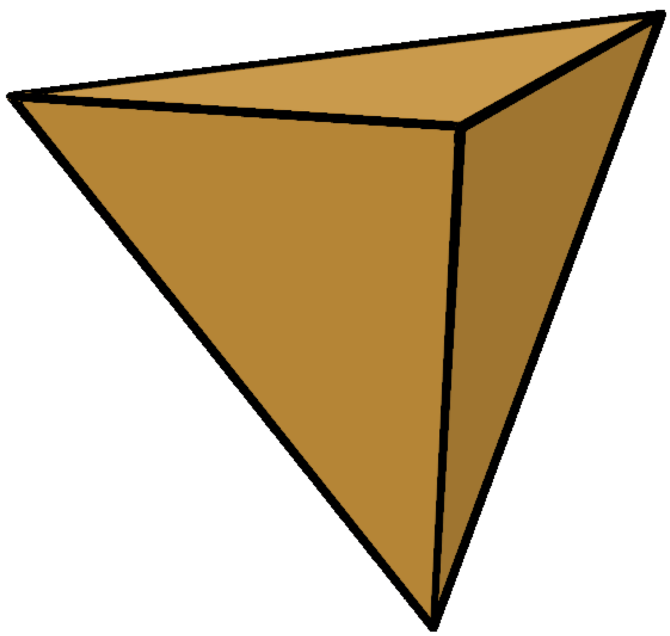

Grupo S_4, simetrías del tetraedro. (pag. 77)

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ARTÍCULOS

La familia de bases de una media continua y la representación de las medias cuasiaritméticas

Lucio R. Berrone* y Gerardo E. Sbérgamo**

Resumen. Como desarrollo de la teoría de iteraciones diádicas de una función, se presenta la noción de familia de medias base de una media continua y simétrica. Además de permitir una solución satisfactoria a la cuestión de falta de unicidad esencial en la representación de las medias cuasiaritméticas, el concepto de familia base ofrece una herramienta adecuada para utilizar la teoría de iteración en la construcción de soluciones continuas para la ecuación funcional de autoconjugación.

Abstract. As development of the theory of dyadic iterations of a function, we present the concept of base family for continuous and symmetric means. Besides allowing a satisfactory answer to the question of lack of essential uniqueness in quasi–arithmetic representation of the mean, the concept of based family provides an appropriate tool to use the theory of iteration for the construction of continuous solutions for the self–conjugate functional equation.

The theory of dyadic iterations of two-variables continuous means is revised and extended in order to introduce the concept of base family of a continuous mean. Besides other results of interest, a new analytic characterization of quasilinear means is obtained by studying the means admitting a unique base mean

1. Introducción

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Sea I un intervalo real. Se dice que una función $M : I \times I \to I$ que satisface la desigualdad

$$x < M(x,y) < y, \quad x, y \in I, \quad x < y, \tag{1}$$

es una función *interna* en I. Si además M es continua en $I \times I$, entonces M es una *media continua en* I. Sigue de (1) que las medias continuas son funciones *reflexivas*; es decir,

$$M(x,x) = x, \ x \in I. \tag{2}$$

Una media continua se dice *simétrica* cuando satisface la igualdad

$$M(x,y) = M(y,x), \ x, y \in I. \tag{3}$$

Dada una media continua M y x_0, $y_0 \in I$, las proyecciones M_{x_0}, $M_{y_0} : I \to I$ respectivamente definidas por $M_{x_0}(s) = M(x_0, s)$ y $M_{y_0}(s) = M(s, y_0)$ no son necesariamente funciones crecientes, un hecho ejemplificado por la media antiarmónica

$$M(x,y) = \frac{x^2 + y^2}{x + y}, x, y > 0.$$

Una media M se dice *reducible a derecha* cuando, para todo $y_0 \in I$,

$$M(s, y_0) = M(t, y_0) \Rightarrow s = t;$$

es decir, cuando M_{y_0} es una función inyectiva cualquiera sea y_0. Análogamente, se dice que M es *reducible a izquierda* si, para cada $x_0 \in I$, M_{x_0} es una función inyectiva. Cuando M es reducible a derecha e izquierda simultaneamente se dice que M es *reducible a ambos lados*. Una media continua es reducible a ambos lados si y sólo si sus proyecciones son funciones estrictamente crecientes. Una prueba de este hecho puede verse en [2].

Una clase importante de medias continuas y simétricas son las medias de la forma

$$\mu_\phi(x,y) = \phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right), \ x, y \in I, \tag{4}$$

donde $\phi : I \to \mathbb{R}$, es una función continua y estrictamente monótona llamada *función generadora* (de la media μ_ϕ). Estas medias se denominan *medias cuasiaritméticas* pues resultan de introducir el cambio de escala ϕ en la media aritmética $A(x,y) = (x+y)/2$. La media geométrica G, la media armónica H, la cuadrática Q y muchas otras medias elementales admiten esta representación (see, for example, [5]) que es, sin embargo, no única. En efecto, es bién conocido (véase, por ejemplo, [5] o [3], pág. 246) el siguiente:

Teorema 1 *Sean μ_ϕ y μ_ψ dos medias cuasiaritméticas respectivamente generadas por $\phi : I \to \mathbb{R}$ y $\psi : I \to \mathbb{R}$; entonces $\mu_\phi = \mu_\psi$ si y sólo si existen α, $\beta \in \mathbb{R}$, $\alpha \neq 0$, tales que $\psi = \alpha\phi + \beta$.*

En realidad, la falta de unicidad en la representación (4) es más profunda de lo que muestra el teorema anterior. Por ejemplo, μ_ϕ puede expresarse como

$$\mu_\phi(x,y) = \psi^{-1}\left(\sqrt{\psi\left(x\right)\psi\left(y\right)}\right),\ x,y \in I, \tag{5}$$

donde $\psi\left(x\right) = e^{\phi(x)}$; y es claro que, introduciendo un apropiado cambio de escala, μ_ϕ admite parecida representación en términos de cualquier otra media cuasiaritmética definida en I. Desde luego, es la función monótona y continua ψ (y no ϕ) la función generadora de la media μ_ϕ en la nueva representación. Es así que la definición de una media cuasiaritmética (4) en términos de la media aritmética A, tal como viene presentándose sistemáticamente en la literatura específica, aparece como inesencial. Deben incluirse entre dicha literatura algunos resultados ya clásicos sobre soluciones continuas de ciertas ecuaciones funcionales de tipo compuesto tales como las ecuaciones de bisimetría o autodistributividad (cfr. [1], [2], [3], [8]; para una exposición abreviada, véase [7]).

En principio, la preferencia por la representación (4) pudiera justificarse mediante razones o bién de naturaleza histórica o bién de economía o simplicidad. Las primeras no resultan convincentes puesto que, aún cuando sea cierto que el estudio de la media aritmética se remonta a la antigüedad, lo mismo ocurre, v.g. con el de la media geométrica $G(x,y) = \sqrt{xy}$, de modo que, sobre un sustento igualmente sólido, es aceptable llamar "cuasigeométricas" a las medias representadas por (5). En lugar de ello, en los párrafos siguientes nos referiremos a (5) como *representación geométrica* de una media cuasiaritmética. Notemos de paso que un resultado correspondiente al Teorema 1 puede probarse para la representación de un media cuasiaritmética en términos de una media cuasiaritmética prefijada. Por ejemplo, para la representación geométrica (5), se cumple el siguiente:

Teorema 2 *Sean μ y ν dos medias cuasiaritméticas dadas por su representación geométrica. Si $\phi : I \to \mathbb{R}^+$ y $\psi : I \to \mathbb{R}^+$ son sus correspondientes funciones generadoras; entonces, $\mu_\phi = \mu_\psi$ si y sólo si existen α, $\beta \in \mathbb{R}$, $\alpha \neq 0$, tales que $\ln\psi = \alpha \ln\phi + \beta$.*

Por otra parte, las razones de economía o simplicidad de la representación (4) son tan insignificantes como la afirmación "A es más simple que G" si es que no se proporciona un criterio de simplicidad o economía concreto. Uno de los resultados principales de este trabajo es la construcción de un tal criterio.

El mencionado criterio se apoya en la teoría de iteraciones diádicas de funciones $F : I \times I \to I$ tal como ha sido presentada en [6] y es recordada en la Sección 2. Mediante el uso de iteraciones reales de una media continua, en la Sección 3 se define el concepto de familia de medias base de una media continua y simétrica. Las medias cuasiaritméticas admiten una familia de medias base integrada por una única media: la media aritmética. Este hecho excepcional no sólo proporciona el criterio de economía buscado, sino que motiva el plantéo del problema de determinar todas las medias que admiten una única media base. Llamaremos *ecuación de autoconjugación* a la ecuación funcional

$$h\left(M\left(u,v\right)\right)=M\left(h\left(u\right),h\left(v\right)\right),\ \ u,v\in K, \tag{6}$$

donde M es una media continua definida en I y K es un subintervalo compacto de I. Las medias que admiten una única base puede caracterizarse (Teorema 9) como aquellas medias M para las que existen soluciones continuas h de la ecuación (6) que satisfacen ciertas condiciones en los extremos de K.

Por último, la sección final reúne observaciones y comentarios que complementan el contenido de las anteriores.

2. Iteraciones diádicas de medias continuas

En lo que sigue, $D\left([0,1]\right)$ denotará al conjunto de *números diádicos* del intervalo $[0,1]$. Más generalmente, el subconjunto $D\left([a,b]\right)$ del intervalo $[a,b]$ definido por

$$D\left([a,b]\right)=\{x\in[a,b]:x=(1-d)a+db,\ d\in D\left([0,1]\right)\}$$

será denominado conjunto de *particiones diádicas* del intervalo $[a,b]$.

Ahora, si F es una función definida de $I\times I$ en I, al igual que en [6] podemos asociar a cada par de puntos $x,\ y\in I$ una familia $\{F^{d}(x,y):d\in D\left([0,1]\right)\}$ de *iteraciones diádicas* de $F(x,y)$ de la siguiente manera. En primer lugar definimos

$$F^{0}\left(x,y\right)=x,\ \ F^{1}\left(x,y\right)=y.$$

Ahora, si asumimos que para $n\in\mathbb{N}$ y $0\leq j\leq 2^{n}$ conocemos $F^{\frac{j}{2^{n}}}\left(x,y\right)$; entonces definimos

$$F^{\frac{k}{2^{n+1}}}\left(x,y\right)=F^{\frac{h}{2^{n}}}\left(x,y\right)$$

cuando $k=2h,\ 0\leq h\leq 2^{n}$; y

$$F^{\frac{k}{2^{n+1}}}\left(x,y\right)=F\left(F^{\frac{h}{2^{n}}}\left(x,y\right),F^{\frac{h+1}{2^{n}}}\left(x,y\right)\right)$$

cuando $k=2h+1,\ 0\leq h\leq 2^{n}-1$.

La familia de iteraciones diádicas de una función F en $[x,y]$ será denotada por $D(F;[x,y])$

Un ejemplo simple de iteraciones diádicas de una función es el de las iteraciones diádicas de la media aritmética A. Un argumento inductivo muestra que si $x,y\in I,\ A^{d}\left(x,y\right)=(1-d)x+dy$, y por lo tanto el conjunto

$D(A;x,y) = D([x,y])$ es denso en $[x,y]$. Más generalmente, las iteraciones diádicas de la media cuasiaritmética μ_ϕ son de la forma

$$\mu_\phi^d(x,y) = \phi^{-1}\left((1-d)\,\phi(x) + d\phi(y)\right),\ x,y \in I,\ d \in D\left([0,1]\right). \tag{7}$$

Una propiedad destacable del conjunto de iteraciones diádicas de una media cuasiaritmética es su cerradez bajo μ_ϕ; concretamente, si $d_1,\ d_2 \in D([0,1])$, se cumple la igualdad

$$\mu_\phi(\mu_\phi^{d_1}(x,y), \mu_\phi^{d_2}(x,y)) = \mu_\phi^{\frac{d_1+d_2}{2}}(x,y). \tag{8}$$

En general, cuando la función F es una media continua, el conjunto de iteraciones diádicas de F tiene la siguiente propiedad.

Teorema 3 *Si $F : I\times I \to I$ es una función interna; entonces, si $x,\ y \in I,\ x < y$, la desigualdad*

$$F^{d_1}(x,y) < F^{d_2}(x,y) \tag{9}$$

se cumple para cada par $d_1, d_2 \in D([0,1])$ tal que $d_1 < d_2$. Más aún, el conjunto $D(F;[x,y])$ es denso en $[x,y]$ cuando F es continua.

Demostración. Ver [6]. ■

Sea M una función interna definida en I. Para cada $\delta \in [0,1]$, consideremos una sucesión $\{d_n\} \subseteq D([0,1])$, tal que $d_n \nearrow \delta$ cuando $n \to \infty$. Si $x,y \in I$, con $x < y$, por el Teorema 3, la sucesión $\left\{M^{d_n}(x,y)\right\}$ es creciente y acotada superiormente por y; luego podemos definir

$$M^\delta(x,y) = \lim_{d_n \nearrow \delta} M^{d_n}(x,y). \tag{10}$$

Observemos que el límite en la definición anterior es independiente de la sucesión $\{d_n\}$ y en consecuencia, la función $\delta \mapsto M^\delta(x,y)$ está bien definida y es estrictamente creciente. En efecto, si $\delta_1 < \delta_2$, existen $d,\ d' \in D([0,1])$, tales que $\delta_1 < d < d' < \delta_2$. Luego, si $\left\{d_n^1\right\}$ y $\left\{d_n^2\right\}$ son dos sucesiones crecientes de números diádicos tales que $d_n^{(i)} \nearrow \delta_i,\ i = 1,2$, las desigualdades $d_n^{(1)} \leq \delta_1 < d < d^{'} < d_m^{(2)} \leq \delta_2$ valen para $n \in \mathbb{N}$ y $m \geq m_0 \in \mathbb{N}$, de modo que por el Teorema 3, se cumple

$$M^{d_n^{(1)}}(x,y) < M^d(x,y) < M^{d'}(x,y) < M^{d_m^{(2)}}(x,y),\ n,m \in \mathbb{N},\ m \geq m_0.$$

Tomando límites cuando $n,\ m \to \infty$ en esta desigualdad, obtenemos

$$M^{\delta_1}(x,y) \leq M^d(x,y) < M^{d'}(x,y) \leq M^{\delta_2}(x,y).$$

Cuando $M : I \times I \to I$ es una media continua, la función $\delta \mapsto M^\delta(x,y)$ es también continua dado que, por el Teorema 3, el conjunto imagen de la misma contiene un subconjunto denso del intervalo $[x,y]$ y por lo tanto la función $\delta \mapsto M^{\delta}(x,y)$ no posee discontinuidades de salto. Sumado a la monotonía, este hecho prueba que la función $\delta \mapsto M^{\delta}(x,y)$ es continua.

Supondremos en lo que sigue que M es una media continua y simétrica. Para cada par de puntos $x,\ y \in I$ con $x < y$, asociamos a M y al subintervalo $J = [x,y] \subseteq I$, el homeomorfismo $f_J : [0,1] \to J$ definido por $f_J(u) = M^u(x,y)$.

Los homeomorfismos f_J están determinados por los valores de M en J^2. Para ver esto observemos que $f_J(0) = x$ y $f_J(1) = y$. Ahora, si asumimos conocido $f_J(h/2^n)$, con $0 \leq h \leq 2^n$, tenemos que, cuando $0 \leq h \leq 2^n - 1$,

$$\begin{aligned} f_J\left(\frac{2h+1}{2^{n+1}}\right) &= M^{\frac{2h+1}{2^{n+1}}}(x,y) \\ &= M\left(M^{\frac{h}{2^n}}(x,y), M^{\frac{h+1}{2^n}}(x,y)\right) \\ &= M\left(f_J\left(\frac{h}{2^n}\right), f_J\left(\frac{h+1}{2^n}\right)\right). \end{aligned}$$

Como $f_J(h/2^n),\ f_J((h+1)/2^n) \in J$, sigue de la identidad anterior que, para todo $d \in D([0,1])$, $f_J(d)$ depende sólo de los valores de de M en J^2.

Observemos que de la discusión previa y de (8) sigue que para una media cuasiaritmética μ_ϕ, la identidad

$$\mu_\phi(\mu_\phi^u(x,y), \mu_\phi^v(x,y)) = \mu_\phi^{\frac{u+v}{2}}(x,y),\ x,y \in I, \tag{11}$$

es valida para $u,v \in [0,1]$. Más aún, las medias cuasiaritméticas son las únicas medias continuas con esta propiedad. Para ver esto, supongamos primero que M es una media continua y simétrica en un intervalo compacto $I = [a,b]$. Entonces, si para todo $u,v \in I$, vale

$$M(M^u(x,y), M^v(x,y)) = M^{\frac{u+v}{2}}(x,y),\ x,y \in I\,; \tag{12}$$

haciendo $x = a$, $y = b$, obtenemos

$$M(f_I(u), f_I(w)) = f_I\left(\frac{u+w}{2}\right),\ u,w \in [0,1],$$

o, equivalentemente

$$M(x,y) = \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right),$$

donde hemos puesto $\phi = f_I^{-1}$. Cuando I no es compacto, podemos considerar una familia de intervalos compactos $\{K_n\}_{n\in\mathbb{N}}$ tal que $I = \underset{n\in\mathbb{N}}{\cup} K_n$ y $K_n \subset K_{n+1}$; de modo que, para cada $n \in \mathbb{N}$, tengamos

$$M(x,y) = \phi_n^{-1}\left(\frac{\phi_n(x)+\phi_n(y)}{2}\right), \quad x,y \in K_n,$$

donde ϕ_n es una función creciente definida en K_n. Ahora bién, es posible definir una familia de funciones crecientes $\varphi_n : K_n \to \mathbb{R}$ tales que φ_n genere a M en K_n y $\varphi_{n+1}|_{K_n} = \varphi_n$. Para esto definamos primero $\varphi_1(x) = \phi_1(x)$, $x \in K_1$. Ahora, si suponemos definida φ_n en K_n, tenemos entonces que

$$\varphi_n^{-1}\left(\frac{\varphi_n(x) + \varphi_n(y)}{2}\right) = \phi_{n+1}^{-1}\left(\frac{\phi_{n+1}(x) + \phi_{n+1}(y)}{2}\right), \quad x, y \in K_n$$

y consecuentemente existen, por el Teorema 1, constantes α_n y β_n, $\alpha_n \neq 0$, tales que

$$\phi_{n+1}(x) = \alpha_n \varphi_n(x) + \beta_n, \ x \in K_n.$$

Definiendo entonces

$$\varphi_{n+1}(x) = \frac{\phi_{n+1}(x) - \beta_n}{\alpha_n}, \ x \in K_{n+1},$$

resulta que la familia de funciones $\{\varphi_n\}_{n\in\mathbb{N}}$ satisface las condiciones deseadas. Finalmente, la función $\phi : I \to \mathbb{R}$ dada por

$$\phi(x) = \varphi_n(x), \ x \in K_n,$$

está bien definida y se cumple la igualdad

$$M(x, y) = \phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right), \ x, y \in I.$$

3. Bases de medias continuas y simétricas

Si M es una media continua y simétrica definida en I y $J = [x, y]$ es un subintervalo de I, la continuidad y la monotonía de $\delta \mapsto M^{\delta}(x, y)$ aseguran que, para cada $u, v \in [0, 1]$, existe un único $\mu_J(u, v) \in [0, 1]$, tal que

$$M(M^u(x, y), M^v(x, y)) = M^{\mu_J(u,v)}(x, y). \tag{13}$$

En términos de los homeomorfismos f_J definidos en la sección anterior, la ecuación (13) puede reescribirse en la forma

$$M(f_J(u), f_J(v)) = f_J(\mu_J(u, v)), \quad u, v \in [0, 1]. \tag{14}$$

Proposición 4 *Sea M una media continua y simétrica definida en I y $J \subseteq I$; entonces μ_J es también una media continua y simétrica.*

Demostración. Sean $u, v \in [0,1]$. Si $u < v$ entonces sigue de la internalidad de M y de (13) que

$$M^u(x,y) < M^{\mu_J(u,v)}(x,y) < M^v(x,y),$$

y consecuentemente

$$u < \mu_J(u,v) < v.$$

Esto prueba que μ_J es una función interna. La continuidad de μ_J se deriva de (14), pues

$$\mu_J(u,v) = f_J^{-1}(M(f_J(u), f_J(v))),$$

y f_J es un homeomorfismo. La simetría es consecuencia de (13) y de la simetría de M. ■

Cuando M es una media continua y simétrica, la familia de medias continuas y simétricas $\{\mu_J\}_{J\subseteq I}$ será denominada *familia de medias base* de M. Cuando μ_J resulte independiente de J; es decir, cuando $\mu_J = \mu$ para todo subintervalo $J \subseteq I$, diremos simplemente que M admite una *media base*.

Si M y N son medias continuas definidas en I_1y I_2 respectivamente, tales que existe un homeomorfismo $h : I_1 \to I_2$ que verifica

$$h(M(x,y)) = N(h(x), h(y)), \ x, y \in I_1,$$

entonces diremos que M y N son *medias conjugadas* y lo denotaremos mediante $M \overset{h}{\sim} N$. Es fácil ver que la conjugación define una relación de equivalencia en la familia de medias continuas.

Proposición 5 *Si M y N son medias continuas definidas en I_1 e I_2 respectivamente y h es una solución continua de la ecuación funcional*

$$h(M(x,y)) = N(h(x), h(y)), \ x, y \in I_1,$$

entonces

$$h(M^u(x,y)) = N^u(h(x), h(y)), \ x, y \in I_1, \tag{15}$$

para todo $u \in [0,1]$.

Demostración. Un argumento inductivo prueba que (15) vale cuando $u \in D([0,1])$ (ver [6]). Cuando $u \in [0,1]$, la validez de (15) sigue de la continuidad de h, M y N. ■

Las familias de medias base de dos medias conjugadas M y N están relacionadas según se establece en la siguiente proposición.

Proposición 6 *Sean M y N dos medias continuas en I tales que $M \overset{h}{\sim} N$ y $\{\mu_J\}_{J\subseteq I}$, $\{\varphi_J\}_{J\subseteq I}$ sus respectivas familias de medias base; entonces $\mu_J = \varphi_{h(J)}$.*

Demostración. Como $\{\mu_J\}_{J\subseteq I}$ es una familia de medias base para M, tenemos que

$$M(M^u(x,y), M^v(x,y)) = M^{\mu_J(u,v)}(x,y).$$

Por la Proposición 6, la aplicación de h al primer miembro de esta igualdad proporciona

$$\begin{aligned} h\left(M(M^u(x,y), M^v(x,y))\right) &= N(N^u(h\left(x\right), h\left(y\right)), N^v(h\left(x\right), h\left(y\right)) \\ &= N^{\varphi_{h(J)}(u,v)}(h\left(x\right), h\left(y\right)); \end{aligned} \tag{16}$$

mientras que aplicando h al segundo miembro se obtiene

$$h\left(M^{\mu_J(u,v)}(x,y)\right) = N^{\mu_J(u,v)}(h\left(x\right), h\left(y\right)). \tag{17}$$

De (16) y (17) se concluye que

$$N^{\varphi_{h(J)}(u,v)}(h\left(x\right), h\left(y\right)) = N^{\mu_J(u,v)}(h\left(x\right), h\left(y\right)),$$

y por lo tanto, $\varphi_{h(J)}\left(u,v\right) = \mu_J\left(u,v\right)$ para cada par $u,v \in [0,1]$. ■

Una consecuencia importante de la proposición anterior es la siguiente: si una media continua M admite una media base μ, entonces μ es también una base para cualquier media conjugada de M. Por otra parte, cuando una media base μ es admitida por M, μ es también una base de $M|_{K\times K}$ para cualquier intervalo compacto $K \subseteq I$. Luego, sigue de (14) que

$$M\left(f_K\left(u\right), f_K\left(v\right)\right) = f_K\left(\mu\left(u,v\right)\right),\ u,v \in [0,1];$$

es decir $\mu \overset{f_K}{\sim} M|_{K\times K}$ y, consecuentemente, toda media base es una media base de sí misma. Este hecho puede expresarse en términos de las funciones f_J asociadas a μ, de la siguiente manera

$$\mu\left(f_J\left(u\right), f_J\left(v\right)\right) = f_J\left(\mu\left(u,v\right)\right),\ u,v \in [0,1]. \tag{18}$$

Observemos que de (12) sigue que toda media cuasiaritmética admite una media base: la media aritmética A. Otra propiedad que distingue a la media aritmética en la la familia de medias cuasiaritméticas es que el homeomorfismo $f\left(u\right) = A^u\left(0,1\right)$ asociado a la media A en el intervalo $[0,1]$ es la función identidad. Una media continua definida en $[0,1]$ con esta propiedad será denominada *media normal.* La media aritmética es la única media normal en la clase de medias cuasiaritméticas. En efecto, si una media cuasiaritmética μ_ϕ definida en el intervalo $[0,1]$ tienen la propiedad de normalidad, entonces tenemos por (7) que

$$u = \mu_\phi^u\left(0,1\right) = \phi^{-1}\left(\left(1-u\right)\phi(0) + u\phi(1)\right),\ \ u \in [0,1],$$

es decir,

$$\phi\left(u\right) = \phi(0) + \left(\phi(1) - \phi(0)\right)u,\ \ u \in [0,1],$$

y como consecuencia del Teorema 1, μ_ϕ es la media aritmética.

Proposición 7 *Sea M una media continua y simétrica definida en $[a, b]$. Si $\Lambda = \{N : N \sim M\}$ es la familia constituida por todas las medias continuas y simétricas conjugadas con M; entonces,*
i)Λ admite un único representante normal;
ii)si M admite una base μ, entonces μ es el representante normal de Λ.

Demostración. Para demostrar $i)$, sea M una media definida en $I = [a, b]$. Consideremos la función m definida en $[0, 1]$ por

$$m(x, y) = f^{-1}(M(f(x), f(y))), \ x, y \in [0, 1],$$

donde $f = f_I$, es el homeomorfismo asociado a M en el intervalo I. Obviamente, m es una media continua y $m \overset{f}{\sim} M$, por lo que $m \in \Lambda$. Ahora bién, si g es el homeomorfismo asociado a m en el intervalo $[0, 1]$, tenemos, por la Proposición 5, que

$$g(u) = m^u(0, 1) = f^{-1}(M^u(a, b)) = f^{-1}(f(u)) = u, \ u \in [0, 1].$$

Esto prueba que m es una media normal. Si suponemos que l es otra media normal en Λ entonces existe un homeomorfismo $h : [0, 1] \to [0, 1]$, tal que

$$h(l(x, y)) = m(h(x), h(y)), \ x, y \in [0, 1].$$

Tomando $x = 0$, $y = 1$ en la identidad anterior y haciendo uso nuevamente de la Proposición 5 podemos concluir que, para todo $u \in [0, 1]$,

$$h(u) = h(l^u(0, 1)) = m^u(h(0), h(1)) = u,$$

y por tanto, que $l = m$.
A fin de probar $ii)$, supongamos que M admite una base μ y denotemos con f al homeomorfismo asociado a μ en el intervalo $[0, 1]$. Sabemos por (18) que

$$f(\mu(u, w)) = \mu(f(u), f(w)), \quad u, w \in [0, 1].$$

Veamos que f es la función identidad. Para esto observemos primero que $f(0) = \mu^0(0, 1) = 0$ y $f(1) = \mu^1(0, 1) = 1$. Si suponemos, por el absurdo, que existe $z \in (0, 1)$ tal que $f(z) \neq z$, por la continuidad de f existiría un intervalo maximal (a, b) tal que $z \in (a, b)$ y $f(x) \neq x$, $x \in (a, b)$. La maximalidad de dicho intervalo asegura que $f(a) = a$ y $f(b) = b$, por lo que, haciendo $u = a$ y $w = b$ en la ecuación anterior, se obtiene

$$\begin{aligned} f(\mu(a, b)) &= \mu(f(a), f(b)) \\ &= \mu(a, b). \end{aligned}$$

Esto es una contradicción, ya que $\mu(a, b) \in (a, b)$. Luego f es la función identidad y μ es normal. ■

No toda media continua admite una base. A continuación exponemos un ejemplo que muestra este hecho. Sea μ_f la media cuasiaritmética generada por la función

$$f(x) = \frac{1-\cos(\pi x)}{2}, \; x \in [0,1].$$

Puesto que $\mu_f(x, 1-x) = A(x, 1-x)$, la función definida por

$$M(x,y) = \begin{cases} \mu_f(x,y) & si\ 0 \le x \le 1, 1-x \le y \le 1 \\ A(x,y) & si\ 0 \le x \le 1, 0 \le y \le 1-x \end{cases}, \tag{19}$$

resulta una media continua y simétrica. Así definida, M no es una media cuasiaritmética ya que si lo fuera y estuviera generada por alguna función $\phi : [0,1] \to \mathbb{R}$, tendríamos que ϕ es una solución monótona y continua de la ecuación funcional

$$\phi\left(\frac{x+y}{2}\right) = \frac{\phi(x)+\phi(y)}{2}, \; (x,y) \in \Delta, \tag{20}$$

que es la bién conocida ecuación funcional de Jensen sobre el dominio

$$\Delta = \{(x,y) \in [0,1] \times [0,1] : \; 0 \le x \le 1, 0 \le y \le 1-x\}. \tag{21}$$

Veamos que, al igual que sucede con la ecuación de Jensen planteada en un intervalo, las soluciones continuas ϕ son también de la forma $\phi(x) = ax + b$ con a, $b \in \mathbb{R}$. Para esto observemos que si ϕ es una solución de (20) en el dominio (21), entonces también es una solución en $[0,1/2] \times [0,1/2]$ y por lo tanto, bajo la hipótesis de continuidad tenemos que $\phi(x) = ax + b$ cuando $x \in [0,1/2]$ ([2], pgs. 44 y 45). Ahora, si $(x,y) \in \Delta$ con $1/2 < x \le 1$, tenemos que y, $(x+y)/2 \in [0,1/2]$ y sigue de (20) que

$$\begin{aligned} \phi(x) &= 2\phi\left(\frac{x+y}{2}\right) - \phi(y) \\ &= 2\left(a\frac{x+y}{2} + b\right) - ay - b \\ &= ax + b. \end{aligned}$$

Esto prueba que $\phi(x) = ax + b$ para todo $x \in [0,1]$ y en consecuencia $M = A$ en $[0,1]^2$, en contradición con (19).

Ahora bien, si suponemos que M admite una base μ, considerando el intervalo $J = [0,1/2]$, tenemos por (14) que, para todo $x, y \in [0,1]$,

$$\mu(x,y) = f_J^{-1}\left(M\left(f_J(x), f_J(y)\right)\right) = A(x,y),$$

esto contradice el hecho de que M no es cuasiaritmética.

Observemos que una media M definida por (19) para una función generadora arbitraria $f \in C^1([0,1])$ no posee derivadas parciales continuas, exceptuando el caso en que f es una función afín (y M es, en consecuencia, la media aritmética). En efecto, si suponemos que M es de clase C^1, fijado un punto $(x_0, 1-x_0)$ sobre la diagonal del cuadrado unitario, tenemos que

$$\frac{1}{2} = \lim_{y \nearrow 1-x_0} \frac{\partial M}{\partial y}(x_0, y) = \lim_{y \searrow 1-x_0} \frac{\partial M}{\partial y}(x_0, y) = \lim_{y \searrow 1-x_0} \frac{\partial \mu_f}{\partial y}(x_0, y).$$

Por otra parte, derivando con respecto a y ambos miembros de la igualdad

$$f(\mu_f(x_0, y)) = \frac{f(x_0) + f(y)}{2},$$

obtenemos

$$f'(\mu_f(x_0, y)) \frac{\partial \mu_f}{\partial y}(x_0, y) = \frac{f'(y)}{2},$$

y por tanto

$$\lim_{y \searrow 1-x_0} f'(\mu_f(x_0, y)) \frac{\partial \mu_f}{\partial y}(x_0, y) = \lim_{y \searrow 1-x_0} \frac{f'(y)}{2}.$$

Como f' es una función continua y $\mu_f(x_0, 1-x_0) = 1/2$, se concluye que

$$f'(1-x_0) = f'\left(\frac{1}{2}\right),$$

y consecuentemente $f(x) = ax + b$.

En lo que resta de la sección, mostraremos que el hecho de que una media continua M definida en I admita una base está vinculado a la existencia de soluciones de la ecuación funcional

$$h(M(x,y)) = M(h(x), h(y)), \; x, y \in I, \tag{22}$$

que satisfacen determinadas condiciones de contorno. Observemos primero que por (14), M admite una media base μ si y sólo si dados dos subintervalos arbitrarios de I, J_1 y J_2

$$f_{J_1}^{-1}(M(f_{J_1}(u), f_{J_1}(v))) = f_{J_2}^{-1}(M(f_{J_2}(u), f_{J_2}(v))), \; u, v \in [0,1], \tag{23}$$

donde las funciones f_{J_1} y f_{J_2} son los homeomorfismos asociados a M en los intervalos J_1 y J_2 respectivamente. Ahora bien, haciendo la sustitución $z = f_{J_1}(u)$ y $w = f_{J_1}(v)$, podemos rescribir (23) como

$$f_{J_2} \circ f_{J_1}^{-1}(M(z, w)) = M\left(f_{J_2} \circ f_{J_1}^{-1}(z), f_{J_2} \circ f_{J_1}^{-1}(w)\right), \; z, w \in J_1, \tag{24}$$

es decir, M admite una base si y sólo si las composiciones $f_{J_2} \circ f_{J_1}^{-1}$ son soluciones de la ecuación funcional

$$h\left(M(z,w)\right) = M(h\left(z\right), h\left(w\right)),\ z,w \in J_1.$$

En particular cuando M está definida en $[0,1]$ y es normal, la condición (24) es equivalente a que los homemorfismos f_J asociados a M en un subintervalo J del intervalo $[0,1]$, sean soluciones de (22).

Hecha esta observación podemos enunciar la siguiente:

Proposición 8 *Una media continua y normal μ es base de alguna clase de medias continuas si y sólo si para todo α, $\beta \in [0,1]$ con $\alpha < \beta$, la ecuación de autoconjugación*

$$h\left(\mu\left(u,v\right)\right) = \mu\left(h\left(u\right), h\left(v\right)\right),\ u,v \in [0,1],$$

admite una solución continua que verifica $h\left(0\right) = \alpha$ y $h\left(1\right) = \beta$.

Demostración. Supongamos primero que μ es base de su clase conjugada, entonces para cada subintervalo $J \subseteq [0,1]$ tenemos que

$$f_J\left(\mu\left(u,v\right)\right) = \mu\left(f_J\left(u\right), f_J\left(v\right)\right),\ u,v \in [0,1]. \tag{25}$$

Considerando $J = [\alpha, \beta]$ tenemos que $f_J\left(u\right) = \mu^u\left(\alpha,\beta\right)$ es una solución de (25) tal que $f_J\left(0\right) = \alpha$ y $f_J\left(1\right) = \beta$. Recíprocamente, si asumimos que para cada par de puntos $\alpha, \beta \in [0,1]$, con $\alpha < \beta$, la ecuación (25) admite una solución h tal que $h\left(0\right) = \alpha$ y $h\left(1\right) = \beta$, entonces, por la normalidad de μ, tenemos que

$$h\left(u\right) = h\left(\mu^u\left(0,1\right)\right) = \mu^u\left(h\left(0\right), h\left(1\right)\right) = \mu^u\left(\alpha,\beta\right) = f_J\left(u\right).$$

Esto prueba que que $h = f_J$ y por lo tanto μ es base de su clase conjugada. ■

Observemos, para finalizar, que dados una media continua M definida en I y K un subintervalo compacto de I, entonces g es una solución de la ecuación funcional

$$h\left(M\left(u,v\right)\right) = M\left(h\left(u\right), h\left(v\right)\right),\ u,v \in K,$$

si y sólo si $f_K^{-1} \circ g \circ f_K$ es solución de la ecuación funcional

$$h\left(\mu\left(u,v\right)\right) = \mu\left(h\left(u\right), h\left(v\right)\right),\ u,v \in [0,1],$$

donde μ es la conjugada normal de M y f_K es el homeomorfismo asociado a M en el intervalo K. Como f_K es una biyección de $[0,1]$ en K, podemos establecer el siguiente:

Teorema 9 *Una media continua y simétrica M definida en un intervalo I admite una base si y sólo si para todo subintervalo compacto $K = [a, b]$ de I y para todo α, $\beta \in K$ con $\alpha < \beta$, la ecuación funcional (6) admite una solución continua que verifica $h(a) = \alpha$ y $h(b) = \beta$.*

Demostración. Si M admite una base μ y $K = [a, b]$ es un subintervalo compacto de I, entonces μ es una base de $M|_{K\times K}$ y de la discusión anterior y la Proposición 8 sigue que (6) tiene una solución continua que verifica $h(a) = \alpha$ y $h(b) = \beta$ cualesquiera sean α y β en K, con $\alpha < \beta$. Recíprocamente, si para todo intervalo compacto $K = [a, b]$ contenido en I y para todo $\alpha, \beta \in K$, con $\alpha < \beta$, existe una solución continua de (6) que verifica $h(a) = \alpha$ y $h(b) = \beta$, entonces, por la Proposición 8, $M|_{K\times K}$ admite una base μ. Además, si $K \subseteq K'$ y ψ es una base de $M|_{K'\times K}$, entonces ψ es también una base de $M|_{K\times K}$ y por lo tanto $\mu = \psi$. Esto prueba que μ es una base de M. ■

Observemos que en el caso en que M es una media cuasiaritmética definida en un intervalo I, con función generadora ϕ, y $[a, b]$ es un subintervalo cerrado de I, entonces la función h, definida por

$$h(x) = \phi^{-1}\left(\frac{\phi(\beta) - \phi(\alpha)}{\phi(b) - \phi(a)}(\phi(x) - \phi(a)) + \phi(\alpha)\right), \ x \in [a, b],$$

es una solución continua de la ecuación de autoconjugación (6) que verifica $h(a) = \alpha$ y $h(b) = \beta$ cualesquiera sean α, $\beta \in I$.

4. Observaciones finales

El concepto de familia de bases introducido en la Sección 3 puede extenderse a medias continuas no simétricas. En [4] pueden encontrarse los desarrollos correspondientes a este caso general. Resultados como el Teorema 9 se extienden a las medias no simétricas sin alterar su enunciado. Las medias definidas por

$$M(x, y) = \phi^{-1}(\alpha\phi(x) + (1 - \alpha)\phi(y)), \quad x, y \in I, \tag{26}$$

donde $\alpha \in (0, 1)$ y $\phi : I \to \mathbb{R}$, es una función continua y estrictamente monótona, son denominadas *medias cuasilineales con peso α (y función generadora ϕ).* Cuando $\alpha = 1/2$ en (26), la media M se reduce a la media cuasiaritmética generada por ϕ. Tal como se muestra en [4], para cada α fijo, una media base es admitida por las medias cuasilineales (26). La recíproca es también cierta asumiendo que la media M es diferenciable, de modo que puede establecerse el siguiente:

Teorema 10 *Sea M una media diferenciable y reducible a ambos lados. Una media base es admitida por M si y sólo si M es una media cuasilineal.*

El Teorema 10, para cuya prueba remitimos a [4], proporciona una nueva caracterización análitica de las medias cuasilineales. Otra de sus consecuencias es el siguiente resultado sobre la solución de la ecuación de autoconjugación.

Teorema 11 *Sea M es una media diferenciable, simétrica y reducible a ambos lados definida en I. La ecuación de autoconjugación (6) admite una solución que pasa por dos puntos arbitrariamente elegidos de I^2, si y sólo si M es una media cuasiaritmética.*

La prueba de este resultado se obtiene de la directa aplicación de los teoremas 9 y 10.

Si la existencia de una media base es o no suficiente para garantizar la cuasilinealidad de una media continua pero no diferenciable constituye un interesante problema abierto.

Referencias

[1] J. Aczél, On mean values, Bull. Amer. Math. Soc. **54**, (1948), 392-400.

[2] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, (1966).

[3] J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.

[4] L. R. Berrone, *A dynamical characterization of quasilinear means*, (to appear).

[5] P. S. Bullen, D. S. Mitrinović, P. M. Vasić, *Means and Their Inequalities*, D. Reidel Publishing Company, Dordrecht, 1988.

[6] L. R. Berrone, A. L. Lombardi, *A note on equivalence of means*, Publ. Math. Debrecen **58**, Fasc. **1-2**, (2001), 49-56.

[7] J. Fodor, M. Roubens, *On meaningfulness of means*, J. Comp. Appl. Math. **64**, (1995), 103-115.

[8] C. Ryll-Nardzewski, *Sur les moyennes*, Studia Math. **11**, (1949), 31-37.

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Numerical solutions for a core–annular film fluid within a circular tube

Rodolfo Gallo and Said Kas–Danouche

Abstract. The nonlinear partial differential equation that models the evolution of the interface between two core–annular fluids within a circular cylinder of radius a and length L, is numerically solved using two finite difference schemes, one implicit and the other one explicit. Also, two pseudo–spectral schemes are used, one with Euler's method and the other one with Runge–Kutta's method of fourth order. The results of these methods are analized and compared considering the absolute error calculated with the infinite norm, the relative error, and the execution time. We find that the pseudo–spectral with Euler's method produces a very good numerical solution to the problem, considering the numerical solution obtained from the pseudo–spectral with Runge Kutta's method of fourth order as the most accurate numerical solution of the problem.

Resumen. La ecuación no lineal en derivadas parciales que modela la evolución de la interfaz entre dos fluidos núcleo–anulares dentro de un cilindro circular de radio a y de longitud L, es resuelto numéricamente utilizando dos esquemas en diferencias finitas, una implícita y la otra explícita. Además, se utilizan dos regímenes seudo–espectrales, uno con el método de Euler y el otro con un método de Runge–Kutta de cuarto orden. Los resultados de estos métodos se analizaron y se compararon considerando el error absoluto calculado con la norma infinito, el error relativo, y el tiempo de ejecución. Nos parece que el esquema pseudo–espectral con el método de Euler produce una muy buena solución numérica al problema, teniendo en cuenta la solución numérica obtenida de el esquema pseudo–espectral con el método de Runge Kutta de cuarto orden como la solución numérica más exacta del problema.

1 Introduction

The interface between two core–annular fluids within a circular cylinder was studied by Hammond [9]. He modeled the problem deducing the nonlinear partial differential equation $H_t = -\frac{1}{3}(H^3(H_{zzz} + H_z))_z$.

In 1934, Taylor [25] proved that the thread of unconfined fluids are decomposed into spheres. A year later, Tomotika [26] studied the case of a cylindrical thread of a viscous liquid suspended into another fluid, and Goren [7], in 1962, analyzed the linear stability of this problem. Hammond, in his work [9], extended Goren's study to a nonlinear regime. Also, he considered the dynamics of the problem, which it was not done before. Gauglitz & Radke [6] developed an alternative approximation based on Hammond's analysis including the exact expression of the curvature in the theory.

On the other hand, Chen, Bai, & Joseph [4] studied core–annular flows in vertical tubes considering the gravity, and including all the effects of viscosity stratification and interfacial tension. Renardy [21] studied core–annular flows of two fluids considering non axisymmetric instability. Kouris & Tsamopoulos [17] analized the dynamics of a flow of two phases of concentric immiscible fluids in a cylindrical tube. Later, Kas–Danouche [12] and Kas–Danouche, Papageorgiou, & Siegel [13], [15] studied the nonlinear interfacial stability of core–annular film flows with a constant pressure gradient and adding surfactants at the interface between the two fluids. In 2007, Kas–Danouche [14] considered the same problem studied by Hammond in [9], but he added insoluble surfactants at the interface between the two fluids, obtaining a new coupled system of two nonlinear partial differencial equations.

In this paper, a sketch of the derivation of the interface equation made by Hammond is presented. This equation is numerically solved using different numerical schemes: finite difference methods, explicit and implicit; also pseudo–spectral methods, one with Euler's method and another one with Runge–Kutta's method of fourth order, which is the relevant part of this work. A comparative analysis is made between the applied methods looking for the most convenient one. In the second section of this paper, the governing equations and the mathematical model developed by Hammond [9] are presented. This model consists in a nonlinear partial differential evolution equation. The numerical schemes are briefly introduced and the way how they are applied to the model is explained in the third section. In the fourth section, the numerical solutions are obtained when the methods previously proposed, are implemented. Several hundreds of numerical experiments were performed. They were mainly analyzed making use of the absolute and relative errors. Finally, we expose the conclusions. The results of this work will be used, in future researches, to validate numerical schemes that may be developed to solve the mathematical model obtained in [14].

2 Mathematical model and governing equations

We consider a film of an annular liquid surrounding a core fluid of length L, both concentric within a circular tube of radius a. The fluid in the film is of viscosity μ, while the core fluid has a viscosity $\lambda\mu$ and initial unperturbed radius b. The gravitational effects are neglected and we assume that no pressure gradients are applied to the system. So, the only force present is due to the interface tension γ which acts at the interface $S(r, z, \phi, t)$ between the two fluids.

The mathematical model that we present, in this article, was developed by Hammond in [9]. We do not try in this article to redo all the derivation of the model, since the reader interested in it can find it in [9]. However, we mention some aspects oriented to the derivation of it.

2.1 Governing equations

Let us denote the velocity and the pressure in the annular fluid (film) as $\vec{u}$ and p, respectively. In the core fluid, the velocity and the pressure will be denoted by $\overrightarrow{U}$ and P. We will use cylindrical coordinates (r, z, ϕ); the velocity components associated to these coordinates will be (u, w, v) in the film, and (U, W, V) in the core. At the interface $S(r, z, \phi, t)$, the radial variable r will take the value

$$R(z, \phi, t) = a - h(z, \phi, t), \tag{1}$$

where h is the thickness of the film.

The governing equations for this problem are the Navier–Stokes and continuity equations

$$\lambda\mu\nabla^2\overrightarrow{U} = \nabla P \tag{2}$$

$$\nabla \cdot \overrightarrow{U} = 0, \tag{3}$$

in the core fluid, where $R > r \geq 0$, and

$$\mu\nabla^2\vec{u} = \nabla p \tag{4}$$

$$\nabla \cdot \vec{u} = 0, \tag{5}$$

in the film, where $a > r > R$.

2.2 Boundary conditions

The imposed boundary conditions satisfy the physical problem to be modeled. They are: No slip condition at the pipe wall

$$\vec{u} = 0 \ \text{ in } r = a,$$

continuity of velocity at the interface

$$\vec{u} = \vec{U} \text{ in } r = R,$$

normal stress balance

$$\vec{n} \cdot \sigma \cdot \vec{n} - \vec{n} \cdot \Sigma \cdot \vec{n} = \vec{n}\gamma\kappa\vec{n},$$

tangential stress balance

$$\vec{t} \cdot \sigma \cdot \vec{n} - \vec{t} \cdot \Sigma \cdot \vec{n} = \vec{t}\gamma\kappa\vec{n},$$

where γ is the interfacial tension coefficient, $\vec{n}$ is the normal vector to the interface and pointing out the annular fluid, $\vec{t}$ is the tangential vector at the interface, σ and Σ are the stress tensors given by

$$\begin{aligned} \sigma &= -pI + \mu(\nabla\vec{u} + \nabla\vec{u}^T), \\ \Sigma &= -PI + \lambda\mu(\nabla\vec{U} + \nabla\vec{U}^T), \end{aligned}$$

with I as the identity matrix of order 3, $.^T$ denotes the transposed, and

$$\kappa = \nabla \cdot \hat{n}, \tag{6}$$

is the mean curvature of the interface, where $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$.

Also, the kinematic condition is required

$$u = -h_t - \vec{u} \cdot \nabla h \quad = \quad -h_t - wh_z - v\frac{h_\phi}{r},$$

which, rearranging terms, it takes the form,

$$h_t = -u - wh_z - v\frac{h_\phi}{a-h} \quad \text{at} \quad S(r, z, \phi, t). \tag{7}$$

This way, the evolution equation for the interface will be completely determined by (7), if the components of the velocity are known.

In what follows, all independent and dependent variables are non–dimensionalized, and asymptotic approximations are used to obtain the evolution equation of the interface. The thickness of the unperturbed film is $a - b$. We introduce the small non–dimensional parameter ϵ

$$\epsilon = \frac{a-b}{a}.$$

Now, it is assumed that $\epsilon << 1$ and $\epsilon\lambda << 1$, $h(z,t)/\epsilon a = O(1)$ and $a\partial/\partial z = O(1)$. Taking into account certain estimations which Hammond, in

[9], introduces; we define the following variables, denoting the non–dimensional variables with asterisks. In the film we take the independent variables

$$z^* = \frac{z}{a}; \quad y^* = \frac{a-r}{\epsilon a}; \quad t^* = \frac{t}{\epsilon^{-3}(a\mu/\gamma)}.$$

So, from (1) the interface can be described by

$$y^* = \frac{h(z,t)}{\epsilon a} = H^*(z^*, t^*),$$

with $H^* = O(1)$. Also, the radial and axial velocities, and the pressure are expressed in non–dimensional form.

Considering the non–dimensional form of (4)–(5) and eliminating the asterisks from the notation, we obtain

$$\begin{aligned} w_z - u_y &= -\epsilon u + O(\epsilon^2), \\ p_y &= -\epsilon^2 u_{yy} + O(\epsilon^3), \\ p_z - w_{yy} &= -\epsilon w_y + O(\epsilon^2). \end{aligned}$$

Similarly, the boundary conditions are non–dimensionalized. In this way, the non–dimensional tangential stress balance takes the following form

$$\begin{aligned} &\epsilon^2\Big((1-\epsilon^2 H_z^2)(-w_y + \epsilon^2 u_z) + 2\epsilon^2 H_z(u_y - w_z)\Big) \\ = \;& \lambda\epsilon^3\Big((1-\epsilon^2 H_z^2)(U_z + W_r) + 2\epsilon H_z(W_z - U_r)\Big), \text{ in } S(r,z,t); \end{aligned}$$

from which we obtain

$$w_y(H,z) = -\epsilon\lambda(U_z + W_r) + O(\epsilon^2, \epsilon^2\lambda).$$

Now, we non–dimensionalize the normal stress balance, obtaining

$$\begin{aligned} &\mid \vec{n} \mid^2 - \epsilon \mid \vec{n} \mid^2 p + 2\Big(-\epsilon^3 u_y + \epsilon^5 H_z u_z - \epsilon^3 H_z w_y + \epsilon^5 H_z^2 w_z\Big) \\ - \;& \lambda\epsilon^3\Big(-\mid \vec{n} \mid^2 P + 2U_r + 2\epsilon H_z U_z + 2\epsilon H_z W_r + 2\epsilon^2 H_z^2 W_z\Big) \\ = \;& |\vec{n}|^2\Big((1+\epsilon^2 H_z^2)^{-\frac{1}{2}}(1-\epsilon H)^{-1} + \epsilon H_{zz}(1+\epsilon^2 H_z^2)^{-\frac{3}{2}}\Big), \text{ in } S \end{aligned}$$

from which results

$$P(H(z,t),z) = -(H + H_{zz}) + O(\epsilon^2, \epsilon^2\lambda).$$

Next, we non–dimensionalize the equation (7), which is the kinematic condition, to obtain:

$$H_t = -u - wH_z. \tag{8}$$

At this point, Hammond [9] introduces asymptotic expansions into the whole problem and consider only the high order terms (O(1)). From that we obtain $\boldsymbol{w}$ and $\boldsymbol{u}$, which when are substituted in (8) give as a result the nonlinear partial differential evolution equation for the interface

$$H_t = -\frac{1}{3}(H^3(H_z + H_{zzz}))_z, \tag{9}$$

which, in this paper, we call it the Hammond's equation. In [14], Kas–Danouche obtained a system of two coupled non linear partial differential equations, one for the interface evolution and the other one for the evolution of the surfactant concentration. When the surfactant concentration is set equal to zero, we obtain the equation (9). So, the results of this paper will help us to validate the numerical results of the system derived in [14], which it will be the core for future researches.

2.3 The rescaled Hammond's equation

With the goal of simplifying the numerical calculations, we rescale z from $[0, L]$ to the interval $[0, 2\pi]$. In order to do this, we consider the change of variables $z = \frac{2\pi}{L}\tilde{z}$ and $t = (\frac{2\pi}{L})^2\tilde{t}$, where $\tilde{z} \in [0, L]$, $\tilde{t} \geq 0$ and the variables with "~" represent the unscaled variables. Thus, we have $z \in [0, 2\pi]$, $t \geq 0$, and (9) expressed in the new variables as

$$H_t = -\frac{1}{3}(H^3(\lambda^2 H_{zzz} + H_z))_z, \tag{10}$$

where $\lambda = \frac{2\pi}{L}$.

The initial condition for the rescaled problem is

$$H(z, 0) = 1 + \beta \cos z, \quad 0 \leq z \leq 2\pi, \tag{11}$$

where $\beta > 0$.

The numerical schemes to be developed in the next section will be applied to the equation (10) with the initial condition (11), in the interval $0 \leq z \leq 2\pi$, at time t.

3 Numerical schemes

In this section, four schemes are developed to be applied to the interface equation (10). Two of them use finite differences for both spatial and temporal variables, one explicit and the other one implicit. The other two schemes are based on pseudo–spectral methods [2], making use of the Euler's method, in one of them, and the Runge Kutta's method of fourth order in the other one.

In order to numerically solve the partial differential equation for the evolution of the interface, certain initial and boundary conditions are imposed. The boundary conditions that Hammond [9] considered are:

$$\frac{\partial^{2j+1} H}{\partial z^{2j+1}} = 0, \quad \text{with} \qquad j = 0, 1, 2, \ldots ; z = 0, \frac{1}{2}L, \tag{12}$$

which correspond to H even, with period L and reflectionally symmetric about $z = \frac{1}{2}L$.

3.1 Finite differences

Here, we suppose that H is sufficiently smooth that admit Taylor's expansions for $H(z+h,t)$ and $H(z-h,t)$ at point (z,t) [1]. Therefore, we can write the finite difference approximations for H_z, H_{zz}, H_{zzz}, and H_{zzzz} as follows

$$H_z \approx \frac{H(z+h,t) - H(z-h,t)}{2h}, \tag{13}$$

$$H_{zz} \approx \frac{H(z-h,t) - 2H(z,t) + H(z+h,t)}{h^2}. \tag{14}$$

From the equations (13) and (14), we obtain

$$\begin{aligned} H_{zzz} = (H_z)_{zz} &\approx \frac{1}{2h^3}\Big(H(z+2h,t) - 2H(z+h,t) \\ &+ 2H(z-h,t) - H(z-2h,t)\Big). \end{aligned} \tag{15}$$

From the equation (14) we have

$$\begin{aligned} H_{zzzz} = (H_{zz})_{zz} &\approx \frac{1}{h^4}\Big((H(z-2h,t) - 4H(z-h,t) + 6H(z,t) \\ &- 4H(z+h,t) + H(z+2h,t)\Big). \end{aligned} \tag{16}$$

On the other hand, for all $k \neq 0$, at the point (z,t) we use finite differences for the time derivative to obtain the expression

$$H(z, t+k) \approx H(z,t) + kH_t, \tag{17}$$

to approximate H_t, where k is the time step.

3.1.1 Explicit scheme In what follows we develop the explicit scheme [23] to be applied to the interface evolution equation (10). We uniformly particionate the intervals $0 \leq z \leq 2\pi$ and $0 \leq t \leq T$, in n and m sub–intervals, respectively. The stepsize in z will be denoted by Δz, and the stepsize in t by Δt.

The rescaled interface evolution equation (10) can be rewritten in an expanded form as

$$H_t = -\frac{1}{3}\Big(3H^2 H_z(\lambda^2 H_{zzz} + H_z) + H^3(\lambda^2 H_{zzzz} + H_{zz})\Big). \tag{18}$$

Substituting the finite difference approximations (13)-(17) for the partial derivatives in (18), we obtain

$$\begin{aligned} H\ (z, t+\Delta t) \approx -\frac{\Delta t \lambda^2}{3(\Delta z)^4} H^2(z,t)\Bigg[\frac{3}{4}\Big(H(z+\Delta z,t) - H(z-\Delta z,t)\Big). \\ \Bigg[H(z+2\Delta z,t) - H(z-2\Delta z,t) + \left(\frac{(\Delta z)^2}{\lambda^2} - 2\right)\Big(H(z+\Delta z,t) - H(z-\Delta z,t)\Big)\Bigg] \\ + \quad H(z,t)\Bigg[H(z-2\Delta z,t) + H(z+2\Delta z,t) + \left(\frac{(\Delta z)^2}{\lambda^2} - 4\right)\Big(H(z-\Delta z,t) \\ + \quad H(z+\Delta z,t)\Big) + \left(6 - 2\frac{(\Delta z)^2}{\lambda^2}\right)H(z,t)\Bigg]\Bigg] + H(z,t). \end{aligned}$$

Re–writing the equation using the notation

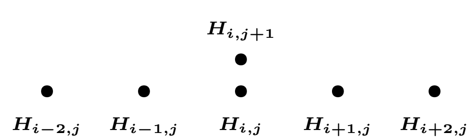
$$H_{i+\tilde{n}, j+\tilde{m}} = H(z_i + \tilde{n}\Delta z, t_j + \tilde{m}\Delta t)$$

, for $\tilde{n}, \tilde{m} = 0, 1, 2$, we have

$$\begin{aligned} H_{i,j+1} \quad = \quad & L_1 H_{i,j}^2 \Big(\frac{3}{4}(H_{i+1,j} - H_{i-1,j})\Big(H_{i+2,j} - H_{i-2,j} + L_2(H_{i+1,j} - H_{i-1,j})\Big) \\ & + H_{i,j}\Big(H_{i-2,j} + H_{i+2,j} + L_3(H_{i-1,j} + H_{i+1,j}) + L_4 H_{i,j}\Big)\Big) + H_{i,j}, \end{aligned} \tag{19}$$

where $L_1 = -\frac{1}{3}\frac{\Delta t \lambda^2}{(\Delta z)^4}$, $L_2 = \frac{(\Delta z)^2}{\lambda^2} - 2$, $L_3 = \frac{(\Delta z)^2}{\lambda^2} - 4$ and $L_4 = 6 - 2\frac{(\Delta z)^2}{\lambda^2}$.

The equation (19) is the explicit representation in finite differences for the interface evolution equation. In order to find the value of H at the point (z_i, t_{j+1}), this scheme requires the values of H at five spatial points z_i at time t_j, as is shown in the following diagram of points



The initial condition (11) gives the values of $H_{i,0}$; i.e.,

$$H_{i,0} = 1 + \beta\cos(z_i), \quad i = 0, \ldots, n-1.$$

From the equation (19) we obtain the values of H at the nodes z_i, $i = 0, ..., n-1$ and times t_{j+1}, $j = 0, 1....$ The boundary conditions suggest that H has to be periodic of period 2π (after rescaling z from $[0, L]$ to $[0, 2\pi]$), and symmetric with respect to $z = 0$. Therefore, at the nodes of the boundary we have

$$\begin{aligned} &H_{-2,j} = H_{2,j}, \; H_{-1,j} = H_{1,j}, \qquad j = 0, \ldots, m \\ &H_{n,j} = H_{0,j}, \; H_{n+1,j} = H_{1,j}, \qquad H_{n+2,j} = H_{2,j}, \quad j = 0, \ldots, m. \end{aligned}$$

3.1.2 Implicit scheme The implicit scheme that we develop here acts on the nonlinear term of higher order of the interface evolution equation; i.e, H_{zzzz} is considered to be found for times $t_{j+\frac{1}{2}}$ and t_{j+1}. For this, is convenient to write the equation (10) in terms of the higher order derivative, as follows

$$H_t = -\frac{1}{3}\lambda^2 H^3 H_{zzzz} - f(H, H_z, H_{zz}, H_{zzz}),$$

where

$$f(H, H_z, H_{zz}, H_{zzz}) = \frac{1}{3}H^3 H_{zz} + \lambda^2 H^2 H_z H_{zzz} + H^2 H_z^2.$$

The equation (10) will be solved numerically in the interval $0 \leq z \leq 2\pi$, under the initial condition

$$H(z, 0) = 1 + \beta\cos(z), \qquad z \in [0, 2\pi], \tag{20}$$

and the boundary conditions

$$\begin{aligned} H_{-1,j} &= H_{n-1,j}, \qquad H_{0,j} = H_{n,j}, \quad H_{1,j} = H_{n+1,j}, \\ H_{2,j} &= H_{n+2,j}, \qquad H_{3,j} = H_{n+3,j}, \end{aligned} \tag{21}$$

which guarantee the periodicity of the solution.

The implicit scheme in finite differences that we propose is based on the implicit scheme proposed by Kas–Danouche in [14]:

$$\frac{H^{j+\frac{1}{2}} - H^j}{\frac{\Delta t}{2}} = -\frac{1}{3}\lambda^2 (H^j)^3 H_{zzzz}^{j+\frac{1}{2}} - f^j(H^j, H_z^j, H_{zz}^j, H_{zzz}^j) \tag{22}$$

$$\begin{aligned} \frac{H^{j+1} - H^j}{\Delta t} &= -\frac{1}{3}\lambda^2 (H^{j+\frac{1}{2}})^3 \Big(\frac{H_{zzzz}^{j+1} + H_{zzzz}^j}{2\Delta z}\Big) \\ &- f^{j+\frac{1}{2}}(H^{j+\frac{1}{2}}, H_z^{j+\frac{1}{2}}, H_{zz}^{j+\frac{1}{2}}, H_{zzz}^{j+\frac{1}{2}}). \end{aligned} \tag{23}$$

It is a scheme of half step that uses forward finite differences for the time integration, where (22) is the predictor and (23) is the corrector. Using (22) we find H computed for time $t_{j+\frac{1}{2}}$. Using (23) we improve H computed in the next half step after the predictor is used. It corresponds to H for time t_{j+1}, using Crank–Nicolson [18] for H_{zzzz} and centered finite differences for all the spatial derivatives.

In the equations (22) and (23), the notation $H^{j+\frac{1}{2}}$, $j = 0, 1$, means H evaluated at $(z, t_{j+\frac{1}{2}})$. Now, we use the notation $H_{i,j+\frac{1}{2}}$, to make reference to H evaluated at $(z_i, t_{j+\frac{1}{2}})$, $j = 0, 1$. Substituting (16) into (22), we obtain

$$\begin{aligned}\frac{H_{i,j+\frac{1}{2}} - H_{i,j}}{\frac{\Delta t}{2}} &= -\frac{\lambda^2 H_{i,j}^3}{3(\Delta z)^4}\Big(H_{i-2,j+\frac{1}{2}} - 4H_{i-1,j+\frac{1}{2}} + 6H_{i,j+\frac{1}{2}} \\ &\quad -4H_{i+1,j+\frac{1}{2}} + H_{i+2,j+\frac{1}{2}}\Big) \\ &\quad -f_i^j(H^j, H_z^j, H_{zz}^j, H_{zzz}^j), \end{aligned} \tag{24}$$

where,

$$\begin{aligned} f_i^j &= \frac{1}{3}H_{i,j}^3 H_{zz}(z_i,t_j)\lambda^2 H_{i,j}^2 H_z(z_i,t_j)H_{zzz}(z_i,t_j) \\ &\quad +H_{i,j}^2 H_z^2(z_i,t_j). \end{aligned}$$

For each fixed j, we denote

$$C_i = -\frac{(\Delta t)\lambda^2 H_{i,j}^3}{6(\Delta z)^4}, \quad i = 0, 1, \ldots, n-1,$$

so that (24) takes the form

$$\begin{aligned} -C_i H_{i-2,j+\frac{1}{2}} &+ 4C_i H_{i-1,j+\frac{1}{2}} + (1-6C_i)H_{i,j+\frac{1}{2}} + 4C_i H_{i+1,j+\frac{1}{2}} \\ &-C_i H_{i+2,j+\frac{1}{2}} = H_{i,j} - \frac{\Delta t}{2} f_i^j(H^j, H_z^j, H_{zz}^j, H_{zzz}^j), \end{aligned} \tag{25}$$

for values of $i = 0, \ldots, n-1$. This equation along with the initial and boundary conditions given by (20) and (21), respectively, can be expressed in the matricial form

$$A^j H^{j+\frac{1}{2}} = b^j, \quad j = 1, \ldots, M, \tag{26}$$

where A^j is the coefficient matrix of the system of equations (25). A^j is of order n, $H^{j+\frac{1}{2}}$ is the column vector of order n, corresponding to the solution we want to calculate for time $t_{j+\frac{1}{2}}$, and b^j is the vector corresponding to the right hand side of the system of equations.

In a similar way, substituting (16) in (23), we have

$$\begin{aligned} -c'_i H_{i-2,j+1} & +4c'_i H_{i-1,j+1} + (1-6c'_i) H_{i,j+1} \\ & +4c'_i H_{i+1,j+1} - c'_i H_{i+2,j+1} \\ = & \ c'_i (H_{i-2,j} - 4H_{i-1,j} + 6H_{i,j} - 4H_{i+1,j} + H_{i+2,j}) \\ & -(\Delta t) f_i^{j+\frac{1}{2}} + H_{i,j}, \end{aligned}$$

where

$$c'_i = -\frac{(\Delta t)\lambda^2 H^3_{i,j+\frac{1}{2}}}{6(\Delta z)^4}, i = 0, \ldots, n-1.$$

This equation, along with the initial and boundary conditions (20) and (21), respectively, produces the matricial system

$$B^{j+\frac{1}{2}} H^{j+1} = a, \quad j = 1, \ldots, M. \tag{27}$$

For each fixed j, $B^{j+\frac{1}{2}}$ is the matrix of order n, which can be obtained substituting C_i by c'_i in the matrix A^j of the system (26). H^{j+1} is the column vector of order n, corresponding to the solution we want to calculate for time t_{j+1}, and a is the vector corresponding to the right hand side of the system of equations.

For each $j = 1, \ldots, M$, the systems of equations (26) and (27) are solved using the method of Woodbury [20], which transforms the almost pentadiagonal matrices to pentadiagonal matrices.

3.2 Pseudo–spectral methods

These methods use the Fourier Transforms for the calculation of the spatial derivatives. For the time derivative a different method, usually finite differences, is used. This is why they are classified as pseudo–spectral methods [2]. The application of these methods is justified, in the first place, because the solution of the interface equation is a smooth function, periodic in the spatial variable and it can be approximated using a finite sum as $H(z,t) = \sum_{n=-N}^{N} h_n(t) e^{inz}$. In the second place, by the periodicity caracter of the initial condition. Computations of the spatial derivatives involved in the nonlinear terms are made in the spectral space. The connection between the spectral space and the physical space is made using the Fast Fourier Transform (FFT) and the Inverse Fast Fourier Transform (IFFT). The errors of the spectral methods are exponentially smalls [2], [8], and this is the main reason why we apply them to our problem.

In this section, two pseudo–spectral schemes are proposed for the integration with respect to time. One scheme uses the Euler's method and the other one uses the Runge Kutta's method of fourth order [1], [5].

In the rescaled Hammond equation (10), let us call

$$\begin{aligned} R(z,t) &= -\frac{1}{3}\Big(H^3(\lambda^2 H_{zzz}+H_z)\Big)_z \\ &= -H^2H_z(\lambda^2 H_{zzz}+H_z)-\frac{1}{3}H^3(\lambda^2 H_{zzzz}+H_{zz}). \quad (28) \end{aligned}$$

Suppose that both $H(z,t)$ and $R(z,t)$ admit each one a developing Fourier series [11]

$$\begin{aligned} H(z,t) &= \sum_{n=-\infty}^{\infty} h_n(t)e^{inz} \quad (29) \\ R(z,t) &= \sum_{n=-\infty}^{\infty} r_n(t)e^{inz}. \end{aligned}$$

In the equation (29), the partial derivative $\frac{\partial H}{\partial t}$, is obtained deriving the series term by term; then the equation (10) can be written in the form

$$\sum_{n=-\infty}^{\infty} \frac{\partial h_n}{\partial t}(t)e^{inz} = \sum_{n=-\infty}^{\infty} r_n(t)e^{inz}.$$

These series coincide if the coefficients of e^{inz} are both equal term by term. Therefore, we obtain the system of ordinary differential equations

$$\frac{dh_n}{dt}(t) = r_n(t). \quad (30)$$

The system (30) is numerically solved for values of $n = \frac{-N}{2}+1,...,\frac{N}{2}$, with N a natural number, using the Euler's method and the Runge Kutta's method of fourth order [1], [5].

3.2.1 Pseudo–spectral with Euler's method For the Euler's method we write

$$h_n(t_{i+1}) = h_n(t_i) + (\Delta t)r_n(t_i), \quad \text{for } i = 0,1,... \quad (31)$$

The coefficients $h_n(t)$ of $H(z,t)$, in (29), are calculated applying the Fast Fourier Transform (FFT) to a set of values of $H(z,t)$ in the interval $[0,2\pi]$ at equidistant points $z_j = j(\Delta z)$, $\Delta z = \frac{2\pi}{N}$, $j = 0,...,N-1$. The value of the partial derivatives $H_z, H_{zz}, H_{zzz}, H_{zzzz}$, at the points z_j, are obtained calculating $\frac{\partial}{\partial z},\ldots,\frac{\partial^4}{\partial z^4}$ at the right hand side of (29) term by term, and applying later the Inverse Fast Fourier Transform (IFFT).

In order to obtain the initial condition, we apply FFT to the initial condition, which is given by $H_0 = H(z,0) = 1+\frac{1}{2}\cos z$, to obtain the $h_n(0)$.

3.2.2 Pseudo–spectral with Runge Kutta's method of fourth order

Here we solve the system of equations (30) using the Runge–Kutta's method of fourth order (RK).

For an initial value problem as

$$y' = f(t, y), \quad y(t_0) = y_0,$$

the RK method is given by

$$y_{k+1} = y_k + \frac{1}{6}(f_1 + 2f_2 + +2f_3 + f_4),$$

where

$$\begin{aligned} f_1 &= (\Delta t) f(t_k, y_k), \\ f_2 &= (\Delta t) f(t_k + \frac{\Delta t}{2}, y_k + \frac{1}{2} f_1), \\ f_3 &= (\Delta t) f(t_k + \frac{\Delta t}{2}, y_k + \frac{1}{2} f_2), \\ f_4 &= (\Delta t) f(t_k + \Delta t, y_k + f_3). \end{aligned}$$

Thus, the following value y_{k+1} is determined by the current value y_k plus a weighted average of slopes. The RK method is a fourth order method which means that the error at each step is of order $(\Delta t)^5$, while the accumulated total error has order $(\Delta t)^4$.

In our case, as r_n is a function obtained applying FFT to

$$R(H, H_z, H_{zz}, H_{zzz}, H_{zzzz})$$

in (28), and H is obtained applying IFFT to h in (29), the equation (30) can be written, eliminating n, as

$$h' = f(h), \tag{32}$$

where

$$f(h) = FFT(R(IFFT(h))).$$

Therefore, we apply RK to the equation (32), with the initial condition

$$h_0 = FFT(H_0).$$

4 Numerical solutions

All the methods used in this article were coded using MatLab. The code for the explicit finite difference method is identified as EXPLI. We have called the code for the implicit finite difference method as IMPLI. We use the name SPEC_EU to identify the code for the pseudo–spectral with Euler's method and SPEC_RK to represent the code for the pseudo–spectral with Runge Kutta's method of fourth order. The codes depend on certain parameters: a) the length L of the tube, the variable z is rescaled from the interval $[0, L]$ to the interval $[0, 2\pi]$, the graphs of the calculated solutions are shown in the interval $[0, 2\pi]$, but the solution in the interval $[0, L]$ can be recovered multiplying z by $\frac{L}{2\pi}$; b) the final time T, the solutions were calculated for $T = 0, 6, 18, 30, 60$, observing that for values of $T > 60$, the solutions do not change too much from that for $T = 60$, c) N is another parameter, which represents the number of sub–intervals, with the same length, in which the interval $[0, 2\pi]$ is sub–divided. For the case of the pseudo–spectral methods, N must be chosen as a power of 2; d) $\lambda = \frac{2\pi}{L}$ is the parameter related to the length of the tube.

The verification of the codes is essential to trust the results obtained from them. In many problems we try to consider cases with exact solutions and compare the solutions computed using the codes with the exact solutions of the problems. In our case, we do not know the exact solution, but we know the solution found and computed by Hammond in [9]. So, we compare all our results with the Hammond's one.

In the subsection 4.2, we check the numerical results and statements that we use in the subsection 4.1, related to the accuracy of the schemes, considering the linear case $H_t = -\frac{1}{3}H_{zzzz}$. This equation with the initial condition $H(z, 0) = 1 + \beta\cos(z)$, has an exact solution expressed by $H(z, t) = 1 + \beta\cos(z)e^{-\frac{1}{3}t}$.

4.1 Numerical experiments

In this section, we present the numerical experiments made using codes for the different numerical schemes previously proposed. It is known that the "FD schemes are somewhat inferior in accuracy compared to spectral schemes" [19], where FD stands for Finite Differences. This is clear from the fact that the pseudospectral methods have exponencial convergence, which makes them by far superior to the FD methods which expose algebraic convergence [10]. The error between a solution and its N–th order truncated Fourier series decays faster than algebraically in 1/N, when such solution is infinitely smooth and periodic with all its derivatives. This is known as the spectral accuracy, or infinite–order accuracy. In this case we say that the series has an infite–order convergence [3].

So first, we numerically solve the nonlinear equation (10) using all the numerical schemes considered in this work. We need to compute the relative errors in order to decide what method gives the best approximation to the exact solution; but we do not know the exact solution for this equation. However, Hammond [9] solved it numerically; so, it would be natural to compare all the numerical solutions that we obtained, with the one already known and obtained by Hammond. In what follows, we present the comparisons.

4.1.1 Comparisons with Hammonds results In this subsection, we present all the numerical results obtained from the methods already explained in previous sections of this article, and we compare them with the numerical results obtained by Hammond. Thus, we solve (10) and (11) for times $\boldsymbol{T = 6, 18, 30, 60}$ and length $\boldsymbol{L = 6\pi}$ for the system, applying the pseudo–spectral methods, one with Euler's method and the other one with Runge–Kutta's method of fourth order. Also, we apply the finite difference methods, one explicit and the other one implicit.

We performed several runs using all the codes (EXPLI, IMPLI, SPEC–EU, and SPEC–RK) applied to Hammond's equation and compared them with the numerical solution obtained by Hammond [9].

In Figure 1, we present the graph corresponding to the modules of the differences between the numerical results obtained by using the implicit finite difference method and the Hammond's solution. We observe that the results are almost the same for all the times considered, except at the extremes where we can note that the differences increase as $\boldsymbol{T}$ increases. Thus, when $\boldsymbol{T = 60}$, the absolute error is **0.03131** and the relative error is by the order of **0.87%** which is good.

For the methods of pseudo–spectral with Euler and pseudo–spectral with Runge–Kutta of fourth order, we have found similar behaviors for the modules of the differences between the numerical results obtained from one of our methods and the Hammond's solution. We have found that the worse case happens for the explicit method when time increases.

In Table 1, we have the absolute and relative errors for the numerical results, obtained from all the methods we study in this article, comparing them with the Hammond's solution. The results are given for final time $\boldsymbol{T = 60}$, $\boldsymbol{N = 2^7}$, and $\boldsymbol{\Delta t = 10^{-4}}$. It can be observed that the pseudo–spectral methods (with Euler and Runge–Kutta of fourth order) produce solutions with relative errors by the order of **3.5%** (see rows **3** and **4**), while the implicit FD method produces a solution with relative error by the order of **0.87%** (see row **2**).

We can conclude that the implicit finite difference method gives the best approximation to the numerical solution obtained by Hammond. But this is a contradiction with the fact that the FD methods are inferior to the pseudospectral methods [19]. So, what is it happening? Can we trust our results? Let us analize the situation. Hammond used the method of lines to obtain the numerical solutions [9], but the method of lines is based on FD methods. Therefore, we can not trust the comparisons.

On the other hand, since the pseudo–spectral methods have infinite–order

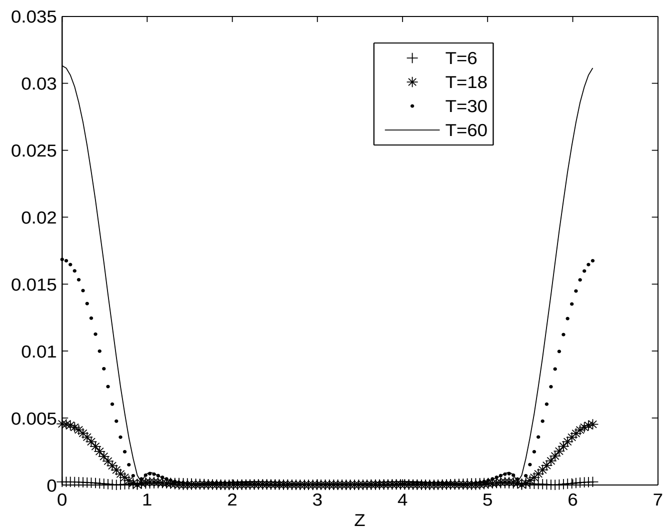


Figure 1: Profiles of the modules of the differences between the numerical results obtained by using the implicit finite difference method and the Hammond's solution.

convergence [3], see section 4.1 for comments, we decide to choose one of the pseudospectral methods considered in this article as the most accurate solution to the problem. It is known that the method of Runge–Kutta of fourth order is more accurate than the method of Euler, which is second order. So, from now on, we take the solution obtained from pseudo–spectral with Runge–Kutta's method of fourth order as the most accurate solution for the initial value problem (10) and (11) in this article, and do the comparisons.

4.1.2 Comparisons with results from pseudo–spectral with Runge–Kutta of fourth order In this subsection, we compare the numerical solutions obtained from our numerical methods, with the solution computed using the pseudo–spectral method with Runge–Kutta of fourth order; for an explanation, see subsection 4.1.1 and introduction to section 4.1.

In Figure 2 we present the profiles of the interface evolution obtained solving (10) and (11) using the pseudo–spectral with Runge–Kutta's method of fourth order. The profiles correspond to times $\boldsymbol{T = 6, 18, 30, 60}$, and $\boldsymbol{L = 6\pi}$. Here, the horizontal axis represents the tube wall. As we can observe in Figure 2, as time increases, the interface perturbation becomes higher and higher. Thus, the perturbed interface may eventually touch the tube wall; in this sense the

Table 1: Comparisons of the numerical solutions obtained applying the methods: explicit and implicit finite differences, pseudo–spectral with Euler and pseudo–spectral with Runge–Kutta of fourth order, with the Hammond's solution [9] for $\boldsymbol{N = 2^7}$, final time $\boldsymbol{T = 60}$ and $\boldsymbol{\Delta t = 10^{-4}}$.

Row	Code	Absolute error	Relative error
1	EXPLI	$\mathbf{1.019}$ x$\mathbf{10^{-1}}$	$\mathbf{9.132}$ x$\mathbf{10^{-1}}$
2	IMPLI	$\mathbf{3.131}$ x$\mathbf{10^{-2}}$	$\mathbf{8.744}$ x$\mathbf{10^{-3}}$
3	SPEC-EU	$\mathbf{8.554}$ x$\mathbf{10^{-2}}$	$\mathbf{3.531}$ x$\mathbf{10^{-2}}$
4	SPEC-RK	$\mathbf{8.482}$ x$\mathbf{10^{-2}}$	$\mathbf{3.535}$ x$\mathbf{10^{-2}}$

system becomes more unstable. So, from the profile for $\boldsymbol{T = 60}$, we may think about the possibility, for some time $\boldsymbol{T}$ in the future, of a contact of the interface with the tube wall. For this case, we could need to do another research and, of course, consider other numerical schemes to be able to capture the rupture of the interface.

In Figure 3, we present the graph corresponding to the modules of the differences between the numerical results obtained by using the pseudo–spectral methods studied in this article. We observe that the results are almost the same for all the times considered, except when $\boldsymbol{T = 60}$. In such a case, near the extremes the differences are by the order of $\boldsymbol{10^{-4}}$, which it is still acceptable. In Table 2, Row 3, we can see the precise values for the errors.

In the literature the implicit methods, in general, have been proved to be more effective than the explicit ones [16] in the sense that convergence happens with bigger steps than the ones required for the explicit methods. Thus, we compare the results obtained from the implicit finite difference method with those obtained by the pseudo–spectral with Runge–Kutta's method of fourth order. Therefore, we present in Figure 4, the graph corresponding to the modules of the differences between the results from the implicit finite difference method and those from the pseudo–spectral with Runge–Kutta's method of fourth order. In this graph we also can observe that bigger differences occur at the extremes, but now by the order of $\boldsymbol{10^{-2}}$. Also see Row 2 of Table 2 for details on the errors calculated for $\boldsymbol{Time = 60}$.

In Figure 5, we plot the modules of the differences between the results from the method of lines and those from the pseudo–spectral with Runge–Kutta's method of fourth order. Looking at this graph we can observe that it is very similar to that in Figure 4, bigger differences occur at the extremes, and by the order of $\boldsymbol{10^{-2}}$ too. Check Row 4 of Table 2 to see the values of the errors.

In Figure 6, we present the relative error of the numerical solution obtained from the pseudo–spectral with Euler's method, considering the solution from the pseudo–spectral with Runge Kutta's method of fourth order as the most accurate numerical solution of the problem, as was discussed in Section 4.1.1.

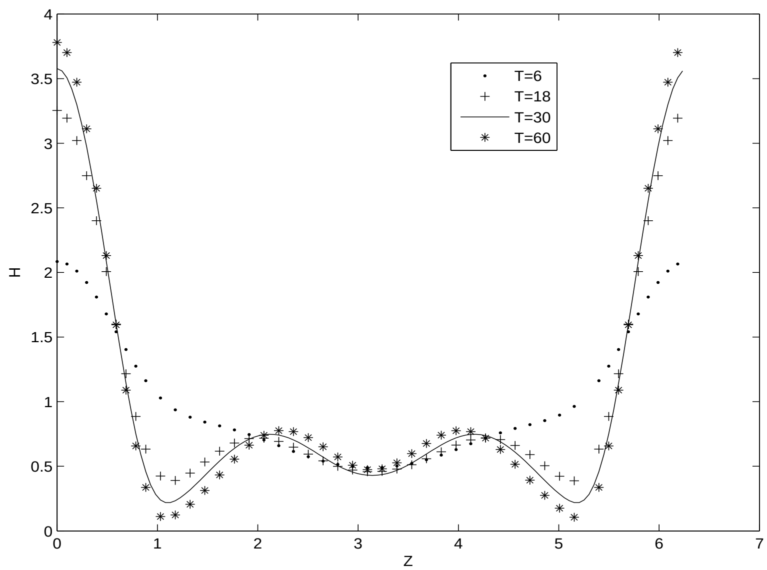


Figure 2: Profiles of the evolution of the interface using the pseudo–spectral with Runge Kutta's method of fourth order.

It can be easily observed that the relative error is by the order of $\mathbf{10^{-4}}$ which is excellent. See Row 3 in Table 2. This tells us that the pseudo–spectral with Euler's method produces a very good numerical solution.

On the other hand, in Figure 7, we plot the relative error of the numerical solution produced by the method of lines, considering, by the same reasons, that the solution from the pseudo–spectral with Runge Kutta's method of fourth order is the most accurate solution. It can be noted that the relative error is by the order of $\mathbf{10^{-2}}$. Also, in Table 2, we can observe that the method of lines (Row 4) and the implicit FD method (Row 2) produce very similar relative errors. The errors are by the order of $\mathbf{10^{-2}}$ which, even though it is also good, it tells us that the pseudo–spectral with Euler's method gives a better solution, with errors by the order of $\mathbf{10^{-4}}$, see Row 3.

All the results observed in Figures 3, 4, 5, 6, and 7, say that the numerical solution produced using the pseudo–spectral with Euler's method approximates the best the solution of the pseudo–spectral with Runge Kutta's method of fourth order. In what follows, we want to check the methods used in this article with some case in which we can have its exact solution. In order to do so, we consider the linear case of the Hammond's equation (10) which involves the higher order derivative of $\boldsymbol{H}$; then we calculate the exact solution to this linear equation. In the next step, we compute the numerical solutions using

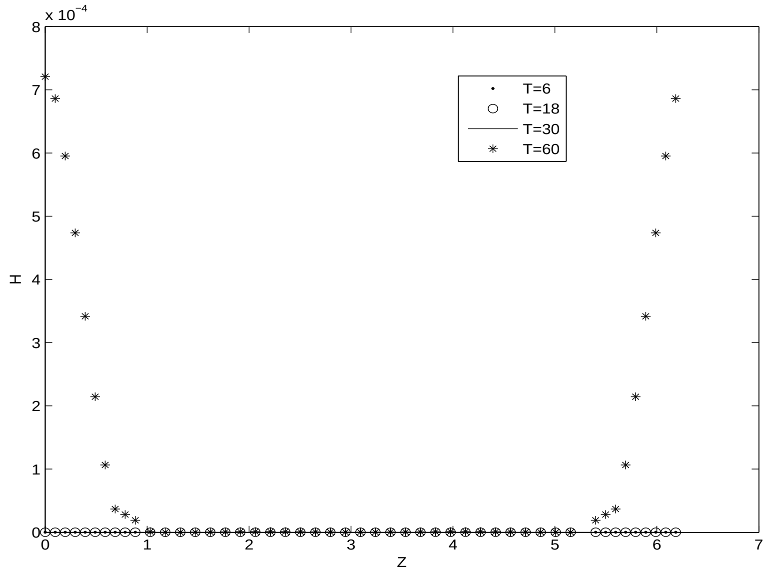


Figure 3: Module of the differences between the solutions from the pseudo–spectral with Euler's method and pseudo–spectral with Runge Kutta's method of fourth order.

all the methods proposed in this article that can be applied to this case, and determine which one of them is the best approximation to the exact solution. We also compare the numerical results with the one obtained by the method of lines.

4.2 Linear case

The linear form of (10) is given by

$$\boldsymbol{H_t = -\frac{1}{3}H_{zzzz}.} \tag{33}$$

We apply the pseudo–spectral with Euler's method (SPEC–EU) and the pseudo–spectral with Runge–Kutta's method of fourth order (SPEC–RK) to (33). From the finite difference methods, we use only the explicit method (EXPLI). The implicit scheme can not be applied to this equation because it does not have any nonlinear part, so the application of (22) and (23) is not possible.

The linear partial differential equation (33) is numerically solved for values of $\boldsymbol{z}$ in the interval $[\boldsymbol{0, 2\pi}]$ and $\boldsymbol{t}$ in the interval $[\boldsymbol{0, T}]$ under the initial condition

$$\boldsymbol{H(z, 0) = 1 + \beta \cos(z).} \tag{34}$$

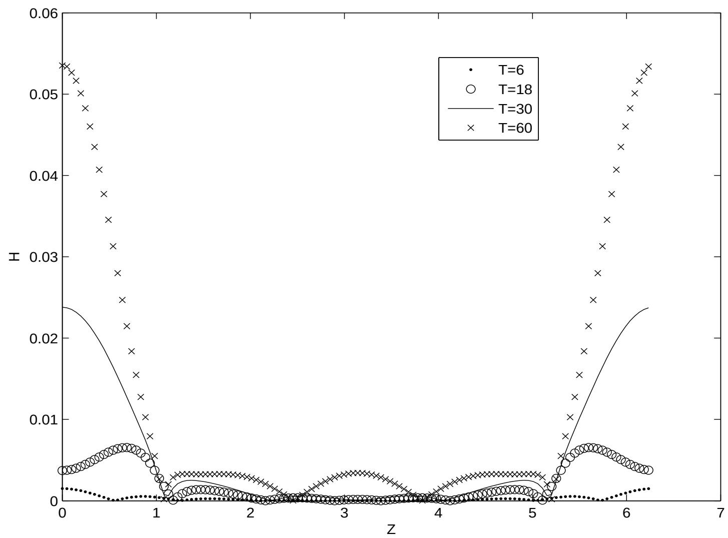


Figure 4: Module of the differences between the solutions from the implicit finite difference method and the pseudo–spectral with Runge Kutta's method of fourth order.

Also, we have verified each one of the codes for each corresponding method, applying them to the linear case. The codes were executed using several data sets and the numerical results obtained were compared with the exact solution of the initial value problem (33) and (34), which was determined using the method of separation of variables. The exact solution is expressed as follows:

$$\boldsymbol{H(z,t) = 1 + \beta\cos(z)e^{-\frac{1}{3}t}.} \tag{35}$$

With the purpose of comparing the numerical solutions with the exact one, we compute the absolute and relative errors. In order to calculate the absolute error, we use the norm $\|\cdot\|_\infty$ of the difference between the exact solution and the numerical solution calculated at each node. We compute the relative errors dividing the absolute errors by the exact solution. We present in Table 3 several results from where it can be observed what method gives the best approximation to the exact solution. This can be seen looking at the columns for the $\|\cdot\|_\infty$ error and the relative error.

From several runs performed in this section, we chose only sixteen of them, which are tabulated in Table 3. Here, we do not present comparative graphs since the differences between the exact solution (35) and the numerical solutions obtained from the pseudo–spectral methods, are undistinguishables. So, we

Table 2: Comparisons of the numerical solutions obtained applying the methods: explicit and implicit finite differences, pseudo–spectral with Euler and the method of lines, with the numerical solution from the method of pseudo–spectral with Runge–Kutta of fourth order for $\boldsymbol{N = 2^7}$, $\boldsymbol{\Delta t = 10^{-4}}$, and final time $\boldsymbol{T = 60}$.

Row	Code	Absolute error	Relative error
1	EXPLI	**1.835** x$\mathbf{10^{-1}}$	**9.335** x$\mathbf{10^{-1}}$
2	IMPLI	**5.351** x$\mathbf{10^{-2}}$	**3.632** x$\mathbf{10^{-2}}$
3	EXPEC-EU	**7.207** x$\mathbf{10^{-4}}$	**1.906** x$\mathbf{10^{-4}}$
4	LINE	**8.482** x$\mathbf{10^{-2}}$	**3.414** x$\mathbf{10^{-2}}$

limit the presentation of the results for this case to comparisons shown in Table 3.

Observing the Table 3, we can note that for the first four rows, which correspond to results for final time $\boldsymbol{T = 6}$, we have the relative errors: **9.919**x$\mathbf{10^{-3}}$ for the explicit FD method, **7.360**x$\mathbf{10^{-4}}$ for the method of lines, **4.838**x$\mathbf{10^{-6}}$ for the pseudo–spectral method with Euler, and **2.419**x$\mathbf{10^{-6}}$ for the pseudo–spectral method with Runge–Kutta of fourth order. For the next four rows, 5, 6, 7, and 8, which correspond to results for $\boldsymbol{T = 18}$, we can observe, once more, that the smallest relative error is obtained from the results given by the pseudo–spectral method with Runge–Kutta of fourth order. For time $\boldsymbol{T = 30}$, it is observed that the relative error using the method of lines is **7.416**x$\mathbf{10^{-6}}$, for the pseudospectral method with Euler, is **4.540**x$\mathbf{10^{-9}}$, and for the pseudospectral method with Runge–Kutta of fourth order, is **7.577**x$\mathbf{10^{-10}}$, which is the smallest relative error for this time too. When we run the codes for all the methods, at final time $\boldsymbol{T = 60}$, we obtain **1.544**x$\mathbf{10^{-3}}$, **2.268**x$\mathbf{10^{-7}}$, **3.440**x$\mathbf{10^{-12}}$, and **0.000**x$\mathbf{10^{-15}}$, for the explicit FD method, the method of lines, the pseudospectral method with Euler, and the pseudospectral method with Runge–Kutta of fourth order, respectively. For this time we can observe the spectral accuracy of the pseudospectral method with Runge–Kutta of fourth order.

On the other hand, for $\boldsymbol{N = 2^5}$, $\boldsymbol{\Delta t = 10^{-3}}$, and $\boldsymbol{T = 60}$, the execution times for the codes EXPLI, SPEC–EU, and SPEC–RK are 1.062, 3.04, and 25.89, respectively. This put the explicit and pseudo–spectral with Euler methods in advantage with respect to the pseudo–spetral method with Runge–Kutta of fourth order. When $\boldsymbol{N = 2^6}$, $\boldsymbol{\Delta t = 10^{-4}}$, and $\boldsymbol{T = 60}$, it is observed that EXPLI continue having the smallest execution time, then SPEC–EU follows with a bigger execution time, and finally SPEC–RK has the biggest one.

Thus, from our results we have checked that the pseudo–spectral with Runge–Kutta's method of fourth order produces the most accurate numerical solution, even though there is a computational cost to pay; it is followed, in accuracy,

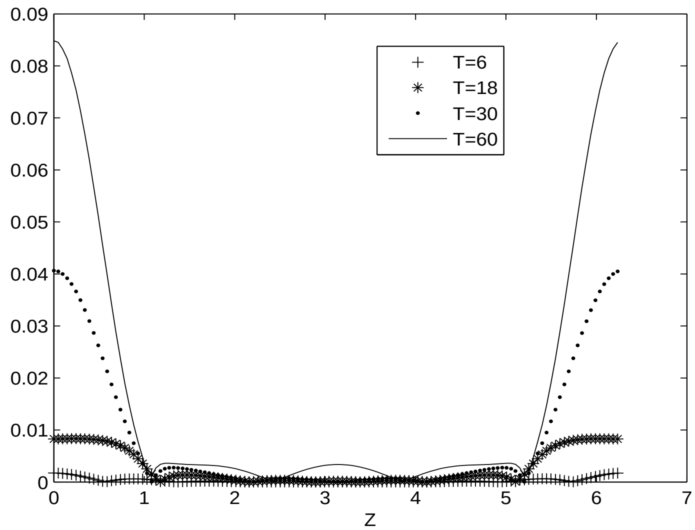


Figure 5: Module of the differences between the solutions from the method of lines and the pseudo–spectral with Runge Kutta's method of fourth order.

by the pseudo–spectral with Euler's method. This is in accordance with the fact that the pseudospectral methods have exponencial convergence [3], which makes them by far more accurate than the FD methods [10], [19], and with the known fact that the method of Runge–Kutta of fourth order produces more accurate solutions than the method of Euler. So, we have checked that there is no problem to consider, in this article, the solution from the pseudo–spectral method with Runge–Kutta of fourth order as the most accurate solution for the problem (10) and (11).

5 Conclusions

Since the pseudo–spectral methods have infinite–order convergence [3], see Section 4.1 for comments, and the method of Runge–Kutta of fourth order is more accurate than the method of Euler, we decided to choose the pseudo–spectral with Runge–Kutta's method of fourth order as the most accurate solution for the initial value problem (10) and (11), in this article.

The horizontal axis represents the tube wall. As we can observe in Figure 2, as time increases, the interface perturbation becomes higher and higher. Thus, the perturbed interface may eventually touch the tube wall; in this sense the

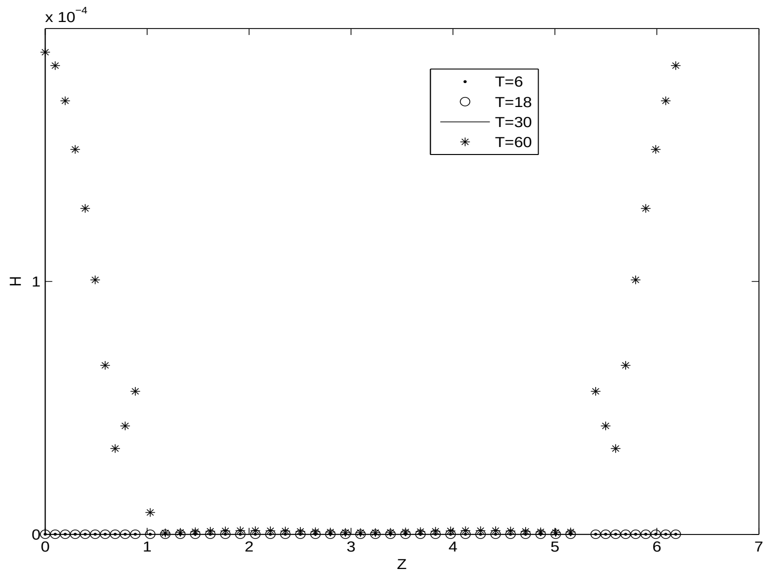


Figure 6: Relative error of the solution obtained by the pseudo–spectral with Euler's method.

system becomes more unstable. Therefore, from the profile for T = 60, we may think about the possibility, for some time T in the future, of a contact of the interface with the tube wall. For this case, we could need to do another research and, of course, consider other numerical schemes to be able to capture the rupture of the interface.

We observe, see Figure 3, that the numerical results obtained by using the two pseudo–spectral methods, used in this article, are almost the same for all the times considered, except when T = 60. In such a case, near the extremes the differences are by the order of $\mathbf{10^{-4}}$, which it is very good.

In the literature the implicit methods, in general, have been proved to be more effective than the explicit ones [16] in the sense that convergence happens with bigger steps than the ones required for the explicit methods. Thus, we compare the results obtained from the implicit finite difference method with those obtained by the pseudo–spectral with Runge–Kutta's method of fourth order. Therefore, in Figure 4, we can observe that bigger differences occur at the extremes, by the order of $\mathbf{10^{-2}}$.

The differences between the results from the method of lines and those from the pseudo–spectral with Runge–Kutta's method of fourth order, see Figure 5, are noted at the extremes, and by the order of $\mathbf{10^{-2}}$ too.

In Figure 6, we present the relative error of the numerical solution obtained

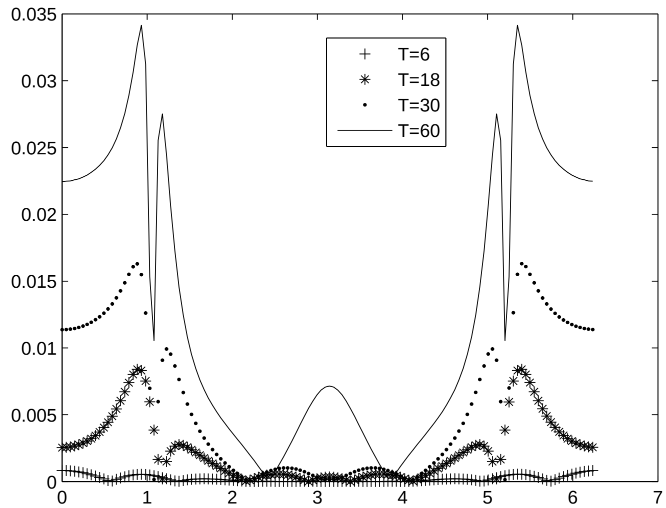


Figure 7: Relative error of the solution obtained by the method of lines.

from the pseudo–spectral with Euler's method, considering the solution from the pseudo–spectral with RungeKutta's method of fourth order as the most accurate numerical solution of the problem. It can be easily observed that the relative error is by the order of $\mathbf{10^{-4}}$ which is excellent. Thus, the pseudo–spectral with Euler's method produces a very good numerical solution.

On the other hand, the relative error of the numerical solution produced by the method of lines is by the order of $\mathbf{10^{-2}}$, considering, by the same reasons, that the solution from the pseudo–spectral with RungeKutta's method of fourth order is the most accurate solution.

Considering the linear form of (10), we apply the pseudo–spectral with Euler's method (SPEC–EU) and the pseudo–spectral with Runge–Kutta's method of fourth order (SPEC–RK) to (33). From the finite difference methods, we use only the explicit method (EXPLI). The implicit finite difference scheme cannot be applied to this equation because it does not have any nonlinear part, so the application of our implicit scheme (22) and (23) is not possible here. We have verified each one of the codes for each corresponding method, applying them to the linear case. The codes were executed using several data sets and the numerical results obtained were compared with the exact solution of the initial value problem (33) and (34), which was determined using the method of separation of variables. With the purpose of comparing the numerical solutions with the exact one, we compute the absolute and relative errors.

Table 3: Comparisons of the exact solution (35) with the numerical solutions obtained applying the following methods: explicit finite differences, lines, pseudo–spectral with Euler and pseudo–spectral with Runge–Kutta of fourth order for $\boldsymbol{N = 2^5}$, $\boldsymbol{\Delta t = 10^{-4}}$ and different final times, $\boldsymbol{T}$.

Row	T	Code	Absolute error	Relative error
1	6	EXPLI	$\mathbf{1.059}$ x$\mathbf{10^{-2}}$	$\mathbf{9,919}$ x$\mathbf{10^{-3}}$
2	6	LINE	$\mathbf{6,863}$ x$\mathbf{10^{-4}}$	$\mathbf{7,360}$ x$\mathbf{10^{-4}}$
3	6	ESPEC–EU	$\mathbf{4,511}$ x$\mathbf{10^{-6}}$	$\mathbf{4,838}$ x$\mathbf{10^{-6}}$
4	6	ESPEC–RK	$\mathbf{2,255}$ x$\mathbf{10^{-6}}$	$\mathbf{2,419}$ x$\mathbf{10^{-6}}$
5	18	EXPLI	$\mathbf{2,002}$ x$\mathbf{10^{-3}}$	$\mathbf{1,999}$ x$\mathbf{10^{-3}}$
6	18	LINE	$\mathbf{9,692}$ x$\mathbf{10^{-6}}$	$\mathbf{9,680}$ x$\mathbf{10^{-6}}$
7	18	ESPEC–EU	$\mathbf{1,652}$ x$\mathbf{10^{-7}}$	$\mathbf{1,654}$ x$\mathbf{10^{-7}}$
8	18	EXPEC–RK	$\mathbf{4,131}$ x$\mathbf{10^{-8}}$	$\mathbf{4,136}$ x$\mathbf{10^{-8}}$
9	30	EXPLI	$\mathbf{1,556}$ x$\mathbf{10^{-3}}$	$\mathbf{1,557}$ x$\mathbf{10^{-3}}$
10	30	LINE	$\mathbf{7,414}$ x$\mathbf{10^{-6}}$	$\mathbf{7,416}$ x$\mathbf{10^{-6}}$
11	30	EXPEC–EU	$\mathbf{4,539}$ x$\mathbf{10^{-9}}$	$\mathbf{4,540}$ x$\mathbf{10^{-9}}$
12	30	EXPEC–RK	$\mathbf{7,566}$ x$\mathbf{10^{-10}}$	$\mathbf{7,577}$ x$\mathbf{10^{-10}}$
13	60	EXPLI	$\mathbf{1,544}$ x$\mathbf{10^{-3}}$	$\mathbf{1,544}$ x$\mathbf{10^{-3}}$
14	60	LINE	$\mathbf{2,268}$ x$\mathbf{10^{-7}}$	$\mathbf{2,268}$ x$\mathbf{10^{-7}}$
15	60	EXPEC–EU	$\mathbf{3,440}$ x$\mathbf{10^{-12}}$	$\mathbf{3,440}$ x$\mathbf{10^{-12}}$
16	60	EXPEC–RK	$\mathbf{0,000}$ x$\mathbf{10^{-15}}$	$\mathbf{0,000}$ x$\mathbf{10^{-15}}$

From the results obtained using all the methods considered in this article, at final time T = 60, we can observe the spectral accuracy of the pseudo–spectral method with Runge–Kutta of fourth order with an error of order $\mathbf{10^{-15}}$. Thus, from our results, we have checked that the pseudo–spectral with Runge–Kutta's method of fourth order produces the most accurate numerical solution, even though there is a computational cost to pay; it is followed, in accuracy, by the pseudo–spectral with Euler's method. This is in accordance with the fact that the pseudo–spectral methods have exponencial convergence [3], which makes them by far more accurate than the finite difference methods [10], [19], and with the known fact that the method of Runge–Kutta of fourth order produces more accurate solutions than the method of Euler.

Therefore, considering that the numerical solution obtained from the pseudo–spectral with Runge Kutta's method of fourth order is the most accurate numerical solution of the problem, we took it as the reference solution for the nonlinear problem, and conclude that the pseudo–spectral with Euler's method produces the best numerical solution to the problem (10) and (11) for times $\boldsymbol{T = 6, 18, 30, 60}$ and length $\boldsymbol{L = 6\pi}$.

Acknowledgments

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References

[1] R. L. BURDEN AND J. D. FAIRES, (1997), *Numerical Analysis, Sixth edition*, Brooks/Cole Publishing Company.

[2] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI AND T. A. ZANG, (1988), *Spectral Methods in Fluid Dynamics*, Springer Series in Computational Physics, Springer–Verlag, USA.

[3] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI AND T. A. ZANG, (2006), *Spectral Methods. Fundamentals in Single Domains*, Scientific Computation, Springer, USA.

[4] K. CHEN, R. BAI AND D. D. JOSEPH, *Lubricated pipelining. Part 3. Stability of core–anular flow,* Journal of Fluid Mechanics, 214, (1990), pp. 251–286.

[5] T. J. CHUNG, (2002), *Computational Fluid Dynamics*, Cambridge University Press.

[6] P. A. GAUGLITZ AND C. J. RADKE, *An extended evolution equation for liquid film breakup in cylindrical capillaries,* Chemical Engineering Science, 43, (1988), pp. 1457–1465.

[7] S. L. GOREN, *The instability of anular thread of fluid*, Journal of Fluid Mechanics, 12, (1962), pp. 309–319.

[8] D. GOTTLIEB AND S. A. ORSZAG, (1993), *Numerical Analysis of Spectral Methods: Theory ans Applications, Sixth printing*, SIAM, Capital City Press.

[9] P. S. HAMMOND, *Nonlinear adjustament of a thin annular film of viscous fluid surrounding a thread of another within a circular cylindrical pipe*, J. Fluid Mech. (1983), 137, pp. 363–384.

[10] H. HOLMÅS, D. CLAMOND AND H. P. LANGTANGEN, *A pseudospectral Fourier method for a 1D incompressible two–fluid model*, International Journal for Numerical Methods in Fluids (2008), 58, pp. 639–658.

[11] H. P. HSU, *Análisis de Fourier* , Addison Wesley Longman.

[12] S. A. KAS–DANOUCHE, (2002), *Nonlinear interfacial stability of core–annular film flows in the presence of surfactans,* Ph.D. Dissertation, New Jersey Institute of Technology and Rutgers, The State University of New Jersey – Netwark.

[13] S. A. KAS–DANOUCHE, D. T. PAPAGEORGIOU AND M. SIEGEL, *A mathematical model for core–annular flows with surfactants,* Divulgaciones Matemáticas, Vol 12, N 2, (2004), pp. 117–138.

[14] S. A. KAS–DANOUCHE, *A mathematical model for core–annular fluids with surfactants,* Boletín de la Asociación Matemática Venezolana, Vol XIV, N 1 y 2, (2007), pp. 27–39.

[15] S. A. KAS–DANOUCHE, D. T. PAPAGEORGIOU AND M. SIEGEL, *Nonlinear dynamics of core–annular film flows in the presence of surfactant,* Journal of Fluid Mechanics, 626, (2009), pp. 415–448.

[16] D. KINCAID AND W. CHENEY, (1996), *Numerical Analysis*, second edition, Brooks/Cole Publishing Company.

[17] C. KOURIS AND J. TSAMOPOULOS, *Dynamics of axisymmetric core–annular flow in a strtaight tube. I. The more viscous fluid in the core, bamboo waves*, Physics of Fluids, 13 (2002), pp. 1011–1029.

[18] K. W. MORTON AND D. F. MAYERS, (1994), *Numerical Solutions of Partial Differential Equations*, Cambridge University Press.

[19] V. NAULIN AND A. NIELSEN, *Accuracy of spectral and finite difference schemes in 2D advection problems,* SIAM Journal Sci. Comput., Vol 25, N 1, (2003), pp. 104–126.

[20] W. H. PRESS, S. A. TEUKOLSKY, W. T. VETERLING, AND B. P. FLANNERY, (1992), *Numerical Recipes In Fortran 77: The art of Scienctific Computing* (Vol 1 of Fortran Numerical Recipes), Cambridge University Press, New York.

[21] Y. Y. RENARDY, *Snakes and corkscrews in core–annulardown–flow of two fluids*, Journal of Fluid Mechanics, 340, (1997), pp. 297–317.

[22] A. RUTHERFORD, (1962), *Vectors, Tensors, and the Basics Equations of Fluid Mechanics*, Dover Publications, Inc, New York.

[23] J. C. STRIKWERDA, (1989), *Finite Difference Schemes and Partial Differential Equations*, Chapman and Hall.

[24] THOMAS AND FINNEY, (1996), *Calculus*, Addison Wesley.

[25] G. L. TAYLOR, Journal of Fluid Mechanics, 10, (1961), p. 161.

[26] S. TOMOTIKA *On the instability of a cylindrical thread of a viscous liquid surrounded by another viscous liquid,* Proc. R. Soc. Lond. A, 150 (1935), pp. 322–337.

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Common fixed point result for weakly compatible mappings

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V. K. Pathak and Nidhi Sharma

Abstract. In this note a common fixed point theorem for weakly compatible mappings in a metric space is proved which, improves the results of Ciric [3], Fisher [4], Pant [8],[9] and Popa et al.[10].

Resumen. En esta nota se prueba un teorema de punto fijo común para las funciones débilmente compatibles en un espacio métrico que mejora los resultados de Ciric [3], Fisher [4], Pant [8],[9] y Popa et al.[10].

1.Introduction

The concept of commuting mappings has proven useful for generalizing fixed point theorems in the context of metric spaces. Sessa [11] defined two selfmaps S and T of a metric space (X, d) to be weakly commuting if and only if $d(STx, TSx) \leq d(Sx, Tx)$ for each x in X. Jungck [5] further weakened this property by introducing the concept of compatibility of mappings, which asserts that S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Jungk et al. [7] have defined S and T to be compatible of type (A) if $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Clearly weakly commuting mappings are compatible of type (A). By [7], Ex.2.2, it follows that this implication is not reversible and it also follows by Ex.2.1 and Ex.2.2 that the notion of compatible mappings and compatible mappings of type (A) are independent. Recently, Jungck and Rhoades [6] introduced the concept of weakly compatible mappings as, two selfmaps S and T of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points; i.e. if $Su = Tu$ for some $u \in X$, then $STu = TSu$. Notice that every commuting pair of maps are weakly commuting, weakly commuting pair of maps are compatible and compatible pair of maps are weakly compatible, but the converse of each need not be true. [see for instance, [1] and [10]].

The main object of the paper is to prove a common fixed point theorem for weakly compatible mappings in a complete metric space which improves several known results.

2. Preliminaries

Recently, Popa et al. [10] proved the following theorem :

Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that
(i) f is isotone, i.e. if $t_1 \leq t_2$, then $f(t_1) \leq f(t_2)$ for all $t_1, t_2 \in \mathbb{R}_+$;
(ii) f is upper semi-continuous;
(iii) $f(t) < t$ for all $t > 0$.

Theorem 2.1 Let A, B, S, and T be self mappings of a complete metric space (X, d) such that

$$(2.1) \qquad A(X) \subset T(X) \quad \textit{and} \quad B(X) \subset S(X),$$

(2.2) the inequality

$$[1+p\, d(Sx,Ty)]d(Ax,By) \leq p\, max\{d(By,Ty)d(Ax,Sx), d(Ax,Ty)d(Sx,By)\}$$

$$+ f\big(\max\{d(Sx,Ty), d(By,Ty), d(Ax,Sx), \frac{1}{2}[d(Ax,Ty)+d(Sx,By)]\}\big)$$

holds for all $x, y \in X$, where $p \geq 0$ and $f \in \mathcal{F}$.
(2.3) one of A, B, S, and T is continuous;
(2.4) A and B weakly commutes with S and T respectively;
then A, B, S and T have a unique common fixed point q in X. Further q is the common fixed point of A and S and of B and T.

Now we can define a sequence $\{y_n\}$ with the help of (2.1), by choosing an arbitrary $x_0 \in X$ as follows :

$$(2.5) \qquad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \textit{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for $n = 1, 2, 3, \ldots$ (see, for instance, [12]).

The following lemmas help us to establish our results :

Lemma 2.2[2] Let $f \in \mathcal{F}$ and $\{\xi_n\}$ be a sequence of non-negative real numbers. If $\xi_{n+1} \leq f(\xi_n\}$ for $n \in N$, then the sequence $\{\xi_n\}$ converges to 0.

Lemma 2.3 If we denote $d_n = d(y_n, y_{n+1})$ then $lim_{n\to\infty} d_n = 0$.
Proof. The inequality (2.2) with $x = x_{2n}$ and $y = x_{2n+1}$ implies

$$[1+pd(y_{2n-1}, y_{2n})]d(y_{2n}, y_{2n+1}) \leq p\, d(y_{2n+1}, y_{2n})d(y_{2n}, y_{2n-1})$$

$$+f\big(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}), \frac{1}{2}[d(y_{2n}, y_{2n})+d(y_{2n-1}, y_{2n+1})]\}\big),$$

which implies

$$d_{2n} \leq f\big(\max\{d_{2n-1}, d_{2n}, \frac{1}{2}d_{2n-1}, \frac{1}{2}d_{2n}\}\big),$$

If $d_{2n} > d_{2n-1}$, then we have

$$d_{2n} \leq f\big(d_{2n})\big) < d_{2n},$$

a contradiction. Thus, we must have $d_{2n} \leq d_{2n-1}$.
Then using this inequality the condition (2.2) yields

$$d_{2n} \leq f(d_{2n-1}). \tag{2.6}$$

for $n = 0, 1, 2\ldots$ Similarly taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (2.2), we get

$$[1+pd(y_{2n+1}, y_{2n})]d(y_{2n+2}, y_{2n+1}) \leq p\, d(y_{2n+1}, y_{2n})d(y_{2n+2}, y_{2n+1})$$

$$+ f\big(\max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n})), d(y_{2n+1}, y_{2n+2}),$$

$$\frac{1}{2}[d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})]\}\big),$$

which implies

$$d_{2n+1} \leq f\big(\max\{d_{2n}, d_{2n+1}, \frac{1}{2}[d_{2n} + d_{2n+1}]\}\big).$$

If $d_{2n+1} > d_{2n}$, then we have

$$d_{2n+1} \leq f\big(d_{2n+1})\big) < d_{2n+1},$$

a contradiction. Thus, we must have $d_{2n+1} \leq d_{2n}$.
Then using this inequality the condition (2.2) yields

$$d_{2n+1} \leq f(d_{2n}). \tag{2.7}$$

for $n = 1, 2, 3, \ldots$ From equations (2.6) and (2.7), we obtain

$$d_{n+1} \leq f(d_n),$$

for $n = 1, 2, 3, \ldots$. And so, by Lemma 2.2, we get $lim_{n\to\infty}\mathrm{d}_n = 0$.

Lemma 2.4 The sequence $\{y_n\}$ defined by (2.5) is a Cauchy sequence.
Proof. Suppose subsequence $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for each even integer 2k, there exist even integers 2m(k) and 2n(k) with $2m(k) > 2n(k)$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon \tag{2.8}$$

For each even integer 2k, let 2m(k) be the least even integer exceeding 2n(k) satisfying (2.8), that is

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon \;\; and \;\; d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{2.9}$$

Then for each even integer 2k, we have

$$\begin{aligned}\epsilon <& d(y_{2n(k)}, y_{2m(k)}) \\ \leq& d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}) \\ \leq& \epsilon + d_{2m(k)-2} + d_{2m(k)-1}.\end{aligned}$$

Hence from Lemma 2.3 and (2.9), it follows that

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.10}$$

By the triangular inequality, we have

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)-1}, y_{2m(k)}),$$

and

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2n(k)+1}).$$

By Lemma 2.2 and equation (2.9), we obtain

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \epsilon \;\; and \;\; lim_{k\to\infty} d(y_{2m(k)+1}, y_{2m(k)-1}) = \epsilon. \tag{2.11}$$

Now using (2.2) with $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$, we have

$$\begin{aligned}[1{+}p\, d(Sx_{2n(k)}, Tx_{2m(k)-1})]d(Ax_{2n(k)}, Bx_{2m(k)-1}) \leq p\, max\{&d(Bx_{2m(k)-1}, Tx_{2m(k)-1}) \\ &d(Ax_{2n(k)}, Sx_{2n(k)}), d(Ax_{2n(k)}, Tx_{2m(k)-1})d(Sx_{2n(k)}, Bx_{2m(k)-1})\} \\ +f\big(\max\{d(Sx_{2n(k)}, Tx_{2m(k)-1}), d(Bx_{2m(k)-1}, &Tx_{2m(k)-1})), d(Ax_{2n(k)}, Sx_{2n(k)}), \\ &\frac{1}{2}[d(Ax_{2n(k)}, Tx_{2m(k)-1}) + d(Sx_{2n(k)}, Bx_{2m(k)-1})]\}\big),\end{aligned}$$

or

$$\begin{aligned}[1 + p\; d(y_{2n(k)-1}, y_{2m(k)-1})]d(y_{2n(k)}, Bx_{2m(k)-1}) \leq p\; max\{&d(y_{2n(k)-1}, Tx_{2n(k)}) \\ d(y_{2m(k)-2}, y_{2m(k)-1}), d(y_{2n(k)-1}, y_{2m(k)-1})&d(y_{2m(k)-1}, y_{2n(k)})\}\end{aligned}$$

$$+ f\big(\max\{d(y_{2n(k)-1}, y_{2m(k)-1}), d(y_{2n(k)-1}, y_{2n(k)})),$$

$$d(y_{2n(k)}, y_{2n(k-1)}), \frac{1}{2}[d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)})]\}\big).$$

Letting $k \to \infty$ and using Lemma 2.2, equations (2.10) and (2.11), we have

$$[1 + p\ \epsilon]\epsilon \leq p\ \epsilon^2 + f\big(\max\{\epsilon, 0, 0, \epsilon\}\big),$$

or

$$\epsilon \leq f\big(\epsilon\big) < \epsilon,$$

a contradiction. Hence $\{y_{2n}\}$ is a Cauchy sequence.

3. Main Results

Now we present our main results. Throughout this section, suppose (X, d) denotes a complete metric space.

Theorem 3.1 Let A, B, S and T be self maps of a complete metric space (X, d) satisfying (2.1) and (2.2) and suppose one of the mappings A, B, S and T is continuous. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point q in X.

Proof. Since X is complete, it follows from Lemma 2.4 that the sequence $\{y_n\}$ converges to a point q in X. On the other hand, the sub sequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to the point q. Now suppose that A is continuous. Then the sequences $\{ASx_{2n}\}$ and $\{A^2x_{2n}\}$ converges to Aq. Since the pair $\{A, S\}$ is weakly compatible, it follows from proposition 2.8[9] that $lim_{n\to\infty} SSx_{2n} = Aq$.
Now using (2.2) we have,

$$[1{+}p\, d(SSx_{2n}, Tx_{2n+1})]d(ASx_{2n}, Bx_{2n+1}) \leq p\, max\{d(Bx_{2n+1}, Tx_{2n+1})$$

$$d(ASx_{2n}, SSx_{2n}), d(ASx_{2n}, Tx_{2n+1})d(SSx_{2n}, Bx_{2n+1})\}$$

$$+ f\big(\max\{d(SSx_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Tx_{2n+1}),$$

$$d(ASx_{2n}, SSx_{2n}), \frac{1}{2}[d(ASx_{2n}, Tx_{2n+1}) + d(SSx_{2n}, Bx_{2n+1})]\}\big).$$

Letting $n \to \infty$, we have

$$[1{+}p\, d(Aq, q)]d(Aq, q) \leq p\, max\{d(q, q)d(Aq, Aq), d(Aq, q)d(Aq, q)\}$$

$$+f\big(\max\{d(Aq, q), d(q, q)), d(Aq, Aq), \frac{1}{2}[d(Aq, q) + d(Aq, q)]\}\big),$$

or

$$d(Aq,q) \leq f\big(d(Aq,q)\big) < d(Aq,q),$$

which implies $Aq = q$.
Since $A(X) \subset T(X)$, there exists a point $u \in X$ such that $Tu = q$. Using the inequality (2.2), we have

$$[1+p\, d(SSx_{2n},Tu)]d(ASx_{2n},Bu) \leq p\, max\{d(Bu,Tu)d(ASx_{2n},SSx_{2n}),$$
$$d(ASx_{2n},Tu)d(SSx_{2n},Bu)\} + f\big(\max\{d(SSx_{2n},Tu),d(Bu,Tu),$$
$$d(ASx_{2n},SSx_{2n}),\frac{1}{2}[d(ASx_{2n},Tu)+d(SSx_{2n},Bu)]\}\big).$$

Letting $n \to \infty$, we have

$$[1+p\, d(q,q)]d(q,Bu) \leq p\, max\{d(Bu,q)d(q,q),d(q,q)d(q,Bu)\}$$
$$+ f\big(\max\{d(q,q),d(Bu,q)),d(q,q),\frac{1}{2}[d(q,q)+d(q,Bu)]\}\big),$$

or

$$d(Bq,q) \leq f\big(d(Bq,q)\big) < d(Bq,q),$$

which implies $Bq = q$.
Since the pair (B,T) is weakly compatible and $Bu = Tu = q$, we have $d(BTu,TBu) = 0$ whenever $Tu = Bu$, and so $Bq = BTu = TBu = Tq$.
From inequality (2.2), we have

$$[1+p\, d(Sx_{2n},Tq)]d(Ax_{2n},Bq) \leq p\, max\{d(Bq,Tq)d(Ax_{2n},Sx_{2n}),$$
$$d(Ax_{2n},Tq)d(Sx_{2n},Bq)\} + f\big(\max\{d(Sx_{2n},Tq),d(Bq,Tq)),$$
$$d(Ax_{2n},Sx_{2n}),\frac{1}{2}[d(Ax_{2n},Tq)+d(Sx_{2n},Bq)]\}\big).$$

Letting $n \to \infty$, we have

$$[1+p\, d(q,Tq)]d(q,Tq) \leq p\, max\{d(Bq,Tq)d(q,q),d(q,Tq)d(q,Tq)\}$$
$$+f\big(\max\{d(q,Tq),d(Tq,Tq)),d(q,q),\frac{1}{2}[d(q,Tq)+d(q,Tq)]\}\big),$$

which implies,

$$d(q,Tq) \leq f\big(d(q,Tq)\big) < d(q,Tq),$$

and so, $Tq = q = Bq$.
Similarly, since $B(X) \subset S(X)$, there exists a point $u' \in X$ such that $Su' = q$, and from (2.2), we have

$$d(Au',q) \leq f\big(d(q,Au')\big) < d(q,Au')$$

which implies that, $Su' = Au' = q$.
Since the pair (A, S) is weakly compatible, we have $d(ASu, SAu) = 0$ whenever $Au = Su$. So $q = Aq = ASu' = SAu' = Sq$. Thus q is a common fixed point of A, B, S and T. The same conclusion holds if we suppose the mapping B is continuous.

For uniqueness of common fixed point. Let A and S have another fixed point z. Then from (2.2), we have

$$[1+p\, d(Sz, Tq)]d(Az, Bq) \leq p\, max\{d(Bq, Tq)d(Az, Sz), d(Az, Tq)d(Sz, Bq)\}$$
$$+ f\big(\max\{d(Sz, Tq), d(Bq, Tq)), d(Az, Sz), \frac{1}{2}[d(Az, Tq) + d(Sz, Bq)]\}\big),$$

or

$$[1+p\, d(z, q)]d(z, q) \leq p\, d(z, q)d(z, q)\}+f\big(\max\{d(z, q), 0, 0, \frac{1}{2}[d(z, q)+d(z, q)]\}\big),$$

or

$$d(z, q) \leq f\big(d(z, q)\big) < d(z, q)$$

It follows from the above that, $z = q$. Similarly q is the unique common fixed point of B and T.

By setting $p = 0$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2 Let A, B, S and T be self maps of a complete metric space (X, d) satisfying (2.1) and

$$d(Ax, By) \leq f\big(\max\{d(Sx, Ty), d(By, Ty)), \frac{1}{2}[d(Ax, Ty)+d(Sx, By)]\}\big) \tag{3.1}$$

holds for all $x, y \in X$, where $f \in \mathcal{F}$. Suppose one of the mappings A, B, S and T is continuous. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Remark 3.3 If we take $A = B$ and $S = T = I_X$ (the identity mapping on X), then the above Corollary improves the result of Pant [8] and [9].

Again by setting $p = 0$ and $f(t) = \lambda t$; $0 < \lambda < 1$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.4 Let A, B, S and T be self maps of a complete metric space (X, d) satisfying (2.1) and

$$d(Ax, By) \leq \lambda \max\{d(Sx, Ty), d(By, Ty)), \frac{1}{2}[d(Ax, Ty) + d(Sx, By)]\} \tag{3.2}$$

holds for all $x, y \in X$. Suppose one of the mappings A, B, S and T is continuous. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Remark 3.5 If we take $A = B$ and $S = T = I_X$ (the identity mapping on X), the above Corollary improves the result of Ciric [3] and Fisher [4].

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References

[1] A. Aliouche and A. Djoudi, A general common fixed point theorem for reciprocally continuous mappings satisfying an implicit relation, *AJMAA*, **2**(2)(2005),1-7.

[2] S. S. Chang, A common fixed point theorem for commuting mappings, *Math. Japon*, **26**(1981),121-129.

[3] L. B. Ciric, A genaralization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45**(1974),267-273.

[4] B. Fisher, Common fixed points for four mappings, *Bull. Inst. Math. Acad. Sinica*, **11**(1983),103-113.

[5] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.*, **9**(1986), 771-779.

[6] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity , *Ind. J. Pure. Appl. Math.*, **29**(3)(1998), 227-238.

[7] G. Jungck, P. P. Murthy and Y. J. Cho, Compatible mappings of type (A) and common fixed points, *Math. Japonica*, **38**(2)(1993), 381-390.

[8] R. P. Pant, Common fixed points of four mappings, *Bull. Cal. Math. Soc.*, **90**(1998),281-286.

[9] R. P. Pant, Common fixed points of sequence of mappings, *Gnita*, **47**(2)(1996),43-49.

[10] V. Popa and H. K. Pathak, Common fixed point theorems for compatible mappings and compatible mappings of type (A), *Math. Bech.*, **49**(1997), 109-114.

[11] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations , *Publ. Inst. Math. (Beograd)*, **32**(46)(1982), 149-153.

[12] E. Zeidler, Nonlinear Functional Analysis and Its Application I, (Fixed Point Theorems), Springer-Verlog, 1986.

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On almost pseudo concircular Ricci symmetric manifolds

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Abstract. The object of the present paper is to study almost pseudo concircular Ricci symmetric manifolds and its decomposibility. Among others it is shown that in a decomposable almost pseudo concircular Ricci symmetric manifold one of the decompositions is Einstein and the other decomposition is concircular Ricci symmetric. The totally umbilical hypersurfaces of almost pseudo concircular Ricci symmetric manifolds are also studied.

Resumen. El objetivo del presente trabajo es estudiar variedades simétricas de Ricci casi–pseudo concirculares y su decomponibilidad. Entre otros, se demuestra que en una variedad de Ricci simétrica descomponible casi–seudo concircular una de las descomposiciones es Einstein y la otra descomposición es concircular simétrica Ricci. También se estudian las hiper–superficies totalmente umbilicales de variedades de Ricci concirculares seudo–simétricas.

1 Introduction

As an extended class of pseudo Ricci symmetric manifolds introduced by Chaki [1], recently Chaki and Kawaguchi [2] introduced the notion of almost pseudo Ricci symmetric manifolds. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X) + B(X)]S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \quad (1)$$

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where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A, B are nowhere vanishing 1-forms such that $g(X,\rho) = A(X)$ and $g(X,\mu) = B(X)$ for all X and ρ, μ are called the basic vector fields of the manifold. The 1-forms A and B are called associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. The almost pseudo Ricci symmetric manifolds have also been studied by Shaikh, Hui and Bagewadi [18].

If, in particular, $B = A$ then (1) reduces to

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \tag{2}$$

which represents a pseudo Ricci symmetric manifold [1].

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [26]. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by [26]

$$\begin{aligned} \tilde{C}(Y,Z,U,V) &= R(Y,Z,U,V) \\ &- \frac{r}{n(n-1)}\big[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big], \end{aligned} \tag{3}$$

where R and r denotes the curvature tensor and the scalar curvature of the manifold respectively.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(Y,V) = \sum_{i=1}^{n} \tilde{C}(Y, e_i, e_i, V), \tag{4}$$

then from (3), we get

$$P(Y,V) = S(Y,V) - \frac{r}{n} g(Y,V). \tag{5}$$

The tensor P is called the concircular Ricci tensor [4], which is a symmetric tensor of type (0,2). The present paper deals with a type of non-flat Riemannian manifold (M^n, g), $n > 2$ (the condition $n > 2$ is assumed throughout the paper), whose concircular Ricci tensor P is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = [A(X) + B(X)]P(Y,Z) + A(Y)P(X,Z) + A(Z)P(Y,X), \tag{6}$$

where A, B and ∇ has the same meaning as before. Such a manifold is called *almost pseudo concircular Ricci symmetric manifold* and an n-dimensional manifold of this kind is denoted by $A(P\tilde{C}RS)_n$.

The paper is organized as follows. Section 2 is devoted to the study of some basic results of $A(P\tilde{C}RS)_n$. In this section, we investigate the nature of scalar curvature of $A(P\tilde{C}RS)_n$ and it is shown that in an $A(P\tilde{C}RS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.

In 1970 Pokhariyal and Mishra [16] were introduced new tensor fields, called W_2 and E tensor fields, in a Riemannian manifold and studied their properties. According to them a W_2-curvature tensor on a manifold (M^n, g), $n > 2$, is defined by [16]

$$\begin{aligned} W_2(X,Y,Z,U) &= R(X,Y,Z,U) \\ &+ \frac{1}{n-1}\big[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)\big]. \end{aligned} \tag{7}$$

In this connection it may be mentioned that Pokhariyal and Mishra ([16], [17]) and Pokhariyal [12] introduced some new curvature tensors defined on the line of Weyl projective curvature tensor.

The W_2-curvature tensor was introduced on the line of Weyl projective curvature tensor and by breaking W_2 into skew-symmetric parts the tensor E has been defined. Rainich conditions for the existence of the non-null electrovariance can be obtained by W_2 and E, if we replace the matter tensor by the contracted part of these tensors. The tensor E enables to extend Pirani formulation of gravitational waves to Einstein space ([14], [15]). It is shown that [16] except the vanishing of complexion vector and property of being identical in two spaces which are in geodesic correspondence, the W_2-curvature tensor possesses the properties almost similar to the Weyl projective curvature tensor. Thus we can very well use W_2-curvature tensor in various physical and geometrical spheres in place of the Weyl projective curvature tensor.

The W_2-curvature tensor have also been studied by various authors in different structures such as De and Sarkar [5], Matsumoto, Ianus and Mihai [9], Pokhariyal ([13], [14], [15]), Shaikh, Jana and Eyasmin [19], Shaikh, Matsuyama and Jana [20], Taleshian and Hosseinzadeh [22], Tripathi and Gupta [23], Venkatesha, Bagewadi and Kumar [24], Yildiz and De [28] and many others.

Section 3 deals with the W_2-curvature tensor of an $A(P\tilde{C}RS)_n$. It is proved that in a W_2-conservative $A(P\tilde{C}RS)_n$, the vector field μ and ξ are co-directional, where ξ is defined by $B(QX) = g(QX, \mu) = D(X) = g(X, \xi)$. In section 4, we study decomposable $A(P\tilde{C}RS)_n$ and it is shown that in a decomposable $A(P\tilde{C}RS)_n$ one of the decompositions is concircular Ricci flat and the other decomposition is concircular Ricci symmetric.

Recently Özen and Altay [10] studied the totally umbilical hypersurfaces of weakly and pseudosymmetric spaces. Again Özen and Altay [11] also studied the totally umbilical hypersurfaces of weakly concircular and pseudo concircular symmetric spaces. In this connection it may be mentioned that Shaikh, Roy and Hui [21] studied the totally umbilical hypersurfaces of weakly conharmonically symmetric spaces. Section 5 deals with the study of totally umbilical hypersurfaces of $A(P\tilde{C}RS)_n$. It is proved that the totally umbilical hypersurface of an $A(P\tilde{C}RS)_n$ is also an $A(P\tilde{C}RS)_n$.

2 Some basic results of $A(P\tilde{C}RS)_n$

Let Q be the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S. Then $g(QX,Y) = S(X,Y)$ for all vector fields X, Y.
Using (5) in (6), we get

$$\begin{aligned}(\nabla_X S)(Y,Z) - \frac{dr(X)}{n}g(Y,Z) &= [A(X)+B(X)][S(Y,Z) - \frac{r}{n}g(Y,Z)] \\ &+ A(Y)[S(X,Z) - \frac{r}{n}g(X,Z)] \\ &+ A(Z)[S(Y,X) - \frac{r}{n}g(Y,X)].\end{aligned} \tag{8}$$

Setting $Y = Z = e_i$ in (8) and then taking summation over i, $1 \leq i \leq n$, we obtain

$$A(QX) = \frac{r}{n}A(X), \tag{9}$$

i.e.

$$S(X,\rho) = \frac{r}{n}g(X,\rho). \tag{10}$$

This leads to the following:

Proposition 2.1. *In an $A(P\tilde{C}RS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.*

Also from (6), we get

$$(\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z) = B(X)P(Y,Z) - B(Y)P(X,Z). \tag{11}$$

Contracting (11) over Y and Z and using (5), we get

$$\frac{n-2}{2n}dr(X) = B(QX) - \frac{r}{n}B(X). \tag{12}$$

If the scalar curvature r is constant, then

$$dr(X) = 0. \tag{13}$$

By virtue of (13), (12) yields

$$B(QX) = \frac{r}{n}B(X), \tag{14}$$

i.e.

$$S(X,\mu) = \frac{r}{n}g(X,\mu). \tag{15}$$

In the other way, we assume that the concircular Ricci tensor of this manifold is Codazzi type [7] then we have

$$(\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z) = 0. \tag{16}$$

Using (16) in (11), we get

$$B(X)P(Y,Z) - B(Y)P(X,Z) = 0. \tag{17}$$

Contracting (17) over Y and Z and using (5), we also obtain

$$B(QX) = \frac{r}{n}B(X). \tag{18}$$

This leads to the following:

Proposition 2.2. *In an $A(P\tilde{C}RS)_n$, if*
(i) *the scalar curvature is constant or*
(ii) *the concircular Ricci tensor is Codazzi type*
then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector μ.

Definition 2.1. *A Riemannian manifold is said to admit cyclic parallel concircular Ricci tensor if*

$$(\nabla_X P)(Y,Z) + (\nabla_Y P)(Z,X) + (\nabla_Z P)(X,Y) = 0. \tag{19}$$

By virtue of (6), (19) yields

$$\lambda(X)P(Y,Z) + \lambda(Y)P(X,Z) + \lambda(Z)P(X,Y) = 0, \tag{20}$$

where $\lambda(X) = 3A(X) + B(X) = g(X,\sigma)$ for all X.
Contracting (20) over Y and Z, we get

$$\lambda(QX) = \frac{r}{n}\lambda(X), \tag{21}$$

i.e.

$$S(X,\sigma) = \frac{r}{n}g(X,\sigma). \tag{22}$$

This leads to the following:

Proposition 2.3. *In an $A(P\tilde{C}RS)_n$ with cyclic parallel concircular Ricci tensor, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector σ defined by $g(X,\sigma) = \lambda(X) = 3A(X) + B(X)$ for all X.*

In terms of local coordinates, the relation (20) can be written as

$$\lambda_i P_{jk} + \lambda_j P_{ki} + \lambda_k P_{ji} = 0. \tag{23}$$

Next, we consider a lemma, which is as follows:

Lemma 2.1. (Walker's Lemma) [25] *If a_{ij} and b_i are numbers satisfying $a_{ij} = a_{ji}$, $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ for $i,\ j,\ k = 1, 2, \cdots, n$, then either all a_{ij} are zero or all the b_i are zero.*

By virtue of Lemma 2.1, it follows from (23) that either $\lambda_k = 0$ or $P_{ij} = 0$. Also by definition of $A(P\tilde{C}RS)_n$, $P_{ij} \neq 0$ and hence $\lambda_k = 0$, i.e.

$$3A_k + B_k = 0. \tag{24}$$

This leads to the following:

Proposition 2.4. *In an $A(P\tilde{C}RS)_n$ with cyclic parallel concircular Ricci tensor, the 1-forms A and B are related in the form* (24).

We assume that $A(P\tilde{C}RS)_n$ is conformally flat. In a conformally flat Riemannian manifold, the following condition holds [6]

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}\big[dr(X)g(Y,Z) - dr(Y)g(X,Z)\big]. \tag{25}$$

From (5) and (25) we find

$$(\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z) = \frac{2-n}{2(n-1)}\big[dr(X)g(Y,Z) - dr(Y)g(X,Z)\big]. \tag{26}$$

If the concircular Ricci tensor is Codazzi type then from (26) we have

$$dr(X)g(Y,Z) - dr(Y)g(X,Z) = 0 \tag{27}$$

Contracting on Y and Z in (27) we obtain that the scalar curvature r is constant. Conversely, from (26), if the manifold is of constant curvature then the concircular Ricci tensor of this manifold is Codazzi type. Thus we can state the following theorem:

Theorem 2.1. *In a conformally flat $A(P\tilde{C}RS)_n$, the concircular Ricci tensor of this manifold is Codazzi type if and only if the scalar curvature of this manifold is constant.*

3 W_2-curvature tensor of an $A(P\tilde{C}RS)_n$

Let us consider a W_2-flat Riemannian manifold. Then from (7), we have

$$R(X,Y,Z,U) + \frac{1}{n-1}\big[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)\big] = 0. \tag{28}$$

Contracting (28) over X and U, we get $P(Y,Z) = 0$ and hence the manifold is not $A(P\tilde{C}RS)_n$. Thus we can state the following:

Theorem 3.1. *There does not exist W_2-flat $A(P\tilde{C}RS)_n$.*

From (7), we obtain

$$\begin{aligned}(div\ W_2)(X,Y)Z &= (div\ R)(X,Y)Z \\ &+ \frac{1}{2(n-1)}\big[dr(Y)g(X,Z) - dr(X)g(Y,Z)\big],\end{aligned} \tag{29}$$

where 'div' denotes the divergence.
Again it is known that in a Riemannian manifold, we have

$$(div\ R)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{30}$$

Consequently by virtue of (30), (29) takes the form

$$\begin{aligned}(div\ W_2)(X,Y)Z &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &+ \frac{1}{2(n-1)}\big[dr(Y)g(X,Z) - dr(X)g(Y,Z)\big].\end{aligned} \tag{31}$$

Let us consider a W_2-conservative $A(P\tilde{C}RS)_n$. Then we have [8]

$$(div\ W_2)(X,Y)Z = 0 \tag{32}$$

and hence (31) yields

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)]. \tag{33}$$

By virtue of (5), (11) and (12), it follows from (33) that

$$\begin{aligned}B(X)P(Y,Z) - B(Y)P(X,Z) &= -\frac{1}{n-1}\Big[\big\{B(QX) - \frac{r}{n}B(X)\big\}g(Y,Z) \\ &- \big\{B(QY) - \frac{r}{n}B(Y)\big\}g(X,Z)\Big].\end{aligned} \tag{34}$$

Putting $Z = \mu$ in (34) we obtain

$$B(X)B(QY) - B(Y)B(QX) = 0. \tag{35}$$

Let $B(QX) = g(QX, \mu) = D(X) = g(X, \xi)$ for all X. Then from (35), we get

$$B(X)D(Y) = B(Y)D(X), \tag{36}$$

which implies that the vector field μ and ξ are co-directional. This leads to the following:

Theorem 3.2. *In a W_2-conservative $A(P\tilde{C}RS)_n$, the vector field μ and ξ are co-directional.*

From (10), we have

$$P(X, \rho) = 0. \tag{37}$$

Setting $Z = \rho$ in (34) and using (37), we get

$$A(Y)D(X) - A(X)D(Y) = \frac{r}{n}\big\{A(Y)B(X) - A(X)B(Y)\big\}. \tag{38}$$

Let the scalar curvature $r \neq 0$. Then (38) implies that the vector fields ρ and ξ are co-directional if and only if the vector fields ρ and μ are co-directional. Thus we can state the following:

Theorem 3.3. *In a W_2-conservative $A(P\tilde{C}RS)_n$ with non-zero scalar curvature, the vector fields ρ and ξ are co-directional if and only if the vector fields ρ and μ are co-directional.*

4 Decomposable $A(P\tilde{C}RS)_n$

A Riemannian manifold (M^n, g) is said to be decomposable manifold [27] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$, that is, in some coordinate neighbourhood of the Riemannian manifold (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + \overset{*}{g}_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{39}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \cdots, x^p (p < n)$ denoted by $\bar{x}$ and $\overset{*}{g}_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \cdots, x^n$ denoted by $\overset{*}{x}$; $a, b, c, \cdots$ run from 1 to p and $\alpha, \beta, \gamma, \cdots$ run from $p{+}1$ to n. The two parts of (39) are the metrics of $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ which are called the decompositions of the decomposable manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$.

Let (M^n, g) be a decomposable Riemannian manifold such that $M^n = M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$. Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by a 'star' is assumed to be from M_2.

Let $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{U}$, $\bar{V} \in \chi(M_1)$ and $\overset{*}{X}$, $\overset{*}{Y}$, $\overset{*}{Z}$, $\overset{*}{U}$, $\overset{*}{V} \in \chi(M_2)$, $\chi(M_i)$ being the Lie algebra of smooth vector fields on M_i, $i = 1, 2$. Let R, $\bar{R}$ and $\overset{*}{R}$ (resp. S, $\bar{S}$ and $\overset{*}{S}$) be the curvature tensor (resp. Ricci tensor) of the manifold M, M_1 and M_2 respectively. Also let P, $\bar{P}$ and $\overset{*}{P}$ be the concircular Ricci tensor of M, M_1 and M_2 respectively. Then we have the following relations [27]:

$$
\begin{aligned}
&R(\overset{*}{X}, \bar{Y}, \bar{Z}, \bar{U}) = 0 = R(\bar{X}, \overset{*}{Y}, \bar{Z}, \overset{*}{U}) = R(\bar{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
&(\nabla_{\overset{*}{X}} R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}) = (\nabla_{\overset{*}{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}),\\
&R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}); \quad R(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}) = \overset{*}{R}(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
&S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}); \quad S(\overset{*}{X}, \overset{*}{Y}) = \overset{*}{S}(\overset{*}{X}, \overset{*}{Y}),\\
&(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}); \quad (\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}),\\
&P(\bar{X}, \bar{Y}) = \bar{P}(\bar{X}, \bar{Y}); \quad P(\overset{*}{X}, \overset{*}{Y}) = \overset{*}{P}(\overset{*}{X}, \overset{*}{Y}),\\
&(\nabla_{\bar{X}} P)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} P)(\bar{Y}, \bar{Z}); \quad (\nabla_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}),\\
&r = \bar{r} + \overset{*}{r},
\end{aligned}
$$

where r, $\bar{r}$, and $\overset{*}{r}$ are the scalar curvature of M, M_1, M_2 respectively.

Let us consider a Riemannian manifold (M^n, g) which is decomposable $A(P\tilde{C}RS)_n$. Then $M^n = M_1^p \times M_2^{n-p}$, $(2 \leq p \leq n-2)$.
Now from (6), we find

$$(\nabla_{\bar{X}} P)(\bar{Y}, \bar{Z}) = [A(\bar{X}) + B(\bar{X})]P(\bar{Y}, \bar{Z}) + A(\bar{Y})P(\bar{X}, \bar{Z}) + A(\bar{Z})P(\bar{Y}, \bar{X}), \quad (40)$$

$$(\nabla_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}) = [A(\overset{*}{X}) + B(\overset{*}{X})]P(\overset{*}{Y}, \overset{*}{Z}) + A(\overset{*}{Y})P(\overset{*}{X}, \overset{*}{Z}) + A(\overset{*}{Z})P(\overset{*}{Y}, \overset{*}{X}), \quad (41)$$

$$[A(\overset{*}{X}) + B(\overset{*}{X})]P(\bar{Y}, \bar{Z}) = 0, \quad (42)$$

$$A(\overset{*}{Z})P(\bar{Y}, \bar{X}) = 0. \quad (43)$$

From (42) and (43), it follows that either M_1 is concircular Ricci flat, i.e., Einstein or $A = 0$, $B = 0$ on M_2 and hence from (41), we have $(\nabla_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}) = 0$, i.e., M_2 is concircular Ricci symmetric. Similarly it can be easily shown that either M_2 is Einstein or M_1 is concircular Ricci symmetric. Thus we can state the following:

Theorem 4.1. *Let (M^n, g) be a Riemannian manifold such that $M^n = M_1^p \times M_2^{n-p}$, $(2 \leq p \leq n-2)$. If (M^n, g) is a $A(P\tilde{C}RS)_n$, then either*
(1) M_1 (resp. M_2) is Einstein or
(2) M_2 (resp. M_1) is concircular Ricci symmetric.

5 Totally umbilical hypersurfaces of $A(P\tilde{C}RS)_n$

Let $(\bar{V}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let (V, g) be a hypersurface of $(\bar{V}, \bar{g})$ defined in a locally coordinate system by means of a system of parametric equation $y^\alpha = y^\alpha(x^i)$, where Greek indices take values $1, 2, \cdots, n$ and Latin indices take values $1, 2, \cdots, (n+1)$. Let N^α be the components of a local unit normal to (V, g). Then we have

$$g_{ij} = \bar{g}_{\alpha\beta} y_i^\alpha y_j^\beta, \tag{44}$$

$$\bar{g}_{\alpha\beta} N^\alpha y_j^\beta = 0, \quad \bar{g}_{\alpha\beta} N^\alpha N^\beta = e = 1, \tag{45}$$

$$y_i^\alpha y_j^\beta g^{ij} = \bar{g}^{\alpha\beta} - N^\alpha N^\beta, \quad y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}. \tag{46}$$

The hypersurface (V, g) is called a totally umbilical hypersurface ([3],[6]) of $(\bar{V}, \bar{g})$ if its second fundamental form Ω_{ij} satisfies

$$\Omega_{ij} = H g_{ij}, \quad y_{i,j}^\alpha = g_{ij} H N^\alpha, \tag{47}$$

where the scalar function H is called the mean curvature of (V, g) given by $H = \frac{1}{n} \sum g^{ij} \Omega_{ij}$. If, in particular, $H = 0$, i.e.,

$$\Omega_{ij} = 0, \tag{48}$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\bar{V}, \bar{g})$.
The equation of Weingarten for (V, g) can be written as $N_{,j}^\alpha = -\frac{H}{n} y_j^\alpha$. The structure equations of Gauss and Codazzi ([3],[6]) for (V, g) and $(\bar{V}, \bar{g})$ are respectively given by

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} B_{ijkl}^{\alpha\beta\gamma\delta} + H^2 G_{ijkl}, \tag{49}$$

$$\bar{R}_{\alpha\beta\gamma\delta} B_{ijk}^{\alpha\beta\gamma} N^\delta = H_{,i}\ g_{jk} - H_{,j}\ g_{ik}, \tag{50}$$

where R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\delta}$ are curvature tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively, and

$$B^{\alpha\beta\gamma\delta}_{ijkl} = B^{\alpha}_{i} B^{\beta}_{j} B^{\gamma}_{k} B^{\delta}_{l}, \quad B^{\alpha}_{i} = y^{\alpha}_{i}, \quad G_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl}. \tag{51}$$

Also we have ([3],[6])

$$\bar{S}_{\alpha\delta} B^{\alpha}_{i} B^{\delta}_{j} = S_{ij} - (n-1) H^2 g_{ij}, \tag{52}$$

$$\bar{S}_{\alpha\delta} N^{\alpha} B^{\delta}_{i} = (n-1) H_{,i}\ , \tag{53}$$

$$\bar{r} = r - n(n-1) H^2, \tag{54}$$

where S_{ij} and $\bar{S_{\alpha\delta}}$ are the Ricci tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively and r and $\bar{r}$ are the scalar curvatures of (V, g) and $(\bar{V}, \bar{g})$ respectively.
By virtue of (52) and (54), we have

$$\bar{P}_{\alpha\delta} B^{\alpha}_{i} B^{\delta}_{j} = P_{ij}, \tag{55}$$

where P_{ij} and $\bar{P_{\alpha\delta}}$ are the concircular Ricci tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively.
In terms of local coordinates the relation (6) can be written as

$$P_{ij,k} = (A_k + B_k) P_{ij} + A_i P_{jk} + A_j P_{ki}. \tag{56}$$

Let $(\bar{V}, \bar{g})$ be an $A(P\tilde{C}RS)_n$. Then we get

$$\bar{P}_{\alpha\beta,\gamma} = (A_\gamma + B_\gamma) \bar{P}_{\alpha\beta} + A_\alpha \bar{P}_{\gamma\beta} + A_\beta \bar{P}_{\alpha\gamma}, \tag{57}$$

where A and B are nowhere vanishing 1-forms.
Multiplying both sides of (57) by $B^{\alpha\beta\gamma}_{ijk}$ and then using (55), we obtain the relation (56). Hence we can state the following:

Theorem 5.1. *The totally umbilical hypersurface of an* $A(P\tilde{C}RS)_n$ *is also an* $A(P\tilde{C}RS)_n$.

Corollary 5.1. *The totally geodesic hypersurface of an* $A(P\tilde{C}RS)_n$ *is also an* $A(P\tilde{C}RS)_n$.

References

[1] Chaki, M. C., *On pseudo Ricci symmetric manifolds*, Bulg. J. Phys., **15** (1988), 526–531.

[2] Chaki, M. C. and Kawaguchi, T., *On almost pseudo Ricci symmetric manifolds*, Tensor N. S., **68** (2007), 10–14.

[3] Chen, B. Y., *Geometry of submanifolds*, Marcel-Deker, New York, **1973**.

[4] De, U. C. and Ghosh, G. C., *On weakly concircular Ricci symmetric manifolds*, South East Asian J. Math. and Math. Sci., **3(2)** (2005), 9–15.

[5] De, U. C. and Sarkar, A., *On a type of P-Sasakian manifolds*, Math. Reports, **11(61)** (2009), 139–144.

[6] Eisenhart, L. P., *Riemannian Geometry*, Princeton University Press, **1949**.

[7] Ferus, D., *A remark on Codazzi tensors on constant curvature space*, Lecture Notes Math., **838**, Global Differential Geometry and Global Analysis, Springer-Verlag, New York, **1981**.

[8] Hicks, N. J., *Notes on differential geometry*, Affiliated East West Pvt. Ltd., **1969**.

[9] Matsumoto, K., Ianus, S. and Mihai, I., *On P-Sasakian manifolds which admit certain tensor fields*, Publ. Math. Debrecen, **33** (1986), 61–65.

[10] Özen, F. and Altay, S., *On weakly and pseudo symmetric Riemannian spaces*, Indian J. Pure Appl. Math., **33(10)** (2001), 1477–1488.

[11] Özen, F. and Altay, S., *On weakly and pseudo concircular symmetric structures on a Riemannian manifold*, Acta Univ. Palacki. Olomuc. Fac. rer. nat. Math., **47** (2008), 129–138.

[12] Pokhariyal, G. P., *Curvature tensors and their relativistic significance III*, Yokohama Math. J., **20** (1972), 115–119.

[13] Pokhariyal, G. P., *Study of a new curvature tensor in a Sasakian manifold*, Tensor N. S., **36** (1982), 222–225.

[14] Pokhariyal, G. P., *Relative significance of curvature tensors*, Int. J. Math. and Math. Sci., **5** (1982), 133–139.

[15] Pokhariyal, G. P., *Curvature tensors on A-Einstein Sasakian manifolds*, Balkan J. Geom. Appl., **6** (2001), 45–50.

[16] Pokhariyal, G. P. and Mishra, R. S., *The curvature tensor and their relativistic significance*, Yokohoma Math. J., **18** (1970), 105–108.

[17] Pokhariyal, G. P. and Mishra, R. S., *Curvature tensor and their relativistic significance II*, Yokohoma Math. J., **19** (1971), 97–103.

[18] Shaikh, A. A., Hui, S. K. and Bagewadi, C. S., *On quasi-conformally flat almost pseudo Ricci symmetric manifolds*, Tamsui Oxford J. Math. Sci., **26(2)** (2010), 203–219.

[19] Shaikh, A. A., Jana, S. K. and Eyasmin, S., *On weakly W_2-symmetric manifolds*, Sarajevo J. Math., **3(15)** (2007), 73–91.

[20] Shaikh, A. A., Matsuyama, Y. and Jana, S. K., *On a type of general relativistic spacetime with W_2-curvature tensor*, Indian J. Math., **50** (2008), 53–62.

[21] Shaikh, A. A., Roy, I. and Hui, S. K., *On totally umbilical hypersurfaces of weakly conharmonically symmetric spaces*, Global J. Science Frontier Research, **10(4)** (2010), 28–30.

[22] Taleshian, A. and Hosseinzadeh, A. A., *On W_2-curvature tensor $N(k)$-quasi Einstein manifolds*, J. Math. and Computer Science, **1(1)** (2010), 28–32.

[23] Tripathi, M. M. and Gupta, P., *On τ-curvature tensor in K-contact and Sasakian manifolds*, Int. Elec. J. Geom., **4** (2011), 32–47.

[24] Venkatesha, Bagewadi, C. S. and Kumar, K. T. Pradeep, *Some results on Lorentzian Para-Sasakian manifolds*, International scholarly research network, doi:10.5402/2011/161523.

[25] Walker, A. G., *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc., **52** (1950), 36–54.

[26] Yano, K., *Concircular geometry I*, Proc. Imp. Acad. Tokyo, **16** (1940), 195–200.

[27] Yano, K. and Kon, M., *Structure on manifolds*, World Scientific Publ., Singapore, **1986**.

[28] Yildiz, A. and De, U. C., *On a type of Kenmotsu manifolds*, Diff. Geom.-Dynamical Systems, **12** (2010), 289–298.

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DIVULGACIÓN MATEMÁTICA

A simple application of the implicit function theorem*

Germán Lozada-Cruz

Abstract. In this note we show that the roots of a polynomial are C^∞ depend of the coefficients. The main tool to show this is the Implicit Function Theorem.

Resumen. En esta nota se muestra el hecho que las raíces de un polinomio son C^∞, depende de los coeficientes. La herramienta principal para mostrar este resultado es el teorema de la función implícita.

1 Motivation

Quadratic equations appear already in antiquity (Babylon, 2000 BC) and the solution of the general equation, even when it appears in different forms in the Euclid's Elements treated by geometric arguments, is obtained algebraically by the Arab mathematician al-Khowarizmi. He is credited with being the first to solve the quadratic equation

$$ax^2 + bx + c = 0, \tag{1}$$

where a, b and c are real numbers with $a \neq 0$.

Denoting by $\Delta = b^2 - 4ac$, we have the the zeros of (1) are given by

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}. \tag{2}$$

Thus, we have
i) distinct roots if $\Delta > 0$
ii) equals roots if $\Delta = 0$ and

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iii) complex roots if $\Delta < 0$.

In the case i) we say that the roots of (1) are simple and in the case ii) we say that the root of (1) is multiple.

A natural question that arises is:

Question 1.1 *If we change slightly the coefficients a, b and c in* (1), *what happens with the root x given by* (1.2)*?*

First, let's look at two simple examples:

Example 1.2 *Find the roots of the following polynomial p given by*

$$p(x) = x^2 + x - 6. \tag{3}$$

Using the formula (2) *we obtain $x = 2$ and $x = -3$. This is the polynomial p has real and distinct roots.*

Example 1.3 *Find the roots of the following polynomial p given by*

$$p(x) = (1,01)x^2 + 0,99x - 6,01. \tag{4}$$

Using the formula (2) *we obtain $x = 1,99$ and $x = -2,97$. This is the polynomial p has real and distinct roots.*

From (3) and (4) we can see that if we change slightly the coefficients of the quadratic function then the roots will also change slightly

With the intention of answering this question, we look to the right side of equation (2) as a function that depends on the coefficients a, b and c. This is

$$x = x(a, b, c). \tag{5}$$

Thus, if we show that x as a function of the coefficients a, b and c is a continuous function, this answer our question posed above.

More generally consider a polynomial function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i, \tag{6}$$

where $a_j \in \mathbb{R}$, $j = 0, 1, \cdots, n$.

The aim of this short note is to show that the simple real zeros of the equation (6) depend C^∞ of the parameter $\lambda = (a_0, a_1, \cdots, a_n) \in \mathbb{R}^{n+1}$ formed by the coefficients of p.

Definition 1.1 *We say that* $c \in \mathbb{R}$ *is a zero of* p *if* $p(c) = 0$.

Theorem 1.1 *If* c *is a zero of* p *then there exists a polinomial function* q *such that*

$$p(x) = (x - c)q(x). \tag{7}$$

Proof. If c is a zero of p then for all x we have

$$p(x) = p(x) - p(c) = \sum_{i=0}^{n} a_i(x^i - c^i) = (x - c)q(x).$$

□

Definition 1.2 *We say that* c *is a simple zero of* p *if* $q(c) \neq 0$.

If $q(c) = 0$ then $q(x) = (x - c)r(x)$, thus $p(x) = (x - c)^2 r(x)$, where r is another polinomial function.

Deriving the equality (7) we have

$$p'(x) = q(x) + (x - c)q'(x). \tag{8}$$

The following result is easy to prove

Theorem 1.2 *The number* c *is simple zero of* p *if only if* $p'(c) \neq 0$.

Proof. Follows from (8). □

2 C^∞ dependence of the zeros of polynomials with respect to the coefficients

The main result of this short note is the following result which is the positive answer to the Question 1.1

Theorem 2.1 *If* c *is a simple zero of* p *then* c *is a function of class* C^∞ *of the coeficients* $a_0, a_1, \cdots, a_n$ *of the polinomial* p.

Proof. Let us define the following function

$$\begin{array}{rcll} f & : \mathbb{R} \times \mathbb{R}^{n+1} & \to & \mathbb{R} \\ & (x, \lambda) & \to & f(x, \lambda) = \sum_{i=0}^{n} a_i x^i \end{array}$$

where $\lambda = (a_0, \cdots, a_n)$.

Clearly we see that f is C^∞ class in the variables x and λ.

Suppose c_0 is a simple zero of polynomial function p^0 corresponding to the particular case $\lambda_0 = (a_0^0, a_1^0, \cdots, a_n^0) \in \mathbb{R}^{n+1}$ then

- $f(c_0, \lambda_0) = \sum\limits_{i=0}^{n} a_i^0 (c_0)^i = p^0(c_0) = 0$ and
- $\dfrac{\partial f}{\partial x}(c_0, \lambda_0) = \dfrac{\partial}{\partial x}\big(\sum\limits_{i=0}^{n} a_i x^i\big)\big|_{(c_0,\lambda_0)} = (p^0)'(c_0) \neq 0.$

By the implicit function theorem (Theorem 3.1) there are open neighborhoods $U \subset \mathbb{R}$ of c_0 and $V \subset \mathbb{R}^{n+1}$ of λ_0 such that for all $\lambda \in V$ there exists an unique $c \in U$ with $f(c, \lambda) = 0$.

Thus, we have a unique map

$$\begin{aligned} c : V &\to U \\ \lambda &\mapsto c(\lambda) \end{aligned}$$

such that $f(c(\lambda), \lambda) = 0$. This tells us that the polynomial function p corresponding to the value of the parameter $\lambda = (a_0, \cdots, a_n) \in \mathbb{R}^{n+1}$ near λ_0, also has exactly a simple zero $c(\lambda) \in \mathbb{R}$ near to c_0.

Since $f \in C^k$ for all $k \in \mathbb{N}$, the function $c(\cdot) \in C^k$ for all $k \in \mathbb{N}$. This tells us that the simple zeros of p are C^∞ dependent of the coefficients. Moreover, the following formula holds

$$\frac{\partial c}{\partial \lambda}(\lambda) = -\left[\frac{\partial f}{\partial c}(c(\lambda), \lambda)\right]^{-1} \cdot \frac{\partial f}{\partial \lambda}(c(\lambda), \lambda).$$

□

Remark 2.1 *We believe that this simple application of the implicit function theorem is appropriate for the undergraduate classroom or homework. Also, using this same theorem we can conclude that the eigenvalues of a matrix* $A = (a_{ij})_{n\times n}$ *depend continuously of the coefficients* a_{ij}.

Remark 2.2 *This positive result should be contrasted with Abel's theorem, which in algebra states that there is not a formula for the zeros of a polynomial of degree* $n \geqslant 5$ *in terms of coefficients.*

Remark 2.3 *For polynomials with more than one variable we have analytical dependence of the zeros, for more details see [2], p.* 362.

3 Implicit function theorem

For completeness of this note we states here the Implicit function theorem. More information about the history, theory and applications of this theorem can be found in [1].

Denoting by

$$\mathcal{L}(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n; T \text{ is linear}\}$$
$$\mathrm{Aut}(\mathbb{R}^n) = \{T \in \mathcal{L}(\mathbb{R}^n) : T \text{ is a bijection}\}.$$

Theorem 3.1 (Implicit Theorem) *Let $W \subset \mathbb{R}^n \times \mathbb{R}^p$ a open set and $f \in C^k(W, \mathbb{R}^n)$. Suppose that for some $(x_0, y_0) \in W$, $f(x_0, y_0) = 0$ and $D_x f(x_0, y_0) \in \mathrm{Aut}(\mathbb{R}^n)$. Then there are open neighborhoods $U \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}^p$ of y_0 such that for all $y \in V$ there exists a unique $x \in U$ with $f(x, y) = 0$.*

Thus we have a unique map

$$\psi : V \to U$$
$$y \mapsto \psi(y)$$

with $\psi(y) = x$ and $f(\psi(y), y) = 0$. Moreover, the derivative of ψ in y, $D\psi(y) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$ is given by

$$D\psi(y) = -[D_x f(\psi(y), y)]^{-1} \circ D_y f(\psi(y), y), \quad y \in V.$$

Proof. See [1]. □

References

[1] Krantz, Steven G.; Parks, Harold R. *The Implicit function theorem. History, theory, and applications.* Birkhäuser Boston, Inc., Boston, MA, 2002

[2] Hörmander, Lars, *The analysis of linear partial differential operators. II. Differential operators with constant coefficients.* Reprint of the 1983 original. Classics in Mathematics. Springer-Verlag, 2005.

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Un estudio geométrico del grupo S_4

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Abstract. We present a geometric study of the symmetry group of the tetrahedron constructing its elements explicitly as transformations in a tridimensional Euclidean space. We also give all its subgroups and linear representations.

Resumen. En este trabajo presentamos un estudio geomtrico del grupo de simetrías del tetraedro construyendo explicitamente sus elementos como transformaciones en el espacio euclídeo tridimensional. Así mismo, también damos todos sus subgrupos y representaciones lineales.

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1 Introducción

Es considerable la cantidad de grupos que aparecen como grupos de isometrías de una figura geometrica regular. Por otro lado, es frecuente en el estudio del grupo simétrico S_3, recurrir a una interpretación geometrica del mismo por medio de las transformaciones geométricas del plano euclideo que conservan la figura (véase [2]). Sin embargo, en el paso inmediatamente superior (S_4) si bien aparece comentado en algunas referencias (véase [1]) no es habitual encontrar un tratamiento geométrico detallado.

En este trabajo realizamos un estudio original y completo del grupo de isometrías del tetraedro encuadrado en el contexto del álgebra geométrica, frente a las descripciones sintéticas de sus elementos que se pueden encontrar en las referencias existentes.

En lo que sigue, hacemos una descripción y estudio de la estructura del grupo de isometrías del tetraedro, clasificando geometricamente sus elementos y explicitando todos sus subgrupos. Para esto construimos en primer lugar un sistema de referencia adaptado a la figura.

2 Unos ejes para el tetraedro

Sean A, B, C, D los cuatro vértices de un tetraedro regular con longitud L en cada una de sus 6 aristas.

Partimos de unos ejes de manera que
el origen esté en el vértice B,
el eje de las x's contenga al lado $[B, C]$,
el plano $z = 0$ contenga a la cara BCD.
Calculando la altura desde el lado $[B, C]$ hasta el vértice D sale el número $\sqrt{3}L/2$, lo que permite escribir

$$B = (0, 0, 0), C = (L, 0, 0), D = (L/2, \sqrt{3}L/2, 0).$$

El vector

$$H = \frac{B + C + D}{3} = (L/2, \sqrt{3}L/6, 0)$$

sitúa al baricentro de la base BCD, el cual se encuentra a una distancia $\sqrt{3}L/3$ de sus vértices. Entonces, la altura del tetraedro mide $\sqrt{6}L/3$, con lo que

$$A = (L/2, \sqrt{3}L/6, \sqrt{6}L/3).$$

Finalmente, el baricentro G del tetraedro vendrá fijado por

$$G = \frac{A + B + C + D}{4} = (L/2, \sqrt{3}L/6, \sqrt{6}L/12).$$

Hacemos ahora una traslación de ejes para dejar al punto G como origen. En esta referencia se tendrán, para los 4 vértices, las coordenadas

$$A = (0, 0, \sqrt{6}L/4) = (0, 0, 3\alpha\beta\gamma),$$

$$B = (-L/2, -\sqrt{3}L/6, -\sqrt{6}L/12) = (-6\gamma, -2\beta\gamma, -\alpha\beta\gamma),$$

$$C = (L/2, -\sqrt{3}L/6, -\sqrt{6}L/12) = (6\gamma, -2\beta\gamma, -\alpha\beta\gamma),$$

$$D = (0, \sqrt{3}L/3, -\sqrt{6}L/12) = (0, 4\beta\gamma, -\alpha\beta\gamma),$$

donde $\alpha = \sqrt{2}, \beta = \sqrt{3}, \gamma = L/12$.

3 Ejes de simetría

Para los puntos medios de cada una de las aristas, se tiene

$$(A+B)/2 = (-3\gamma, -\beta\gamma, \alpha\beta\gamma)$$

$$(C+D)/2 = (3\gamma, \beta\gamma, -\alpha\beta\gamma)$$

$$(A+C)/2 = (3\gamma, -\beta\gamma, \alpha\beta\gamma)$$

$$(B+D)/2 = (-3\gamma, \beta\gamma, -\alpha\beta\gamma)$$

$$(A+D)/2 = (0, 2\beta\gamma, \alpha\beta\gamma)$$

$$(B+C)/2 = (0, -2\beta\gamma, -\alpha\beta\gamma)$$

Los dos primeros son simétricos respecto del origen y determinan una recta b con vector director unitario

$$\mathbf{w}_1 = \frac{1}{\alpha\beta}(-\beta, -1, \alpha)$$

Las rectas $< A, B >$ y $< C, D >$, dirigidas por los vectores

$$< A, B >: (-6\gamma, -2\beta\gamma, -4\alpha\beta\gamma), < C, D >: (-6\gamma, 6\beta\gamma, 0)$$

son ambas perpendiculares a b, luego en la simetría axial de eje b se intercambian A con D y B con C, lo que nos indican que b es un eje de simetría para el tetraedro. Igual ocurre con la segunda pareja, que determinan una recta c con vector director unitario

$$\mathbf{w}_2 = \frac{1}{\alpha\beta}(\beta, -1, \alpha)$$

la cual es eje de simetría que cambia A en C y B en D. Finalmente, la tercera pareja determina una recta d con vector director unitario

$$\mathbf{w}_3 = \frac{1}{\alpha\beta}(0, 2, \alpha)$$

eje de simetría que aplica A en D y B en C.

4 Referencia canónica para el tetraedro

Se observa enseguida que la terna $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ es ortonormal de orientación positiva. Manteniendo como origen el baricentro G, tenemos un sistema de referencia *canónico* para el tetraedro. Las matrices de cambio desde el anterior son las

$$P = \frac{1}{\alpha\beta}\begin{bmatrix} -\beta & \beta & 0 \\ -1 & -1 & 2 \\ \alpha & \alpha & \alpha \end{bmatrix}, P^{-1} = P^t = \frac{1}{\alpha\beta}\begin{bmatrix} -\beta & -1 & \alpha \\ \beta & -1 & \alpha \\ 0 & 2 & \alpha \end{bmatrix}$$

Usando la inversa, pasamos de las viejas coordenadas a las nuevas, resultando, para los vértices, que

$$A = \delta(1, 1, 1)$$
$$B = \delta(1, -1, -1)$$
$$C = \delta(-1, 1, -1)$$
$$D = \delta(-1, -1, 1)$$

donde hemos escrito $\delta = 3\alpha\gamma = \sqrt{2}L/4$. Las de los puntos medios de las aristas pasan a ser

$$(A + B)/2 = \delta(1, 0, 0)$$
$$(C + D)/2 = \delta(-1, 0, 0)$$
$$(A + C)/2 = \delta(0, 1, 0)$$
$$(B + D)/2 = \delta(0, -1, 0)$$
$$(A + D)/2 = \delta(0, 0, 1)$$
$$(B + C)/2 = \delta(0, 0, -1)$$

5 Rotaciones del tetraedro

Tenemos en primer lugar la identidad

$$1)\ I = (A, B, C, D) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A continuación, las 3 simetrías axiales (rotaciones de π radianes) que denotaremos con las mismas letras que sus ejes:

Eje $b : y = z = 0$

$$2)\ b = (B, A, D, C) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Eje $c : z = x = 0$

$$3)\ c = (C, D, A, B) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Eje $d : x = y = 0$

$$4)\ d = (D, C, B, A) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Éstas son isometrías de orden 2. Con eje la recta que une un vértice con el baricentro de la cara opuesta, hay dos rotaciones con amplitudes de $2\pi/3$ y $4\pi/3$ radianes. Así, se obtienen 8 isometrías de orden 3 que indicamos a continuación:

Eje: $x = y = z$, orientado desde el baricentro hacia el vértice A

$$5)\ g_A = (A, C, D, B) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = g$$

$$6)\ g_A^2 = (A, D, B, C) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = g^2$$

Eje: $x = -y = -z$, orientado desde el baricentro hacia el vértice B

$$7)\ g_B = (D, B, A, C) = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = gd$$

$$8)\ g_B^2 = (C, B, D, A) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} = g^2c$$

Eje: $-x = y = -z$, orientado desde el baricentro hacia el vértice C

$$9)\ g_C = (B, D, C, A) = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = gb$$

$$10)\ g_C^2 = (D, A, C, B) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = g^2 d$$

Eje: $-x = -y = z$, orientado desde el baricentro hacia el vértice D

$$11)\ g_D = (C, A, B, D) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = gc$$

$$12)\ g_D^2 = (B, C, A, D) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} = g^2 b$$

6 Estructura del grupo de rotaciones

Estas 12 isometrías constituyen el **grupo de las rotaciones del tetraedro** (isomorfo al grupo alternado $\mathcal{A}_4$), denotado como $\mathcal{T}^+$ y generado con los elementos g, b, c, d, si bien los dos últimos son superfluos porque

$$d = bc, \quad c = g^{-1}bg.$$

No obstante, mantrendremos en la escritura las transformaciones c y d.

Operando, bien con las matrices, bien con las permutaciones, es evidente que $K = \{I, b, c, d\}$ es un subgrupo isomorfo al cuártico de Klein $\mathcal{K}$. Como las simetrías axiales son los únicos elementos de orden 2, si a una le aplicamos un automorfismo, debe obtenerse otra simetría axial, es decir, el subgrupo K es característico y, por ello, es normal. Otro subgrupo a destacar, éste de orden 3 (luego isomorfo $\mathcal{C}_3$) es el $G = \{I, g, g^2\}$. Siendo claro que

$$\mathcal{T}^+ = GK, \quad G \cap K = \{I\},$$

podemos escribir el producto semidirecto

$$\mathcal{T}^+ = G[K].$$

La operatividad del grupo, además de las tablas de G y K, depende de las fórmulas

$$g^{-1}bg = c, \quad g^{-1}cg = d, \quad g^{-1}dg = b.$$

7 Subgrupos del grupo $\mathcal{T}^+$

Dentro de este grupo tenemos los siguientes subgrupos:

1) Uno de orden 1:

$$\{I\}$$

2) Tres de orden 2:

$$\{I,b\} \simeq \{I,c\} \simeq \{I,d\} \simeq \mathcal{C}_2$$

Son conjugados entre sí porque

$$g^{-1}\{I,b\}g = \{I,c\}, \quad g^{-2}\{I,b\}g^2 = \{I,d\}.$$

3) Cuatro de orden 3:

$$\{I,g,g^2\} \simeq \{I,gd,g^2c\} \simeq \{I,gb,g^2d\} \simeq \{I,gc,g^2b\} \simeq \mathcal{C}_3$$

Son conjugados entre sí porque,

$$b^{-1}\{I,g,g^2\}b = \{I,gd,g^2c\},$$

$$c^{-1}\{I,g,g^2\}c = \{I,gb,g^2d\},$$

$$d^{-1}\{I,g,g^2\}d = \{I,gc,g^2b\}.$$

4) Uno de orden 4:

$$\{I,b,c,d\} \simeq \mathcal{K}$$

Es característico y normal.

5) Se comprueba que no hay subgrupos de orden 6. De haberlos, no puede ser cíclico pues el grupo carece de elementos de orden 6, luego sería isomorfo a $\mathcal{D}_3$. En tal caso, contendría un subgrupo normal de orden 3, por ejemplo el $\{I,g,g^2\}$, y alguno de los elementos b,c,d de orden 2. Esto es imposible porque

$$b^{-1}gb = gd, \quad g^{-1}cg = gb, \quad g^{-1}dg = gc,$$

son distintos de g^2.

6) Uno de orden 12:

$$\mathcal{T}^+.$$

8 Representaciones lineales del grupo $\mathcal{T}^+$

Las órbitas de conjugación en este grupo son las cuatro siguientes:

$$\mathcal{O}(I) = \{I\}$$

$$\mathcal{O}(b) = \{b, c, d\}$$

$$\mathcal{O}(g) = \{g, gd, gb, gc\}$$

$$\mathcal{O}(g^2) = \{g^2, g^2c, g^2d, g^2b\}$$

Para los grados de las representaciones irreducibles se cumplirá que

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = 12 \Leftrightarrow n_1 = n_2 = n_3 = 1, n_4 = 3$$

La primera será la trivial σ^1. Las dos siguientes, de grado 1, se definen por las fórmulas

$$\sigma^2(k) = 1, \sigma^2(g) = \omega,$$

$$\sigma^3(k) = 1, \sigma^3(g) = \omega^2,$$

donde $\omega = -1/2 + i\sqrt{3}/2$. La de grado 3 la da el propio grupo. La tabla de caracteres resultante es la

		χ^1 1	χ^2 1	χ^3 1	χ^4 3
I	1	1	1	1	3
b	3	1	1	1	-1
g	4	1	ω	ω^2	0
g^2	4	1	ω^2	ω	0

9 Planos de simetría

El tetraedro admite 6. El espejo de cada uno de ellos es el plano que pasa por dos vértices y el punto medio de los otros dos. Por ejemplo, el plano que pasa por los vértices A y D y por el punto medio de B y C. Esta simetría la denotaremos como s. Operando, se tiene, entonces, que

13) Simetría de espejo $x - y = 0$ $s = (A, C, B, D) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

14) Simetría de espejo $z - x = 0$

$$sg = (A, D, C, B) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

15) Simetría de espejo $y - z = 0$ $sg^2 = (A, B, D, C) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

16) Simetría de espejo $x + y = 0$ $sd = (D, B, C, A) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17) Simetría de espejo: $z + x = 0$ $sgc = (C, B, A, D) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

18) Simetría de espejo: $y + z = 0$ $sg^2b = (B, A, C, D) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

10 Isometrías impares del tetraedro

Además de las seis simetrías especulares, que son isometrías impares de orden 2, habrá otras seis, con orden 4, las cuales serán simetrías rotatorias. Las especulares se han obtenido componiendo s con las seis rotaciones I, g, g^2, d, gc, g^2b; las rotatorias se obtendrán componiendo s con las restantes rotaciones:

19) Simetría rotatoria dirigida por $(0, 0, 1)$ y ángulo de $3\pi/2$

$$sb = (B, D, A, C) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

20) Simetría rotatoria dirigida por $(0, 0, 1)$ y ángulo de $\pi/2$

$$sc = (C, A, D, B) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

21) Simetría rotatoria dirigida por $(0, 1, 0)$ y ángulo de $3\pi/2$

$$sgd = (D, A, B, C) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

22) Simetría rotatoria dirigida por $(0,1,0)$ y ángulo de $\pi/2$

$$sgb = (B, C, D, A) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

23) Simetría rotatoria dirigida por $(1,0,0)$ y ángulo de $3\pi/2$

$$sg^2c = (C, D, B, A) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

24) Simetría rotatoria dirigida por $(1,0,0)$ y ángulo de $\pi/2$

$$sg^2d = (D, C, A, B) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

11 El grupo del tetraedro

Las 24 isometrías

$$1)\, I = (A, B, C, D)$$

$$2)\, b = (B, A, D, C),\; 3)\, c = (C, D, A, B),\; 4)\, d = (D, C, B, A)$$

$$5)\, g = (A, C, D, B),\; 6)\, g^2 = (A, D, B, C)$$

$$7)\, gd = (D, B, A, C),\; 8)\, g^2c = (C, B, D, A)$$

$$9)\, gb = (B, D, C, A),\; 10)\, g^2d = (D, A, C, B)$$

$$11)\, gc = (C, A, B, D),\; 12)\, g^2b = (B, C, A, D)$$

$$13)\, s = (A, C, B, D),\; 14)\, sg = (A, D, C, B),\; 15)\, sg^2 = (A, B, D, C)$$

$$16)\, sd = (D, B, C, A),\; 17)\, sgc = (C, B, A, D),\; 18)\, sg^2b = (B, A, C, D)$$

$$19)\, sb = (B, D, A, C),\; 20)\, sc = (C, A, D, B)$$

$$21)\, sgd = (D, A, B, C),\; 22)\, sgb = (B, C, D, A)$$

$$23)\, sg^2c = (C, D, B, A),\; 24)\, sg^2d = (D, C, A, B)$$

que hemos descrito constituyen el **grupo del tetraedro**, isomorfo al grupo simétrico $\mathcal{S}_4$ y denotado como $\mathcal{T}$. Lo generamos con el conjunto $\{g, b, c, d, s\}$, aunque bastaría hacerlo con el $\{g, b, s\}$.

El grupo $K = \{I, b, c, d\}$ sigue siendo normal: al aplicar cualquier automorfismo interno a los elementos b, c y d, debe salir una transformación de orden 2 y par, es decir, otra de ellas. Comprobando que $s^{-1}gs = g^2$, vemos que los elementos g y s generan un subgrupo

$$SG = \{I, g, g^2, s, sg, sg^2\} \simeq \mathcal{D}_3$$

Entonces, se llega a la descomposición

$$\mathcal{T} = SG[K].$$

La operatividad en $\mathcal{T}$ resulta de las tablas de estos subgrupos y de las fórmulas

$$g^{-1}bg = c,\ g^{-1}cg = d,\ g^{-1}dg = b,$$

$$s^{-1}bs = c,\ s^{-1}cs = b,\ s^{-1}ds = d,$$

$$s^{-1}gs = g^{-1}.$$

12 Subgrupos del grupo $\mathcal{T}$

Posee los siguientes subgrupos:

1) Uno de orden 1:

$$\{I\}$$

2) Nueve de orden 2:

$$\{I, b\} \simeq \{I, c\} \simeq \{I, d\} \simeq \mathcal{C}_2,$$

$$\{I, s\} \simeq \{I, sg\} \simeq \{I, sg^2\} \simeq \{I, sd\} \simeq \{I,sgc\} \simeq \{I, sg^2b\} \simeq \mathcal{C}_2.$$

Los tres primeros, ya lo sabemos, son conjugados entre sí. También lo son los otros seis porque

$$g^{-1}\{I, s\}g = \{I, sg^2\},\ g^{-2}\{I, s\}g^2 = \{I, sg\},\ b^{-1}\{I, s\}b = \{I, sd\},$$

$$(gd)^{-1}\{I, s\}(gd) = \{I, sg^2b\},\ (g^2d)^{-1}\{I, s\}(g^2d) = \{I,sgc\}.$$

3) Cuatro de orden 3:

$$\{I, g, g^2\} \simeq \{I, gd, g^2c\} \simeq \{I, gb, g^2d\} \simeq \{I, gc, g^2b\} \simeq \mathcal{C}_3.$$

Ya sabemos que son conjugados entre sí.

4) Tres subgrupos cíclicos de orden 4:

$$\{I, sb, d, sc\} \simeq \{I,sgd,c,sgb\} \simeq \{I, sg^2c, b, sg^2d\} \simeq \mathcal{C}_4$$

Son conjugados entre sí porque

$$g^{-1}\{I, sb, d, sc\}g = \{I, sg^2c, b, sg^2d\},\ \ g^{-2}\{I, sb, d, sc\}g^2 = \{I{,}sgd{,}c{,}sgb\}.$$

5) Otros cuatro de orden 4:

$$\{I, b, c, d\} \simeq \{I, s, d, sd\} \simeq \{I, sg, c{,}sgc\} \simeq \{I, sg^2, b, sg^2b\} \simeq \mathcal{K}$$

El primero es normal. Los otros son conjugados entre sí ya que

$$g^{-1}\{I, s, d, sd\}g = \{I, sg^2, b, sg^2b\},\ \ g^{-2}\{I, s, d, sd\}g^2 = \{I, sg, c{,}sgc\}.$$

6) Hay cuatro de orden 6:

$$\{I, g, g^2, s, sg, sg^2\} \simeq \{I, gd, g^2c, sd, sg^2{,}sgc\} \simeq$$

$$\simeq \{I, gb, g^2d, sg, sd, sg^2b\} \simeq \{I, gc, g^2b, s, sg^2b{,}sgc\} \simeq \mathcal{D}_3$$

Son conjugados entre sí porque

$$b^{-1}\{I, g, g^2, s, sg, sg^2\}b = \{I, gd, g^2c, sd{,}sgc{,}sg^2\},$$

$$c^{-1}\{I, g, g^2, s, sg, sg^2\}c = \{I, gb, g^2d, sd, sg, sg^2b\},$$

$$d^{-1}\{I, g, g^2, s, sg, sg^2\}d = \{I, gc, g^2b, s{,}sgc{,}sg^2b\}.$$

7) De orden 8 hay tres:

$$\mathcal{Z}(d) = \{I, sb, d, sc, b, sd, c, s\} \simeq$$

$$\simeq \mathcal{Z}(c) = g^{-2}\mathcal{Z}(d)g^2 = \{I{,}sgb{,}c{,}sgd{,}d, sg, b{,}sgc\} \simeq$$

$$\simeq \mathcal{Z}(b) = g^{-1}\mathcal{Z}(d)g = \{I, sg^2c, b, sg^2d, c, sg^2b, d, sg^2\} \simeq \mathcal{D}_4$$

Son los 2-subgrupos de Sylow.

8) Hay un subgrupo (normal) de orden 12:

$$\{I, b, c, d, g, g^2, gd, g^2c, gb, g^2d, gc, g^2b\} = \mathcal{T}^+ \simeq \mathcal{A}_4$$

13 Representaciones lineales del grupo $\mathcal{T}$

Las órbitas de conjugación en este grupo son las cuatro siguientes:

$$\mathcal{O}(I) = \{I\}$$

$$\mathcal{O}(b) = \{b, c, d\}$$

$$\mathcal{O}(g) = \{g, gd, gb, gc, g^2, g^2c, g^2d, g^2b\}$$

$$\mathcal{O}(s) = \{s, sd, sg^2, sg, sg^2b, sgc\}$$

$$\mathcal{O}(sb) = \{sb, sc, sg^2c, sgd, sg^2d, sgb\}$$

Para los grados de las representaciones irreducibles se cumplirá que

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 24 \to n_1 = n_2 = 1, n_3 = 2, n_4 = n_5 = 3$$

La primera representación es la trivial

$$\sigma^1(s^x g^y k) = 1.$$

La segunda es la función *signo*

$$\sigma^2(s^x g^y k) = 1 \text{ si } x = 0,\ \sigma^2(s^x g^y k) = -1 \text{ si } x = 1.$$

La función

$$\sigma^3(s^x g^y k) = s^x g^y$$

es un morfismo de $\mathcal{T}$ sobre $\mathcal{D}_3$, con lo cual σ^3 se materializa como el grupo del triángulo. Como σ^4podemos tomar la identidad del propio grupo $\mathcal{T}$. Finalmente, σ^5 se define como

$$\sigma^5(s^x g^y k) = s^x g^y k \text{ si } x = 0,\ \sigma^5(s^x g^y k) = -s^x g^y k \text{ si } x = 1,$$

tratándose en realidad del grupo de las rotaciones del cubo. La tabla de caracteres resultante es la

		χ^1 1	χ^2 1	χ^3 2	χ^4 3	χ^4 3
I	1	1	1	2	3	3
b	3	1	1	2	-1	-1
g	8	1	1	-1	0	0
s	6	1	-1	0	1	-1
sb	6	1	-1	0	-1	1

14 Interpretación geométrica de algunos subgrupos

Interpretación geométrica del subgrupo $\{I, b, c, d\} \simeq \mathcal{K}$:
Los cuatro puntos

$$(A+B)/2 = \delta(1,0,0)$$

$$(B+D)/2 = \delta(0,-1,0)$$

$$(D+C)/2 = \delta(-1,0,0)$$

$$(C+A)/2 = \delta(0,1,0)$$

situados en el plano $z = 0$, forman un cuadrilátero cuyos lados los marcan los vectores

$$(B+D)/2 - (A+B)/2 = \delta -1, -1, 0$$

$$(D+C)/2 - (B+D)/2 = \delta(-1,1,0)$$

$$(C+A)/2 - (D+C)/2 = \delta(1,1,0)$$

$$(A+B)/2 - (C+A)/2 = \delta(1,-1,0)$$

todos de igual longitud, por lo que forman un rombo (de hecho es un cuadrado porque cada lado es perpendicular a su contiguo). Restringiéndonos al mismo b y c son las simetrías respecto a las diagonales, mientras que d es la simetría central. El subgrupo $\{I, s, d, sd\}$ actúa sobre este cuadrado, considerado ahora como rectángulo, de manera que s y sd son las simetrías respecto de las mediatrices y d sigue siendo la simetría central.

$$(A+B)/2 = \delta(1,0,0)$$

$$(B+C)/2 = \delta(0,0,-1)$$

$$(C+D)/2 = \delta(-1,0,0)$$

$$(D+A)/2 = \delta(0,0,1)$$

Considerado como rombo, su grupo vuelve a ser el $\{I, b, c, d\}$, actuando ahora como simetría central la transformación c. Considerado como rectángulo su grupo es el $\{I, sg, c, sgc\}$.

$$(A+C)/2 = \delta(0,1,0)$$

$$(C+B)/2 = \delta(0,0,-1)$$

$$(B+D)/2 = \delta(0,-1,0)$$

$$(D+A)/2 = \delta(0,0,1)$$

De nuevo $\{I, b, c, d\}$ es el grupo de este rombo, si bien ahora la simetría central la ejecuta b. Como rectángulo, su grupo es el $\{I, sg^2, b, sg^2b\}$.

Interpretación de los grupos $\mathcal{C}_4$:

Los cuatro puntos $(A+B)/2, (B+D)/2, (D+C)/2, (C+A)/2$, forman un cuadrado. La actuación sobre el mismo del subgrupo $\{I, sb, d, sc\}$ es el grupo de los giros de dicho cuadrado.

Interpretación de los grupos $\mathcal{D}_4$:

La transformación b es una de las simetrías del cuadrado, luego el subgrupo $\{I, sb, d, sc, b, sd, c, s\}$ es el diédrico del mismo.

15 Tabla del grupo $\mathcal{T}$

Terminamos mostrando en su totalidad la tabla del grupo $\mathcal{T}$.

$\mathcal{T}$(Superior izquierda $= \mathcal{A}_4$)

	I	b	c	d	g	gb	gc	gd	g^2	g^2b	g^2c	g^2d
I	I	b	c	d	g	gb	gc	gd	g^2	g^2b	g^2c	g^2d
b	b	I	d	c	gc	gd	g	gb	g^2d	g^2c	g^2b	g^2
c	c	d	I	b	gd	gc	gb	g	g^2b	g^2	g^2d	g^2c
d	d	c	b	I	gb	g	gd	gc	g^2c	g^2d	g^2	g^2b
g	g	gb	gc	gd	g^2	g^2b	g^2c	g^2d	I	b	c	d
gb	gb	g	gd	gc	g^2c	g^2d	g^2	g^2b	d	c	b	I
gc	gc	gd	g	gb	g^2d	g^2c	g^2b	g^2	b	I	d	c
gd	gd	gc	gb	g	g^2b	g^2	g^2d	g^2c	c	d	I	b
g^2	g^2	g^2b	g^2c	g^2d	I	b	c	d	g	gb	gc	gd
g^2b	g^2b	g^2	g^2d	g^2c	c	d	I	b	gd	gc	gb	g
g^2c	g^2c	g^2d	g^2	g^2b	d	c	b	I	gb	g	gd	gc
g^2d	g^2d	g^2c	g^2b	g^2	b	I	d	c	gc	gd	g	gb

$\mathcal{T}$(Superior derecha)

	s	sb	sc	sd	sg	sgb	sgc	sgd	sg^2	sg^2b	sg^2c	sg^2d
I	s	sb	sc	sd	sg	sgb	sgc	sgd	sg^2	sg^2b	sg^2c	sg^2d
b	sc	sd	s	sb	sgd	sgc	sgb	sg	sg^2b	sg^2	sg^2d	sg^2c
c	sb	s	sd	sc	sgc	sgd	sg	sgb	sg^2d	sg^2c	sg^2b	sg^2
d	sd	sc	sb	s	sgb	sg	sgd	sgc	sg^2c	sg^2d	sg^2	sg^2b
g	sg^2	sg^2b	sg^2c	sg^2d	s	sb	sc	sd	sg	sgb	sgc	sgd
gb	sg^2c	sg^2d	sg^2	sg^2b	sd	sc	sb	s	sgb	sg	sgd	sgc
gc	sg^2b	sg^2	sg^2d	sg^2c	sc	sd	s	sb	sgd	sgc	sgb	sg
gd	sg^2d	sg^2c	sg^2b	sg^2	sb	s	sd	sc	sgc	sgd	sg	sgb
g^2	sg	sgb	sgc	sgd	sg^2	sg^2b	sg^2c	sg^2d	s	sb	sc	sd
g^2b	sgc	sgd	sg	sgb	sg^2d	sg^2c	sg^2b	sg^2	sb	s	sd	sc
g^2c	sgb	sg	sgd	sgc	sg^2c	sg^2d	sg^2	sg^2b	sd	sc	sb	s
g^2d	sgd	sgc	sgb	sg	sg^2b	sg^2	sg^2d	sg^2c	sc	sd	s	sb

$\mathcal{T}$(Inferior izquierda)

	I	b	c	d	g	gb	gc	gd	g^2	g^2b	g^2c	g^2d
s	s	sb	sc	sd	sg	sgb	sgc	sgd	sg^2	sg^2b	sg^2c	sg^2d
sb	sb	s	sd	sc	sgc	sgd	sg	sgb	sg^2d	sg^2c	sg^2b	sg^2
sc	sc	sd	s	sb	sgd	sgc	sgb	sg	sg^2b	sg^2	sg^2d	sg^2c
sd	sd	sc	sb	s	sgb	sg	sgd	sgc	sg^2c	sg^2d	sg^2	sg^2b
sg	sg	sgb	sgc	sgd	sg^2	sg^2b	sg^2c	sg^2d	s	sb	sc	sd
sgb	sgb	sg	sgd	sgc	sg^2c	sg^2d	sg^2	sg^2b	sd	sc	sb	s
sgc	sgc	sgd	sg	sgb	sg^2d	sg^2c	sg^2b	sg^2	sb	s	sd	sc
sgd	sgd	sgc	sgb	sg	sg^2b	sg^2	sg^2d	sg^2c	sc	sd	s	sb
sg^2	sg^2	sg^2b	sg^2c	sg^2d	s	sb	sc	sd	sg	sgb	sgc	sgd
sg^2b	sg^2b	sg^2	sg^2d	sg^2c	sc	sd	s	sb	sgd	sgc	sgb	sg
sg^2c	sg^2c	sg^2d	sg^2	sg^2b	sd	sc	sb	s	sgb	sg	sgd	sgc
sg^2d	sg^2d	sg^2c	sg^2b	sg^2	sb	s	sd	sc	sgc	sgd	sg	sgb

$\mathcal{T}$(Inferior derecha)

	s	sb	sc	sd	sg	sg	sgc	sgd	sg^2	sg^2b	sg^2c	sg^2d
s	I	b	c	d	g	gb	gc	gd	g^2	g^2b	g^2c	g^2d
sb	c	d	I	b	gd	gc	gb	g	g^2b	g^2	g^2d	g^2c
sc	b	I	d	c	gc	gd	g	gb	g^2d	g^2c	g^2b	g^2
sd	d	c	b	I	gb	g	gd	gc	g^2c	g^2d	g^2	g^2b
sg	g^2	g^2b	g^2c	g^2d	I	b	c	d	g	gb	gc	gd
sgb	g^2c	g^2d	g^2	g^2b	d	c	b	I	gb	g	gd	gc
sgc	g^2b	g^2	g^2d	g^2c	c	d	I	b	gd	gc	gb	g
sgd	g^2d	g^2c	g^2b	g^2	b	I	d	c	gc	gd	g	gb
sg^2	g	gb	gc	gd	g^2	g^2b	g^2c	g^2d	I	b	c	d
sg^2b	gc	gd	g	gb	g^2d	g^2c	g^2b	g^2	b	I	d	c
sg^2c	gb	g	gd	gc	g^2c	g^2d	g^2	g^2b	d	c	b	I
sg^2d	gd	gc	gb	g	g^2b	g^2	g^2d	g^2c	c	d	I	b

References

[1] M. A. Armstrong: Groups and Symmetry, Springer-Verlag, 1988

[2] Fraleigh, John B. Algebra abstracta. Adison- Wesley Iberoamericana 1987.

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INFORMACIÓN NACIONAL

La esquina olímpica

En homenaje a Jorge Salazar

Rafael Sánchez Lamoneda

A comienzos de Junio finalizó la Olimpiada Juvenil de Matemáticas 2012, (OJM). Participaron en la primera fase, conocida como el Canguro Matemático, 61857 jóvenes provenientes de 24 ciudades del país y 22 estados. Esta prueba se realizó a nivel nacional el día 15 de Marzo, conjuntamente con los otros 50 países que organizaron el Canguro 2012. La segunda fase fue el 5 de Mayo. En ella participaron 4365 estudiantes y la prueba se realizó en cada una de las ciudades involucradas en la OJM. La Final Nacional fue el 2 de Junio, en la Facultad de Ciencias de la Universidad de Carabobo, FACYT, donde disfrutamos de una excelente atención por los miembros del Departamento de Matemáticas. Un especial agradecimiento a nuestro colega Luis Rodríguez, coordinador de la OJM en el estado Carabobo, y a José Marcano, Decano de FACYT, por su amable hospitalidad. En esta etapa, los participantes fueron 150 estudiantes provenientes de 20 ciudades del país. Con esta prueba cerramos la actividad nacional para el año escolar 2011-2012 y ahora comenzamos con los eventos internacionales.

Del 15 al 23 de Junio en San Salvador se llevará a cabo la XIV Olimpiada Matemática de Centroamérica y el Caribe, en San Salvador, El Salvador. Estaremos representados por un equipo de tres estudiantes, Luis Ruiz, del colegio Las Colinas de Barquisimeto, José Guevara, colegio Bella Vista, Maracay y Rafael Aznar, colegio Los Arcos, de Caracas. La tutora de la delegación es Estefanía Ordaz, estudiante de la licenciatura en Matemáticas en la USB y el jefe de delegación, ya veterano en este papel, el prof José Heber Nieto de LUZ.

La Olimpiada Internacional de Matemáticas será esta vez en Mar del Plata, Argentina, del 4 al 16 de Julio, nuestro equipo está integrado por la joven Rubmary Rojas, colegio San Vicente de Paúl, Barquisimeto, Diego Peña, colegio Los Hipocampitos, Altos Mirandinos y Sergio Villarroel, colegio San Lázaro, Cumaná. La tutora de la delegación es la profesora Laura Vielma, de la Academia Washington y el jefe de delegación, quién escribe, Rafael Sánchez Lamoneda.

Esperamos que ambas delegaciones tengan un buen papel.

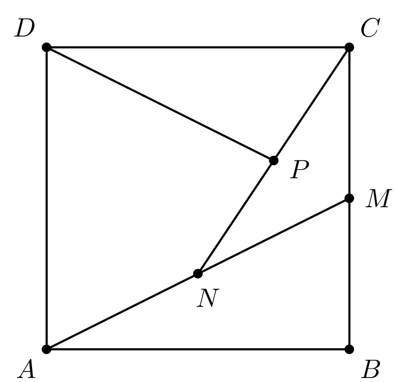
Antes de finalizar esta Esquina Olímpica, un agradecimiento a nuestros patrocinadores y amigos, Fundación Empresas Polar, Banco Central de Venezuela, Acumuladores Duncan, la Fundación Cultural del Colegio Emil Friedman, MRW, la Academia Ciencias Físicas, Matemáticas y Naturales, las universidades UCV, USB, Carabobo, LUZ, URU, ULA y UDO. Muchas gracias por seguir con nosotros otro año más.

Para terminar les ofrecemos los exámenes propuestas en la Final Nacional, para primero y quinto año, todas las pruebas de la OJM 2012, las pueden ver en www.acm.ciens.ucv.ve. La duración de la prueba fue de 4 horas y cada problema tiene un valor de 7 puntos.

OLIMPÍADA JUVENIL DE MATEMÁTICA
Prueba Nacional — Valencia, 2 de junio de 2012
Primer Año

Problema 1. Un dígito k es un *unidivi* de un número natural n si k es la cifra de las unidades de algún divisor de n. Por ejemplo, los divisores de 50 son 1, 2, 5, 10, 25 y 50, por lo tanto sus unidivis son 0, 1, 2 y 5. Halle el menor número natural que tenga 10 unidivis.

Problema 2. El lado del cuadrado $ABCD$ mide 4 cm. M es el punto medio de BC, N es el punto medio de AM y P es el punto medio de NC. Calcule el área del cuadrilátero $ANPD$.



Problema 3. (a) ¿Es posible repartir los números 1^2, 2^2, 3^2, 4^2, 5^2, 6^2 y 7^2 en dos grupos, de manera que la suma de los números de cada grupo sea la misma? (b) ¿Y para los números 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2 y 9^2?

Problema 4. Una calculadora tiene dos teclas especiales A y B. La tecla A transforma el número x que esté en la pantalla en $\frac{1}{x}$. La tecla B transforma el número x que esté en la pantalla en $1-x$. Diego comenzó a pulsar las teclas A, B, A, B,...en forma alternada. Luego de realizar 2012 pulsaciones, en la pantalla quedó el número 0,875. ¿Qué número estaba inicialmente en la pantalla?

OLIMPÍADA JUVENIL DE MATEMÁTICA
Prueba Nacional — Valencia, 2 de junio de 2012
Quinto Año

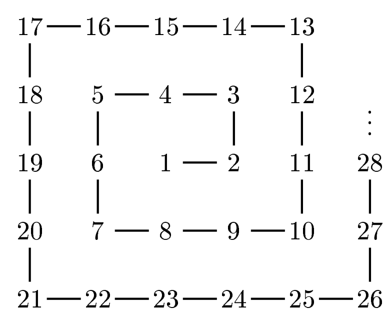
Problema 1. Encuentre todos los enteros a diferentes de cero y de 4, tales que el número $\dfrac{a}{a-4}+\dfrac{2}{a}$ también es un entero.

Problema 2. (a) Pruebe que para todo n se cumple

$$n^2-(n+1)^2-(n+2)^2+(n+3)^2-(n+4)^2+(n+5)^2+(n+6)^2-(n+7)^2=0.$$

(b) En el pizarrón están escritos los cuadrados de los números del 1 al 2012: 1^2, 2^2, 3^2,..., 2012^2. Hay que escribir delante de cada número un signo + ó − de manera que, al realizar la suma algebraica de los 2012 números, se obtenga el menor valor positivo que sea posible. Determine cuál es ese mínimo e indique una manera de distribuir los signos para lograrlo.

Problema 3. Consideremos los puntos con ambas coordenadas enteras en el plano cartesiano, en el origen (0,0) se coloca el 1, en (1,0) se coloca el 2, en (1,1) se coloca el 3, y así sucesivamente se van colocando los enteros positivos en espiral alrededor del origen (ver figura). Determine las coordenadas del punto donde se colocará el 2012.



Problema 4. Sea ABC un triángulo equilátero y P un punto interior tal que $PA=5$, $PB=4$, $PC=3$. ¿Cuánto mide el ángulo $\angle BPC$?

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Asociación Matemática Venezolana
Escuela Venezolana de Matemáticas

XXV Escuela Venezolana de Matemáticas

Emalca Venezuela

Universidad de Los Andes, Mérida, 2 al 7 de septiembre, 2012

Comité Organizador:
Carlos Di Prisco, Instituto Venezolano de Investigaciones Científicas.
Neptalí Romero (Coordinador), Universidad Centroccidental Lisandro Alvarado.
Oswaldo Araujo, Universidad de Los Andes.

Comité Científico:
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Mariela Castillo, Universidad Central de Venezuela.
Neptalí Romero, Universidad Centroccidental Lisandro Alvarado.
Carlos Uzcátegui, Universidad de Los Andes.
Ennis Rosas, Universidad de Oriente.
Vladimir Strauss, Universidad Simón Bolívar.

Desde su inicio, el Comité Científico de la Escuela Venezolana de Matemática ha estado integrado por los coordinadores de los postgrados de matemáticas en las universidades del país; ello en virtud de que el evento nació, y continúa siendo, como una actividad conjunta de esos postgrados.

Cursos de la XXV EVM – Emalca Venezuela 2012

Curso I
Perturbaciones estocásticas de ecuaciones en derivadas parciales: una introducción a través de la ecuación de Allen–Cahn.
Stella Brassesco, Instituto Venezolano de Investigaciones Científicas, Caracas, Venezuela.
Motivación y Objetivos:
Brindar una introducción al estudio de perturbaciones estocásticas de ecuaciones de tipo parabólico en dimensión 1, a través del análisis del ejemplo de una ecuación no lineal con un potencial de dos pozos, con un término aditivo aleatorio dado por un ruido blanco espacio–tiempo. El modelo determinístico fue

propuesto en metalurgia por Allen y Cahn, también aparece bajo otros nombres en diversas áreas, para modelar fenómenos donde hay coexistencia de dos estados estables. Es también el ejemplo mejor estudiado de un sistema dinámico en dimensión infinita.

El curso requiere que el participante esté familiarizado con conceptos básicos de: Probabilidades y Procesos Estocásticos, Ecuaciones Diferenciales Ordinarias y preferiblemente un curso de Ecuaciones Estocásticas.

Contenido:

Preliminares:

- Algunas propiedades de procesos Gaussianos. Ruido Blanco.
- Planteamiento de la ecuación.
- Soluciones “mild” y soluciones del problema lineal.

Resultados generales:

- Resultados de unicidad y existencia para el problema no lineal.
- Propiedades de regularidad de las soluciones.
- Dependencia continua en la condición inicial.

Pequeñas perturbaciones en el caso de volumen finito:

- Clasificación de puntos críticos para la ecuación determinística.
- Funcional de acción y grandes desvíos para pequeñas perturbaciones.
- Aplicaciones al estudio de tiempo y lugar de escape del dominio de atracción.

Dinámica de la interfase en volumen infinito:

- Estabilidad de los puntos críticos para la ecuación determinística.
- Linealización del problema alrededor del frente, y fórmula de Feynamnn–Kac.
- Aproximación lineal del problema perturbado.
- Determinación de la ecuación para la dinámica de la interfase cuando la intensidad del ruido tiende a cero.

Bibliografía:

- S. Brassesco, A. de Masi y E. Presutti. *Brownian Fluctuations of the Interface in the D=1 Ginzburg–Landau Equation with Noise*, Ann. Inst. Henri Poincaré, Vol. 31 (1995) 81–118.
- Dalang, R.C., Khoshnevisan, D., Mueller, C., Nualart, D., Xiao, Y. Khoshnevisan, Davar; Rassoul–Agha, Firas (Eds.) *A Minicourse on Stochastic Partial Diferential Equations*, Lecture Notes in Mathematics, Vol. 1962, Springer Verlag (2009).
- W. G. Faris and G. Jona–Lasinio. *Large fluctuations for a nonlinear heat equation with noise*, J. Phys. A: Math. Gen. 15 (1982) 3025.
- M.I. Freidlin and A.D. Wentzell. *Random Perturbations of Dynamical Systems* Springer–Verlag (1998).
- D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, Vol. 840 Springer–Verlag (1981).

Curso II

Teoría de Ramsey y dinámica de grupos topológicos.

José Gregorio Mijares, Universidad Javeriana, Bogotá, Colombia.

Motivación y Objetivos:

La idea es estudiar la relación entre la Teoría de Ramsey y acciones de grupos sobre espacios compactos. En general, dado un grupo G, se estudiará la propiedad de oscilación finitamente estable, para funciones uniformemente continuas definidas sobre G–espacios uniformes. A partir de allí se discutirá la propiedad de punto fijo sobre compactos para grupos topológicos, el concepto de flujo minimal y la relación de estas nociones con la propiedad de Ramsey para clases de estructuras de Fraïssé. Posteriormente, se hará notar como lo anterior conecta a la Teoría de Ramsey con fenómenos como el problema de la distorsión en la Geometría de Espacios de Banach o la propiedad de concentración de medida, entre otros.

El curso está pensado para estudiantes avanzados de la Licenciatura en Matemáticas y para estudiantes de Postgrado en Matemáticas. Haber cursado asignaturas básicas de Topología, Teoría de Grupos, Análisis Real y Análisis Funcional es recomendable. Conocimientos básicos de Lógica y Combinatoria serán de utilidad, si bien no son indispensables.

Contenido:

- El Teorema de Ramsey y temas preliminares.
- Espacios uniformes.
- Funciones de oscilación finitamente estable: el fenómeno Ramsey–Dvoretzky–Milman.

- La propiedad de punto fijo sobre compactos. Grupos extremadamente dóciles.
- Flujos minimales.
- Teoría de Ramsey para clases de estructuras.
- El espacio de Urysohn y su grupo de isometrías.
- Concentración de medida.
- Distorsión en espacios de Banach.

Bibliografía:

- S.A. Argyros and S. Todorcevic. *Ramsey Methods in Analysis*, Birkhauser Basel, 1999.
- C.A. Di Prisco, *Combinatoria: Un panorama de la teoría de Ramsey*, Editoral Equinoccio, 2009.
- W.T. Gowers. *Lipschitz functions on classical spaces*, European J. Combin. 13 (1992), 141–151.
- R. Graham, B. Rothschild, J. Spencer. *Ramsey Theory*, Weily and sons. 1980.
- I. M. James. *Topological and uniform spaces*, Springer–Verlag, 1987.
- J. Kelley. *General Topology*, Springer; 1955.
- E. Odell and T. Schlumprecht. *The distortion problem*, Acta Math. 173 (1994), 259 – 281.
- V. Pestov. *Dynamics of infinite–dimensional groups*, ULECT 40, AMS, 2006.

Curso III
Introducción al estudio de las geodésicas en superficies.
Rafael Oswaldo Ruggiero, PUC, Rio de Janeiro, Brasil.
Motivación y Objetivos:
El objetivo del curso es demostrar resultados básicos e importantes de la teoría de las geodésicas en superficies que no se mencionan normalmente en los cursos regulares de geometría. El minicurso tendría dos partes. La primera trataría geodésicas en superficies del espacio Euclideano, donde es posible transmitir una intuición física y geométrica más palpable de estos objetos. El resultado principal de esta primera parte es el siguiente: una superficie en el espacio Euclideano cuyas geodésicas son todas curvas planas es de hecho un subconjunto de una

esfera redonda o de un plano. Las herramientas necesarias para demostrar este resultado provienen del cálculo diferencial a varias variables. Esta primera parte del minicurso es elemental y profunda al mismo tiempo, muestra de forma eficaz el alcance del cálculo en geometría. La segunda parte trataría de geodésicas en superficies abstractas, con la propiedad de ser globalmente minimizantes (el equivalente variacional en superficies Riemannianas de las rectas euclideanas y las geodésicas hiperbólicas). Se expodrán ejemplos diferentes al plano Euclideano de superficies que posseen estas geodésicas: plano hiperbólico y superficies de revolución en el toro. El objetivo final de esta segunda parte seria demostrar el famoso resultado de Hedlund: el levantamiento en el recubrimiento universal de toda geodésica globalmente minimizante de una métrica Riemanniana en el toro es "sombreada" por una recta Euclideana en el plano. Esta segunda parte es más avanzada, algunas nociones de topología previas (recubrimiento universal, grupo fundamental) facilitarían la comprensión del contenido. Este célebre trabajo de Hedlund de 1938, inspirado en un trabajo anterior de Morse de 1924 en el contexto de superficies de género superior es el germen de lo que hoy se llama teoría de Aubry–Mather en dinámica conservativa. Para la primera parte del curso sería apenas necesario haber cursado cálculo a varias variables y ecuaciones diferenciales ordinarias. Para la segunda parte del curso es conveniente alguna familiaridad con geometría Riemanniana básica de superficies y con el concepto de recubrimiento universal, aunque lo mínimo necesario para entender la exposición será explicado en las charlas.

Contenido:

Primera parte.

Demostración del siguiente resultado:

Teorema: *Si toda geodésica de una superficie diferenciable conexa es una curva plana entonces la superficie es un subconjunto de un plano o de una esfera de curvatura constante.*

- Repaso de las definiciones fundamentales de la teoría de las superficies en el espacio Euclideano: superficies parametrizadas, diferenciabilidad en superficies, plano tangente, campo normal, aplicación normal de Gauss.

- Derivación covariante en superficies, geodésicas desde el punto de vista de la mecánica clásica y desde el punto de vista variacional. Identificación de geodésicas en superficies con simetrías, superficies de revolución.

- Operadores de forma: aplicación de Weingarten, segunda forma fundamental, curvatura de curvas en una superficie, curvatura Gaussiana, líneas de curvatura, puntos umbílicos. Curvatura de superficies de revolución.

- Demostración del siguiente resultado: si todas las geodésicas de una superficie que pasan por un punto x son curvas planas, entonces la recta normal a la superficie por x es un eje de revolución para la misma. Además, el punto x es un punto umbílico de la superficie.

- Caracterización de la superficies totalmente umbílicas: son subconjuntos de planos o esferas de curvatura constante. La demostración del teorema principal es consecuencia de estas dos últimas afirmaciones.

Segunda parte.
Demostración del siguiente resultado:
Teorema (Hedlund, 1938) *Dada una métrica Riemanniana en el toro existe una constante $C > 0$ tal que cada levantamiento en el recubrimiento universal de una geodésica globalmente minimizante de dicha métrica está contenido en la vecindad tubular de una recta Euclideana (que depende de la geodésica).*

- Definición de superficie diferenciable abstracta, ejemplos, métricas Riemannianas en superficies.
- Fórmula de la primera variación, geodésicas en superficies Riemannianas, ejemplos: plano hiperbólico.
- Recubrimiento universal, espacio cociente, grupo fundamental, ejemplos.
- Geodésicas minimizantes entre pares de puntos y geodésicas globalmente minimizantes, ejemplos: rectas, geodésicas em el plano hiperbólico y em toros de revolución.
- Completitud del conjunto de geodésicas globalmente minimizantes, direcciones asintóticas de curvas en el plano Euclideano, construcción de geodésicas hiperbólicas. Geodésicas globalmente minimizantes en toros de revolución.
- Propiedades de intersección entre geodésicas globalmente minimizantes, lema del atajo (shortcut) y consecuencias: geodésicas cerradas globalmente minimizantes son curvas simples, dos geodésicas globalmente minimizantes no se intersectan em mas de um punto, construcción variacional de geodésicas heteroclínicas.
- Demostración del siguiente resultado de Hedlund: dada una métrica Riemanniana em el toro existe una constante $C > 0$ tal que para todo par de puntos x, y en el recubrimiento universal se tiene que toda geodésica minimizante entre x, y está a distancia menor o igual a C del segmento de recta que une x, y.

Bibliografía:

- Apostol. T. Calculus Vol. 2, *Multi–variable calculus and linear algebra with applications to differential equations probability*, Second Edition, John Wiley & Sons, 1969.
- Bangert, V. *Mather sets for twist maps and geodesics on tori*, Dynamics reported, vol. 1, 1–56 (1988) Dynamical Systems and applications, 1, Wiley, Chichester.
- Hicks, N. *Notas de geometria diferencial*, Van Nostrand Mathematical Studies n. 3, Van Nostrand, Princeton, N.J., 1965.
- Massey, W. S. *A basic course in algebraic topology*, Graduate texts in Mathematics, vol. 127. Springer–Verlag, 1991.
- Ruggiero, R. *On the generic nonexistence of rational geodesic foliations in the torus, Mather sets and Gromov hyperbolic spaces*, Boletim da Sociedade Brasileira de Matemática, vol. 31, 1, (2000) 93–111.

Curso IV
Cálculo diferencial combinatorio.
Miguel Méndez, Instituto Venezolano de Investigaciones Científicas, Caracas, Venezuela.
Motivación y Objetivos:
El cálculo diferencial combinatorio se inició con Désiré André y Percy A. MacMahon en el siglo XIX. El próposito de este curso hacer una presentación históricamente motivada de muchos resultados dispersos en la literatura relacionados con la noción combinatorial e intuitiva de la derivada, introducida por MacMahon, y luego formalizada por A. Joyal en 1980. Estudiaremos la relación entre la enumeración de árboles crecientes con la solución de una ecuación diferencial autónoma de primer orden. Construiremos aplicaciones efectivas al problema del ordenamiento normal de operadores bosónicos de creación y aniquilación en la mecánica cuántica. En la parte final del curso introduciremos el estudio, desde el punto de vista combinatorial, de una clase de ecuaciones diferenciales autónomas de primer orden que dependen un número numerable de parámetros, y cuyo lado derecho está modificado por un desplazamiento (shift), en dichos parámetros. Las soluciones combinatorias a este tipo de ecuaciones son importantes en el estudio estadístico de la profundidad de nodos en los árboles de búsqueda, y otros tipos de estructuras de datos en teoría de la computación. El curso está pensado para estudiantes avanzados de la Licenciatura en Matemáticas y para estudiantes de Postgrado en Matemáticas. Conocimientos básicos de Combinatoria y de Teoría de Categoría será de utilidad, pero no indispensables. Solo lo será la inefable madurez matemática.

Contenido:

- Introducción histórica: La noción de derivada de MacMahon. La ecuación diferencial de D. André y la interpretación combinatoria de su solución.
- Series formales en una, varias, e infinitas variables. Producto, suma, substitución y derivada de series formales. Desplazamiento (shift) en una serie formal con variables indexadas en $\mathbb{N}$. Pletismo de desplazamiento.
- Elementos de teoría de categorías. La noción combinatoria de derivada según A. Joyal. Series formales y funciones generatrices. Demostraciones combinatorias de la regla de la cadena, la regla del producto, la regla de Leibniz, y la fórmula de Faá di Bruno. Polinomios de Bell y fórmulas recursivas.
- Solución combinatorial a una ecuación diferencial autónoma $y\prime = \phi(y)$. Fórmula de Lie–Groebner–Taylor.
- El problema general del orden normal de operadores de creación y aniquilación. Bichos y colonias en la solución del problema. Equivalencia con el problema de torres que no se atacan mutuamente, en un tablero sesgado.
- Ecuaciones diferenciales autónomas modificadas de la forma $y\prime = S\phi(y)$, donde y es una serie formal que depende de un número infinito de variables (parámetros) y S es el desplazamiento de las infinitas variables. Solución Combinatoria. Fórmula de Lie–Groebner–Taylor generalizada. Aplicaciones al conteo de árboles crecientes de acuerdo al parámetro que indica la altura de las hojas. Fórmulas asintóticas. árboles de búsqueda y estructura de datos.

Bibliografía:

- D. André. (1879) *Sur les permutations alternées*, J. Math. Pures Appl. 5, 31–46.
- F. Bergeron, L. Leroux, G. Labelle, (1998) *Combinatorial species and tree–like structures*, Encyclopedia of mathematics and applications, Volume 67.
- D. Foata, M.P. Schtzenberger, (1970) *Théorie géometrique des polynomes Eulériens*, Lecture Notes in Mathematics 138, Springer–Verlag, Berlin, Heidelberg, and New York.
- D. Foata, (1974) *La serie Génératrice Exponentielle dans les Problèmes d' Ènumèration*, Presses de l'Université Montréal. Quebec.
- J. Françon. (1977) *Arbres binaire de recherche: Propriétés combinatoires et applications*, RAI–RO Informatique Théorique, 4, 155–169.

- J. Françon. (1984) *Sur le nombre le registres nécessaires à l'éveluation d'une expression arithmétique*, RAIRO Informatique Théorique, 18, 355–364.
- A. Joyal, (1981) *Une théorie combinatoire des series formelles*, Adv. Math. 4, 21–82.
- MacMahon, Percy A. *Combinatory analysis. Vol. I, II* (bound in one volume). Reprint of *An introduction to combinatory analysis* (1920) and *Combinatory analysis. Vol. I, II* (1915, 1916). Dover Phoenix Editions. Dover Publications, Inc., Mineola, NY, 2004. ii+761 pp. ISBN: 0–486–49586–8
- M. Méndez, P. Blasiak and K. A. Penson (2005) *Combinatorial approach to generalized Bell and Stirling numbers and boson normal ordering problem*, Journal of mathematical physics 46,083511.
- P. Leroux and X. Viennot, *Combinatorial Resolution of systems of differential Equations, I. Ordinary differential Equations*, Lecture Notes in Mathematics, 1986, Volume 1234/1986, 210–245.
- M.P. Schtzenberger, (1975) *Solutions non commutatives d'une équation differentielle classique. New concepts and technologies in parallel Information Processing*, M. Caianello, NATO Advanced Studies Institute Series E: Applied Sciences, 9, 381–401.

Como de costumbre, el primer día del evento se llevará a cabo la conferencia inaugural, la cual ha estado a cargo de destacados matemáticos, generalmente de la región. Para la vigésimo quinta edición aún no se tiene el nombre de esa distinguida persona. El promedio de participantes en los cursos de las ediciones de la Escuela Venezolana de Matemáticas (1988–2011) es superior a cien personas, en la recién concluida edición 2011 el número de participantes fue de 73 personas. Hasta el 2011 un total de 100 cursos han sido dictados, muchos de los libros sobre los cuales se soportaron estos cursos están digitalizados y disponibles en el sitio web `http://cea.ivic.gob.ve/evm`

En la actualidad el financiamiento de la Escuela Venezolana de Matemáticas se obtiene de diversas fuentes; por ejemplo, para las dos últimas ediciones se han recibido fondos de diversos organismos venezolanos (Instituto Venezolano de Investigaciones Científicas, Universidad Centro Occidental Lisandro Alvarado, Universidad de Los Andes, Academia de Ciencias Físicas, Matemáticas y Naturales de Venezuela, Banco Central de Venezuela) y por supuesto ha contado con el apoyo económico de UMALCA y CIMPA, con lo cual ha sido posible cubrir gastos de transporte y estadía de algunos estudiantes provenientes de exterior, especialmente de Centroamérica y el Caribe, y cubrir similares gastos de algunos de los profesores de los cursos dictados y provenientes del exterior.

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