part to $x < 0$ is showed in order to see what happen with the other two roots. The function g has the unique positive root $\alpha = 3.46686$ and the negative roots $\beta = -0.740625$, $\gamma = -2.766235$.

8. If $0 \leq a < b$ then $f_b(x) = f_a(x) = \beta$ at a unique point $x = \alpha$ where α is the positive root of $x^3 - abx - (a+b) = 0$.

   Furthermore $a + b = \alpha\beta$.

   Proof:
$$\frac{b^2+x^2}{bx+1} = \frac{a^2+x^2}{ax+1} \Rightarrow x^3 - abx - (a+b) = 0$$
   On the other side $\dfrac{b^2+\alpha^2}{b\alpha+1} = \dfrac{a^2+\alpha^2}{a\alpha+1} = \beta \Rightarrow a + b = \alpha\beta$

9. **PROBLEM 6 (IMO 1988).-** Let a and b positive integers such that $ab+1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer.

   SOLUTION: With $a < b$ ($a = b$ would give $1 < k < 2$ where $\frac{a^2+b^2}{ab+1} = k$) consider the functions f_a and f_b so, $k = f_b(a) = f_a(b)$ as in (1).

   When $k = a^2$ there is nothing to prove. Suppose $f_a(b) = k > a^2$. There exists always a real $c \neq b > a$ such that $k = \frac{a^2+b^2}{ab+1} = \frac{a^2+c^2}{ac+1}$ from which, as in the proof of (6), we have $b + c = ak$ hence c is an integer. On the other hand, when $k > a^2$, it is easily seen that $-\frac{1}{a} < c < 0$. This is a contradiction and therefore we consider only $k < a^2$. *

   We know, by (3) and (4), that $f_a(x)$ is increasing at $x = b$ because $b > a > a_0$ where a_0 is the unique point in which f_a takes its minimum. Applying

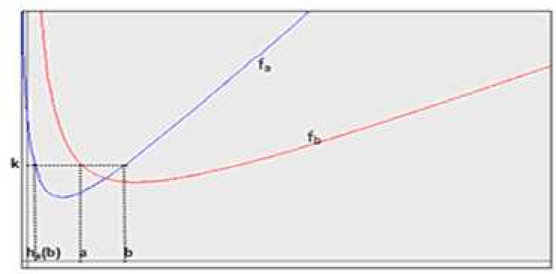
*This is indeed the proposition (13) given below but stated otherwise.

(5) and (6) we obtain the integers $k = f_a(b) = f_a(a_1) = f_{a_1}(a)$ where $0 < a_1 = h_a(b) < a_0 < a < b$ and obviously $a^2 > a_1^2$. Now $f_{a_1}(x)$ is increasing at $x = a$ which implies $0 < a_2 = h_{a_1}(a) < a_1 < a_0 < a < b$ and so on, continuing this way we obtain

$$k = f_{a_n}(a_{n+1}) = f_{a_{n+1}}(a_n) = f_{a_{n+1}}(a_{n+2})$$

where $a_{n+2} = h_{a_{n+1}}(a_n)$

and $b^2 > a^2 > a_1^2 > a_2^2 > a_3^2 > a_n^2 > \cdots \cdots \geq k$



Construction of the a_n

Consequently because of we are dealing with integers, we must have $k = a_n^2$ for a certain index n. The desired result follows.

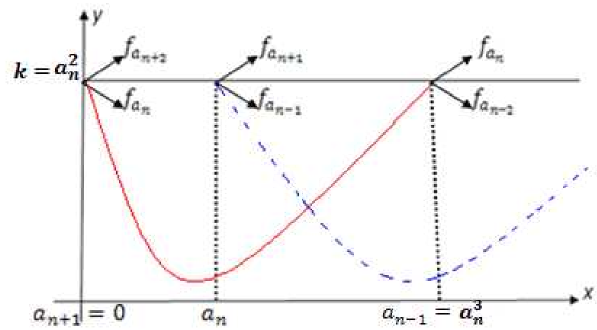
(*) This indeed the proposition (13) given below but stated otherwise.

NOTE: Paragraph (9) gives a third solution which in addition to the two previously known to the author, the first given by the Bulgarian participant in IMO 1988 Emmanuel Atanasiov and the second by the Australian Professor J. Campbell, University of Canberra (see [1], page 65).

The following figure charts the end of the reasoning used in (9) which together with (8) and (1) provides a means of finding integer solutions of the equation $\frac{x^2+y^2}{xy+1} = n^2$

The two curves, f_{a_n} and $f_{a_{n-1}}$ are distorted for practical reasons (the real graphs very quickly stick to the y-axis as can be seen in the figure above where two real graphs are shown).

As $a_n^2 = f_{a_n}(a_{n-1}) = f_{a_n}(0) = f_{a_n}(a_n^3)$ then, by (5), $a_{n-1} = a_n^3$; on the other hand, (8) gives $a_n + a_{n-2} = a_n^3 x\ a_n^2 = a_n^5$, i. e., $a_{n-2} = a_n^5 - a_n$ Continuing in the same way we get integers (by ascent, and not, as in (9), by descent) that are solutions of the proposed equation.



SOLUTIONS OF $\frac{x^2+y^2}{xy+1} = n^2$

10. Thus, given f_n and the trivial point with integer coordinates (n^3, n^2), we consider this point as the intersection of f_n with another curve f_m whose index $m > n$, according to (8), is given by $n + m = n^3\ n^2 = n^5$, i. e. $m = n^5 - n$ (which also goes for the rest solving the equation

$$f_n^3(m) = n^2 \text{ which gives } m = \frac{n^5 + \sqrt{(n^{10} - 4n^6 + 4n^2}}{2} = n^5 - n).$$

The iterated application of the procedure gives the recurrence equation

$$x_{k+2} = n^2 x_{k+1} - x_k, \qquad (x_0, x_1) = (0, n)$$

whose solutions satisfy the condition $f_{x_k}(x_{k+1}) = n^2$ for all $k \geq 1$. The solutions of this equation are given by

$$2^k x_k = \frac{n[(n^2+\alpha)^k - (n^2-\alpha)^k])}{\alpha}$$

where $\alpha = \sqrt{n^4 - 4}$, this is,

$$2^{k-1} x_k = n \sum_i \binom{k}{i} n^{2(k-i)} \alpha^{i-1}$$

where the indexes are the positive odds $i \leq k$.

We finally have

$$2^{k-1} x_k = n \sum_{j=0}^{[k_1]} \binom{k}{i} n^{2(k-2j-1)} (n^4 - 4)^j$$

where $[k_1]$ denotes the integer part of $k_1 = \frac{k-1}{2}$ and moreover

$$F(x_k, x_{k+1}) = \frac{x_k^2 + x_{k+1}^2}{x_k x_{k+1} + 1} = n^2; \qquad k = 1, 2, 3, \ldots\ldots\ldots$$

11. By construction of the integers x_k, the sum in its general definition must be divisible by 2^{k-1} which is clear if n is odd and easily verified in each of the summands if n is even. Therefore each x_k is a multiple of n and moreover, a simple induction using the recurrence equation that defines them proves that n is the greatest common divisor of each pair of consecutive (x_k, x_{k+1}) in that succession.

12. **EXAMPLES:**

$n = 3 \to n^3 = \mathbf{27} \to n^5 - n = \mathbf{240} \to n^7 - 2n^3 = \mathbf{2133} \to n^9 - 3n^5 + n = \mathbf{18957} \to n^{11} - 4n^7 + 3n^3 = \mathbf{168480} \to n^{13} - 5n^9 + 6n^5 - n = \mathbf{1497363} \to \cdots$

$3^2 = 9 = \frac{3^2+27^2}{3*27+1} = \frac{27^2+240^2}{27*240+1} = \frac{240^2+2133^2}{240*2133+1} = \frac{2133^2+18957^2}{2133*18957+1} =$

$= \frac{18957^2+168480^2}{18957*168480+1} = \frac{168400^2+1497363^2}{168480*1497363+1} = \cdots$

13. $f_n(x)$ is not an integer for all integer $x > n^3$.

Proof: Suppose x is an integer with $x > n^3$. If $f_n(x)$ is an integer, by (9) it must be the square of an integer clearly greater than n, then for some integer $h \geq 1$ we have $f_n(x) = (n+h)^2$ which gives the equation $n^2 + x^2 = (nx+1)(n+h)^2$ whose discriminant, $n^2(n+h)^4 + 4(2nh+h^2)$, should be a perfect square. Then there exists an integer $k \geq 1$ such that

$$2n(n+h)^2k + k^2 = 4(2nh + h^2)$$

$$\text{i.e.} \quad 2kn^3 + k^2 + (kn-2)(4nh + 2h^2) = 0$$

This is clearly impossible if $(kn-2) \geq 0$ and then $kn = 1$, but then we have $2h^2 + 4h - 3 = 0$ which gives h irrational. This completes the proof.

Let $[|n|]$ denotes the infinite set of solutions, generated by n, of the recurrence equation $x_{k+2} = n^2 x_{k+1} - x_k$, $(x_0, x_1) = (0, n)$ solved in (10).

14. If $f_n(x) = b^2;\ b \in \mathbb{N};\ x \in \mathbb{N};\ 0 < x < n^3$, then $n \in [|b|]$, i.e. n is one of the solutions in (10) generated by b.

Proof: Suppose $a \in \mathbb{N};\ 0 < a < n^3$ and $f_n(a) \in \mathbb{N}$. By (10) we have $f_n(a) = m^2 < n^2$. By the involutive function of (5) we can choose a such that f_n be decreasing in a which means $0 < a < n_0$ (by (3), (5) and (6)). Then there exists, by (7) and (8), a function f_m increasing in a such that $f_m(a) = f_n(a) = k^2;\ k^2 < m^2 < n^2$ and moreover $m = ak^2 - n$

(Note that n, a and m satisfy the recurrence equation of (10) for the coefficient k^2). We repeat the procedure, now with f_m applied to the point $h_m(a)$ making a descent, as in (9), which should end with f_b such that $f_b(0) = f_b(b^3) = k^2 = b^2$ and then $n \in [|b|]$.

15. **Theorem.-** If $p > 0$ is a prime number, then the unique integer solutions (x, z) of the equation $f_p(x) = z$ are the trivial ones $(0, p^2)$ and (p^3, p^2).

    Proof: It is a consequence of (11), (13) and (14).

CONCLUSION.- Let us denote $A = \{m \in \mathbb{N};\ m > n^3\}$. So far we have obtained the following:

► $f_n(A) \cap \mathbb{N} = \emptyset$ for all natural n ◄

► $f_p(\mathbb{N}) \cap \mathbb{N} = \{p^2\}$ for all prime $p > 0$ ◄

more generally, by (14), we can deduce without difficulty

► $f_n(\mathbb{N}) \cap \mathbb{N} = \{n^2\}$ for all n which does not belong to $[|b|]$ for any non trivial divisor b of n ◄

We know $f_n(\mathbb{N}) \cap \mathbb{N}$ trivially contains $\{n^2\}$. The discussion above leads to conjecture it contains at most one non trivial element.

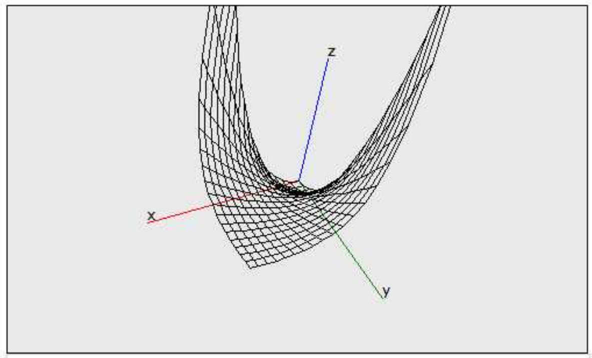
►CONJECTURE◄

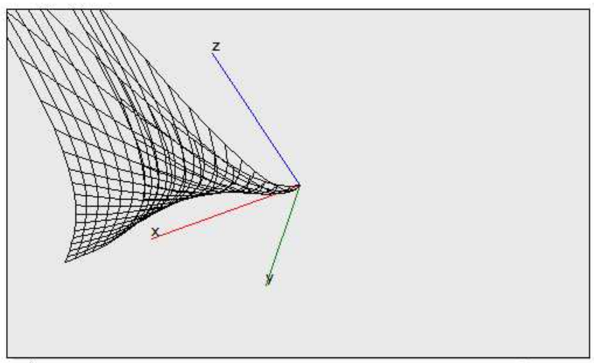
For all $n > 0$, $f_n(\mathbb{N}) \cap \mathbb{N} = \{n^2\}$ or $\{n^2, b^2\}$; ($b < n$ and, by (14), $n \in [|b|]$ therefore, by (11), b divides n).

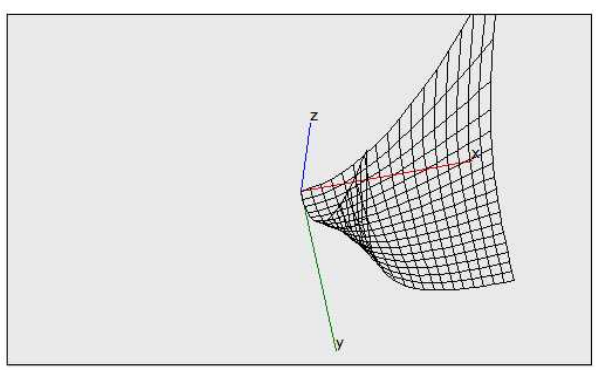
Referencias

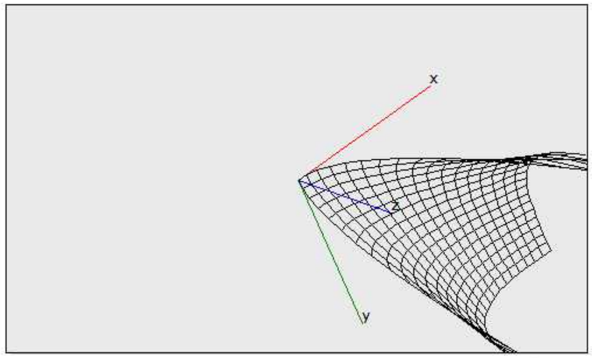
[1] Francisco Bellot Rosado, Ascensión López Ch. *Cien Problemas de Matemáticas.* ICE, Valladolid, 1994

FOUR VIEWS OF THE SURFACE OF EQUATION $z = \frac{x^2+y^2}{xy+1}$









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INFORMACIÓN NACIONAL

La esquina olímpica

Rafael Sánchez Lamoneda

La Esquina Olímpica del primer número de este volumen del Boletín de la AMV, se escribió cuando estaba comenzando la actividad internacional del año 2011. En esa oportunidad informamos sobre los eventos por venir. Ahora podemos resumir toda la experiencia de este año en el ámbito internacional.

El 21 de Mayo se organizó en 6 ciudades del país, Porlamar, Maracaibo, Valencia, Barquisimeto, Puerto Ordaz y Caracas, la Olimpiada de Mayo, con la participación de más de 200 alumnos menores de 16 años. De este grupo de jóvenes se seleccionaron los 10 mejores de cada nivel, de acuerdo a las reglas de la competencia y sus calificaciones, así como las pruebas de los alumnos ubicados en los lugares, primero, tercero y séptimo de cada nivel, fueron enviadas a Buenos Aires, donde está instalado el Comité Organizador de esa olimpiada. Los resultados fueron muy buenos, nuestros muchachos ganaron 17 premios, una medalla de oro, una de plata, cuatro de bronce y 11 menciones honoríficas. Los resultados se pueden ver en nuestra página web, www.amc.ciens.ucv.ve, también pueden tener una información completa en la página oficial de la Olimpiada Matemática Argentina.

Del 16 al 26 de Junio se celebró en Colima, México, la XIII Olimpiada Matemática de Centroamérica y El Caribe, OMCC, con la asistencia de 12 países y 33 estudiantes. Cuba fue el único ausente a la cita y Jamaica participó por segundo año consecutivo. Allí también se destacaron nuestros estudiantes. Rubmary Rojas del colegio Divina pastora de Barquisimeto ganó medalla de plata y Sergio Villarroel, del colegio San Lázaro de Cumaná, ganó medalla de bronce.

La Olimpiada Internacional de Matemáticas, IMO, se llevó a cabo en Amsterdam del 12 al 24 de Julio y como reseñamos en el número anterior asistimos con dos alumnos, Diego Peña del colegio Los Hipocampitos, estado Miranda y Carlos Lamas del colegio Independencia de Barquisimeto. Ambos jóvenes ganaron mención honorífica. Asistieron 101 países y 564 estudiantes y se contó con la participación de dos países como observadores, Senegal y Uganda.

Finalmente del 23 de Septiembre al 1 de Octubre estuvimos presentes en la XXIV Olimpiada Iberoamericana de Matemáticas, OIM. A la cita concurrieron 20 países y un total de 78 estudiantes. Faltaron República Dominicana y Cuba. Nuestro equipo estuvo integrado por Diego Peña, Rubamry Rojas y Sergio Villarroel, solo tres de los cuatro posibles, pues a última hora Carlos Lamas no pudo viajar. La jefe de delegación fue la profesora Laura Vielma de la Academia Washington y la tutora Estefanía Ordaz, alumna de la licenciatura en Matemáticas en la Universidad Simón Bolívar. Otra vez los tres jóvenes fueron galardonados, Diego obtuvo medalla de bronce y Rubmary y Sergio menciones honoríficas.

Como se puede observar, ha sido un buen año para nuestros estudiantes. Cerramos esta esquina con los problemas de la XIII OMCC, la 52ª IMO y la XXIV OIM..

XIII Olimpiada Matemática de Centroamérica y El Caribe.

Primer Día

Problema 1. En cada uno de los vértices de un cubo hay una mosca. Al sonar un silbato, cada una de las moscas vuela a alguno de los vértices del cubo situado en una misma cara que el vértice de donde partió, pero diagonalmente opuesto a éste. Al sonar el silbato, ¿de cuántas maneras pueden volar las moscas de modo que en ningún vértice queden dos o más moscas?

Problema 2. Sean ABC un triángulo escaleno, D el pie de la altura desde A, E la intersección del lado AC con la bisectriz del $\angle ABC$, y F un punto sobre el lado AB. Sea O el circuncentro del triángulo ABC y sean X, Y, Z los puntos donde se cortan las rectas AD con BE, BE con CF, CF con AD, respectivamente. Si XYZ es un triángulo equilátero, demuestra que uno de los triángulos OXY, OYZ, OZX es un triángulo equilátero.

Problema 3, *Aplicar un desliz* a un entero $n \geq 2$ significa tomar cualquier primo p que divida a n y reemplazar n por $\frac{n+p^2}{p}$. Se comienza con un entero cualquiera mayor o igual a 5 y se le aplica un desliz. Al número así obtenido se le aplica un desliz, y así sucesivamente se siguen aplicando deslices. Demuestra que, sin importar los deslices aplicados, en algún momento se obtiene el número 5.

Segundo Día

Problema 4 Encuentra todos los enteros positivos p, q y r, con p y q números primos, que satisfacen la igualdad

$$\frac{1}{p+1}+\frac{1}{q+1}-\frac{1}{(p+1)(q+1)}=\frac{1}{r}.$$

Problema 5. Los números reales positivos x, y, z son tales que

$$x+\frac{y}{z}=y+\frac{z}{x}=z+\frac{x}{y}=2.$$

Determine todos los valores posibles de $x+y+z$.

Problema 6. Sea ABC un triángulo acutángulo y sean D, E y F los pies de las alturas desde A, B y C, respectivamente. Sean Y y Z los pies de las perpendiculares desde B y C sobre FD y DE, respectivamente. Sea F_1 la reflexión de F con respecto a E y sea E_1 la reflexión de E con respecto a F. Si $3EF = FD + DE$, demuestre que $\angle BZF_1 = \angle CYE_1$.

Nota: La *reflexión* de un punto P respecto a un punto Q es el punto P_1 ubicado sobre la recta PQ tal que Q queda entre P y P_1, y $PQ = QP_1$.

52ª Olimpiada Internacional de Matemáticas.

Primer Día

Problema 1. Para cualquier conjunto $A = \{a_1, a_2, a_3, a_4\}$ de cuatro enteros positivos distintos se denota la suma $a_1+a_2+a_3+a_4$ por s_A. Sea n_A el número de parejas (i, j) con $1 \leq i < j \leq 4$ para las cuales a_i+a_j divide a s_A. Encontrar todos los conjuntos A de cuatro enteros positivos distintos para los cuales se alcanza el mayor valor posible de n_A.

Problema 2. Sea $\mathcal{S}$ un conjunto finito de dos o más puntos del plano. En $\mathcal{S}$ no hay tres puntos colineales. Un *remolino* es un proceso que empieza con una recta ℓ que pasa por un único punto $P \in \mathcal{S}$, al cual llamaremos pivote. Se rota ℓ en el sentido de las manecillas del reloj con centro en el pivote P hasta que la recta encuentre por primera vez otro punto de $\mathcal{S}$ al cual llamaremos Q. Con Q como nuevo centro, (pivote), se sigue rotando la recta en el sentido de las manecillas del reloj hasta que la recta encuentra otro punto de $\mathcal{S}$. Este proceso continúa indefinidamente, siendo siempre el centro de rotación un punto de $\mathcal{S}$. Demostrar que se puede elegir un punto $P \in \mathcal{S}$ y una recta ℓ que pasa por P tales que el remolino que resulta usa cada punto de $\mathcal{S}$ como un centro de rotación un número infinito de veces.

Problema 3. Sea f una función del conjunto de los números reales en si mismo que satisface

$$f(x+y) \leq yf(x) + f(f(x))$$

para todos los números reales x, y. Demostrar que $f(x) = 0$ para toda $x \leq 0$.

Segundo Día

Problema 4. Sea Sea $n > 0$ un entero. Se dispone de una balanza de dos platillos y de n pesas cuyos pesos son 2^0, $2^1, \ldots, 2^{n-1}$. Se coloca cada una de las pesas en la balanza, de una en una, mediante una sucesión de n movimientos. En el primer movimiento se elige una pesa y se coloca en el platillo izquierdo. En cada uno de los movimientos siguientes se elige una de las pesas restantes y se coloca en el platillo de la izquierda o en el de la derecha. Determinar el número de formas de llevar a cabo estos n movimientos de manera tal que en ningún momento el platillo de la derecha tenga más peso que el platillo de la izquierda.

Problema 5. Sea f una función del conjunto de los enteros al conjunto de los enteros positivos. Se supone que para cualesquiera dos enteros m y n, la diferencia $f(m) - f(n)$ es divisible por $f(m-n)$. Demostrar que para todos los enteros m y n con $f(m) \leq f(n)$, el número $f(n)$ es divisible por $f(m)$.

Problema 6. Sea ABC un triángulo acutángulo cuya circunferencia circunscrita es Γ. Sean ℓ una recta tangente a Γ, y sean ℓ_a, ℓ_b y ℓ_c las rectas que se obtienen al reflejar ℓ con respecto a las rectas BC, CA y AB, respectivamente. Demostrar que la circunferencia circunscrita del triángulo determinado por las rectas ℓ_a, ℓ_b y ℓ_c es tangente a la circunferencia Γ.

XXIV Olimpiada Iberoamericana de Matemáticas.

Primer Día

Problema 1. En la pizarra está escrito el número 2. Ana y Bruno juegan alternadamente, comenzando por Ana. Cada uno en su turno sustituye el número escrito por el que obtiene al aplicar exactamente una de las siguientes operaciones: multiplicarlo por 2, o multiplicarlo por 3, o sumarle 1. El primero que obtenga un resultado mayor o igual que 2011 gana. Hallar cuál de los dos tiene una estrategia ganadora y describir dicha estrategia.

Problema 2. Encontrar todos los enteros positivos n para los cuales existen tres números enteros no nulos x, y, z tales que

$$x + y + z = 0 \quad y \quad \tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z} = \tfrac{1}{n}.$$

Problema 3. Sea ABC un triángulo y sean X, Y, Z los puntos de tangencia de su circunferencia inscrita con los lados BC, CA, AB, respectivamente. Suponga que C_1, C_2 y C_3 son circunferencias con cuerdas YZ, ZX, XY, respectivamente, tales que C_1 y C_2 se corten sobre la recta CZ y que C_1, C_3 se corten sobre la recta BY. Suponga que C_1 corta a las cuerdas XY y ZX en J y M respectivamente, que C_2 corta a las cuerdas YZ y XY en L e I, respectivamente, y que C_3 corta alas cuerdas YZ y ZX en K y N, respectivamente. demostrar que I, J, K, L, M y N están sobre una misma circunferencia.

Segundo Día

Problema 4. Sea ABC un triángulo acutángulo, con $AB \neq BC$, y sea O su circuncentro. Sean P y Q puntos tales que $BOAP$ y $COPQ$ son paralelogramos. Demostrar que Q es el ortocentro de ABC.

Problema 5. Sean $x_1, \ldots, x_n$ números reales positivos. Demostrar que existen $a_1, \ldots, a_n \in \{-1, 1\}$ tales que

$$a_1x_1^2 + \cdots + a_nx_n^2 \quad \geq \quad (a_1x_1 + \cdots + a_nx_n)^2.$$

Problema 6. Sean k y n números enteros positivos, con $k \geq 2$. En una línea recta se tienen kn piedras de k colores diferentes de tal forma que hay n piedras de cada color. Un *paso* consiste en intercambiar de posición dos piedras adyacentes. Encontrar el menor entero positivo m tal que siempre es posible lograr, con a lo sumo m *pasos*, que las n piedras de cada color queden seguidas si:

- n es par.
- n es impar y $k = 3$.

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Asociación Matemática Venezolana
XXV Jornadas Venezolanas de Matemáticas
Universidad de Oriente
Cumaná, 26 al 29 de marzo de 2012

Segundo Anuncio: llamado a presentación de trabajos

Las Jornadas Venezolanas de Matemáticas son organizadas anualmente por la Asociación Matemática Venezolana en colaboración con algunas universidades nacionales. Se trata de un evento de carácter científico que sirve como mecanismo de registro y divulgación de los resultados de las investigaciones en Matemática que se realizan en el país. Constituyen un foro de encuentro de matemáticos y diversos profesionales interesados en conocer, analizar y debatir los temas más actuales en investigación matemática.

La vigésima quinta edición de las Jornadas Venezolanas de Matemáticas se llevará a efecto en la Universidad de Oriente en la ciudad de Cumaná entre el 26 y 29 de Marzo de 2012. En esta ocasión las sesiones temáticas sobre las que se articula el evento, así como sus coordinadores, son:

Sesión	Coordinadores
Análisis	Ramón Bruzual (ramon.bruzual@ciens.ucv.ve) M. Domínguez(marisela.dominguez@ciens.ucv.ve)
Análisis No Lineal y Teoría de Funciones de Variación Acotada Generalizada	José Giménez (jgimenez@ula.ve) Edward Trousselot (eddycharles2007@hotmail.com)
Ecuaciones Diferenciales Parciales, Análisis de Clifford y Física Matemática	Antonio Di Teodoro (aditeodoro@usb.ve) Daniel Alayón (danieldaniel@gmail.com)
Educación Matemática	Milagros Rodriguez (melenamate@hotmail.com) Manuel Centeno (manuelcenteno11@gmail.com)
Grafos y Combinatoria	Felicia Villaroel(feliciavillarroel@gmail.com) Daniel Brito (dbrito@sucre.udo.edu.ve)
Lógica Matemática	Ramón Pino (pinoperez@gmail.com) Jesús Nieto (jnieto@usb.ve)
Probabilidad y Estadística	Ricardo Rios (rricardorios@gmail.com) Luis Rodriguez (larodriguez@uc.edu.ve)
Modelación Matemática y Análisis Numérico	Zenaida Castillo (zenaida.castillo@ciens.ucv.ve) Said Kas.Danaoushe (sak0525@gmail.com)
Sistemas Dinámicos Continuos y Discretos	Alexander Carrasco (acarrasco@ucla.edu.ve) Ramón Vivas(ramon.alberto.vivas@gmail.com)

Teoría de Números y Álgebra	Pedro Berrizbeitia (pberrizbeitia@gmail.com) Aurora Olivieri (aurora.olivieri@gmail.com)
Topología y Geometría	Ennis Rosas (ennisrafael@gmail.com) Jorge Vielma (vielma@ula.ve)

Las personas interesadas en participar como expositores en cualquiera de las sesiones temáticas deberán atender a los siguientes requerimientos:

1. La propuesta de toda comunicación oral en cualquier sesión temática tiene que ser hecha ante la coordinación de la correspondiente sesión. El resumen de la misma, de no más de una cuartilla, deber ser enviado, en archivos .tex y .pdf, a las direcciones electrónicas de los coordinadores de la sesión. La fecha límite para ello es 16 de enero de 2012.

2. La selección de las comunicaciones orales de cada sesión es competencia exclusiva de la coordinación de la misma. Los proponentes recibirán, antes del 6 de febrero de 2012, la notificación sobre la aceptación o no de su propuesta.

3. Toda comunicación oral en las sesiones tendrá una duración de 20 minutos. El número máximo de ellas es de 30; sus horarios serán establecidos por el Comité Organizador, mientras que el orden de presentación será competencia de la coordinación de cada sesión.

4. Los resúmenes deberán ser sometidos siguiendo, preferiblemente, el siguiete formato LATEX:

```
\documentclass[12pt]{amsart}
\usepackage[spanish]{babel}
\begin{document}
\title{TITULO DE LA COMUNICACION ORAL}
\author{Autor 1, \underline{Autor 2}} %% subrayado el expositor
\address{INSTITUCION del Autor 1}
\email{xxx@uni.ve} %% direcci\'on electr\'onica del Autor 1
\address{INSTITUCION del Autor 2}
\email{xxx@uni.ve} %% direcci\'on electr\'onica del Autor 2
\maketitle
\section*{Resumen}
ACA VIENE EL CONTENIDO DEL RESUMEN
\begin{thebibliography}{99}
\bibitem{zw}
Z. Zhou and J. Wu. Attractive Periodic Orbits in Nonlinear Discrete-time
Neural Networks with Delayed Feedback. \textit{J. Difference. Equ. and Appl.}
Vol. {\bf 8}, (2001) 467--483.
\end{thebibliography}
\end{document}
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