Strong Barrelledness Properties in Lebesgue-Bochner Spaces

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Abstract

If (Ω, Σ, μ) is a finite atomless measure space and X is a normed space, we prove that the space $L_p(\mu, X)$, $1 \leq p \leq \infty$ is a barrelled space of class $\aleph_0$, regardless of the barrelledness of X. That enables us to obtain a localization theorem of certain mappings defined in $L_p(\mu, X)$.

By "space" we mean a "real or complex Hausdorff locally convex space". Given a dual pair (E, F), as usual $\sigma(E, F)$ denotes the weak topology on E. If B is a subset of a linear space E, $\langle B \rangle$ will denote its linear hull.

Let s be a positive integer, then a family $W = \{E_{m_1 m_2 \ldots m_p}, m_r \in \mathbb{N}, 1 \leq r \leq p \leq s\}$ of subspaces of E is said to be an s-net in E if $\{E_{m_1}, m_1 \in \mathbb{N}\}$ is an increasing covering of E and $\{E_{m_1 m_2 \ldots m_j}, m_j \in \mathbb{N}\}$ is an increasing covering of $E_{m_1 m_2 \ldots m_{j-1}}$, for $2 \leq j \leq s$.

A space E is suprabarrelled, or barrelled space of class 1, [17], if given an increasing covering of subspaces of E there is one of them which is dense and barrelled. For a given natural number $s \geq 2$, E is said to be barrelled of class s, [11] and [14], if given an increasing covering of subspaces of E there is one of them which is barrelled of class $s - 1$. If E is barrelled of class s for each $s \in \mathbb{N}$ it is said that E is barrelled of class $\aleph_0$. It is shown in [11] that each non-normable Fréchet space has a dense subspace of class $s - 1$ which is not barrelled of class s for all $s \geq 2$. On the other hand, if Σ denotes any σ-algebra of subsets of a set Ω, the space $\ell_0^\infty(\Omega, \Sigma)$ of all scalar Σ-simple functions defined on Ω with the supremum norm is a barrelled space of class $\aleph_0$, [9]. Since each barrelled space of class s is Baire-like, [15], it

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follows easily that a space E is barrelled of class s if and only if given an s-net $W = \{E_{m_1 m_2 \dots m_p}, m_r \in \mathbb{N}, 1 \leq r \leq p \leq s\}$ in E there is a subspace $E_{q_1 q_2 \dots q_s}$ which is barrelled and dense in E. This means that every $\sigma(E', E_{q_1 q_2 \dots q_s})$-bounded subset of E' is equicontinuous. When E is metrizable, the Amemiya-Kōmura theorem (see for instance [13, 8.2.12]) enables us to avoid all preceding density conditions.

Barrelled spaces of class s, $s \in \mathbb{N}$, allow to extend the Nikodym boundedness theorem and are useful domain classes for the closed graph theorem, a fact which has some incidence in vector measure theory [10].

Throughout this paper (Ω, Σ, μ) will denote, unless another thing is explicitely stated, an atomless finite measure space, X will be a normed space and $L_p(\mu, X)$ will stand for the space of all (equivalence classes of) p-Bochner integrable X-valued functions defined in Ω if $1 \leq p < \infty$, or the space of all (equivalence classes of) X-valued μ-measurable functions on Ω which are essentially bounded if $p = \infty$, both provided with its usual norms. We refer the reader to the monographs [3], [13] and [18].

This paper has been motivated by the three following barrelledness theorems (see [5] for a proof of the first result in a more general context, as well as references [7] and [8] for the measure free case).

Theorem A. (L. Drewnowski, M. Florencio and P.J. Paúl, [4]). *Let (Ω, Σ, μ) be a finite measure space and X be a normed space. If the measure μ is atomless, then $L_p(\mu, X)$ with $1 \leq p < \infty$ is barrelled.*

Theorem B. (J.C. Díaz, M. Florencio and P.J. Paúl, [1]). *Let (Ω, Σ, μ) be a finite measure space and X be a normed space. If the measure μ is atomless, then $L_\infty(\mu, X)$ is barrelled.*

Theorem C. (J.C. Ferrando and L.M. Sánchez Ruiz, [12]). *Suppose that (Ω, Σ, μ) is a finite measure space and X be a normed space. If X is barrelled of class n, then the space $L_\infty(\mu, X)$ is barrelled of class n, for each $n \in \mathbb{N}$.*

The first two results exhibit that the space $L_p(\mu, X)$ may have better barrelledness properties that X has. Indeed, the argument of the proof of Theorems A and B may be easily adapted to show that each $L_p(\mu, X)$ is ultrabornological. This has been noted in [2] and in [6]. In the last reference it has been shown that if $p > 1$ and X' has the Radon-Nikodym property with respect to μ, then $L_p(\mu, X)$ is the locally convex hull of all its Banach subspaces with a basis.

Our purpose here is to prove that if μ is atomless, each space $L_p(\mu, X)$, $1 \leq p \leq \infty$, is barrelled of class n for every $n \in \mathbb{N}$. In the following two theorems we use the well-known fact that if μ is atomless there is a set $A \in \Sigma$ such that $\mu(A) = \mu(\Omega)/2$, and in both cases the basis of the proof is the construction of a certain Banach disk. In Theorem 1 we open up an induction process in order to show that $L_p(\mu, X)$ is suprabarrelled (barrelled of class 1) and then, in Theorem 2, we give a proof of the fact that $L_p(\mu, X)$ being barrelled of class s implies that it is also barrelled of class $s+1$. Therefore, our main theorem is the following.

Main theorem. *Let (Ω, Σ, μ) be an atomless finite measure space, X a normed space and $1 \leq p \leq \infty$. Then $L_p(\mu, X)$ is barrelled of class $\aleph_0$.*

This result enables us to obtain a localization theorem for certain mappings defined on $L_p(\mu, X)$ that have closed graphs.

Theorem 1. *$L_p(\mu, X)$ is suprabarrelled.*

Proof. Assume that $L_p(\mu, X)$ is not suprabarrelled. Then, can find an increasing sequence of non-barrelled subspaces $\{E_n, n \in \mathbb{N}\}$ which covers $L_p(\mu, X)$.

For each n, there is a barrel T_n in E_n such that it is not a neighborhood of the origin in E_n. Let B_n denote the closure of T_n in $L_p(\mu, X)$, and $H_n = \cap\{\langle B_m \rangle, m \geq n\}$.

In what follows a sequence $(\Omega_k, \Sigma_k, \mu_k)$ of atomless measure spaces is to be constructed satisfying the following properties:

(a) $\Omega_k \in \Sigma_{k-1}$, $\mu(\Omega_k) = \mu(\Omega_{k-1})/2$, $\Sigma_k = \Sigma_{k-1|\Omega_k}$, $\mu_k = \mu_{k-1|\Sigma_k}$.

(b) The closed unit ball V_k of $L_p(\mu_k, X)$ is not contained in H_n, for all k and n in $\mathbb{N}$.

The only non-trivial step of the inductive process is the first one, which we reproduce here.

Since μ has no atoms, we split $L_p(\mu, X)$ in two halves, namely $L_p(\lambda, X)$ and $L_p(\lambda', X)$, corresponding to the atomless measure spaces $(A, \mathcal{A}, \lambda)$ and $(\Omega \backslash A, \mathcal{A}', \lambda')$, where $A \in \Sigma$ and $\mu(A) = \mu(\Omega)/2$. If W and W' denote the closed unit balls of $L_p(\lambda, X)$ and $L_p(\lambda', X)$ respectively, we have that either W is not contained in H_n for all n, or W' is not contained in H_n for all n. Otherwise, $L_p(\mu, X) \subset H_{n_0}$, for some $n_0 \in \mathbb{N}$, and thus $L_p(\mu, X) \subset \langle B_{n_0} \rangle$. Hence, making use of Theorems A and B depending on if $1 \leq p < \infty$ or if $p = \infty$, respectively, it is clear that B_{n_0} is a neighborhood of the origin in $L_p(\mu, X)$ and $T_{n_0} = B_{n_0} \cap E_{n_0}$ is a neighborhood of the origin in E_{n_0}, a contradiction. Now, we know how $(\Omega_1, \Sigma_1, \mu_1)$ has to be choosen.

For each $k \in \mathbb{N}$ select an element $g_k \in V_k$, $g_k \notin H_k$. As the sequence (g_k) is contained in the unit ball of $L_p(\mu, X)$ and the support of g_k is a subset of Ω_k, for each $\xi \in \ell_1$ the function $\Sigma_k \xi_k g_k$ takes its values in X outside the zero measure set $\bigcap_k \Omega_k$. Thus $\Sigma_k \xi_k g_k$ is really an element of $L_p(\mu, X)$, and the continuity of the mapping $(\xi_k) \to \Sigma_k \xi_k g_k$ turns the set $D = \{\Sigma_k \xi_k g_k : (\xi_k) \in B_{\ell_1}\}$, where B_{ℓ_1} is the closed unit ball of ℓ_1, into a Banach disk contained in $L_p(\mu, X)$.

Since the subspaces H_n cover $L_p(\mu, X)$, there is an index n_0 such that for each $n \geq n_0$ the subspace $H_n \cap \langle D \rangle$ is barrelled and dense in $\langle D \rangle$, this later provided with the Banach disk norm. Hence, since $H_n \subset \langle B_n \rangle$, we have that $\langle B_n \rangle \cap \langle D \rangle$ is barrelled and dense in $\langle D \rangle$, for each $n \geq n_0$. Since B_n is closed in the L_p-topology, we have that $D \subset \langle B_n \rangle$ for $n \geq n_0$ and $D \subset H_{n_0}$. This is a contradiction, since $g_{n_0} \in D \backslash H_{n_0}$. □

Theorem 2. *$L_p(\mu, X)$ is barrelled of class s, for each positive integer s.*

Proof. We have just seen that $L_p(\mu, X)$ is barrelled of class 1 (suprabarrelled). Given an integer $s \geq 2$, assume that $L_p(\mu, X)$ is barrelled of class $s - 1$. We are going to show that it is barrelled of class s.

If this were not the case, we would have an increasing sequence $\{E_{n_1}, n_1 \geq 1\}$ of dense subspaces, each one of them barrelled of class $s-2$ but not of class $s-1$, covering $L_p(\mu, X)$.

For each positive integer n_1, there is an increasing sequence $\{E_{n_1 n_2}, n_2 \geq 1\}$ of dense subspaces, each one being barrelled of class $s-3$ but not of class $s-2$, which cover E_{n_1}. Repeating this argument an appropriate number of times we have that for each integer $(s-1)$-tuple $(n_1, n_2, \ldots, n_{s-1})$ of positive integers there is an increasing sequence $\{E_{n_1,n_2,\ldots,n_s}, n_s \geq 1\}$ of dense subspaces, none of them barrelled, covering $E_{n_1,n_2,\ldots,n_{s-1}}$. Thus for each $(n_1, n_2, \ldots, n_s) \in \mathbb{N}^s$ we can find a barrel $T_{n_1,n_2,\ldots,n_s}$ in $E_{n_1,n_2,\ldots,n_s}$ which is not a neighborhood of the origin in $E_{n_1,n_2,\ldots,n_s}$. Let $B_{n_1,n_2,\ldots,n_s}$ denote the closure of $T_{n_1,n_2,\ldots,n_s}$ in $L_p(\mu, X)$ and $Z_{n_1,n_2,\ldots,n_s} = \langle B_{n_1,n_2,\ldots,n_s} \rangle$. Going backwards, we define the following subspaces

$$H_{n_1,n_2,\ldots,n_{s-1},n_s} = \cap\{Z_{n_1,n_2,\ldots,n_{s-1},m}, m \geq n_s\}$$

and, for $j = s-1, s-2, \ldots, 1$,

$$Z_{n_1,n_2,\ldots,n_j} = \cup\{H_{n_1,n_2,\ldots,n_j,m}, m \geq 1\}$$

and

$$H_{n_1,n_2,\ldots,n_j} = \cap\{Z_{n_1,n_2,\ldots,n_{j-1},m}, m \geq 1\}.$$

It is plain that for $j = 1, 2, \ldots, s$, $E_{n_1,n_2,\ldots,n_j} \subset H_{n_1,n_2,\ldots,n_j}$ and the increasing sequence $\{H_{n_1,n_2,\ldots,n_j}, n_j \geq 1\}$ covers $E_{n_1,n_2,\ldots,n_{j-1}}$, taking $L_p(\mu, X)$ as E_{n_0}.

As before, our main point is to achieve a sequence $(\Omega_k, \Sigma_k, \mu_k)$ of atomless measure spaces with the same requirements as in the previous theorem, the last one still being that V_k is not contained in H_n, for all k, $n \in \mathbb{N}$. Again, we only give the first step of the induction process. With identical notation to the one used in Theorem 1, we want to show that either W is not contained in H_n for all $n \in \mathbb{N}$, or W' is not contained in H_n for all n. Otherwise, there is n_1 such that $L_p(\mu, X) \subset H_{n_1} \subset Z_{n_1} = \cup\{H_{n_1,m}, m \geq 1\}$. The induction hypothesis then guarantees the existence of n_2 for which $H_{n_1 n_2}$ is dense and barrelled of class $s-2$, and because $H_{n_1 n_2} \subset Z_{n_1 n_2}$, the same can be said of $Z_{n_1 n_2} = \cup\{H_{n_1,n_2,m}, m \geq 1\}$. By continuing this selection argument we can assert there is an s-tuple $(n_1, n_2, \ldots, n_s)$ of positive integers for which $H_{n_1,n_2,\ldots,n_s}$ is dense and barrelled, hence $Z_{n_1,n_2,\ldots,n_s}$ is also dense and barrelled, and $B_{n_1,n_2,\ldots,n_s}$ is a neighborhood of the origin in $Z_{n_1,n_2,\ldots,n_s}$. But then $T_{n_1,n_2,\ldots,n_s} = B_{n_1,n_2,\ldots,n_s} \cap E_{n_1,n_2,\ldots,n_s}$ would be a neighborhood of the origin in $E_{n_1,n_2,\ldots,n_s}$, which is a contradiction.

For each $k \in \mathbb{N}$ we choose an element $g_k \in V_k \backslash H_k$. Using again our old notation, the set $D = \{\Sigma_k \xi_k g_k : (\xi_k) \in B_{\ell_1}\}$ is a Banach disk contained in $L_p(\mu, X)$.

There is a j_0 such that, for each $n_1 \geq j_0$, $H_{n_1} \cap \langle D \rangle$ is barrelled of class $s-1$ and dense in $\langle D \rangle$, with its associated norm. Since $\{H_{n_1,n_2}, n_2 \geq 1\}$ is an increasing covering of H_{n_1}, there is j_{n_1} such that, for each $n_2 \geq j_{n_1}$, $H_{n_1,n_2} \cap \langle D \rangle$ is barrelled of class $s-2$ and dense in $\langle D \rangle$. An appropriate number of repetitions tells us that, whenever $n_1 \geq j_0$, $n_2 \geq j_{n_1}$, $n_3 \geq j_{n_1 n_2}$, ..., $n_{s-1} \geq j_{n_1 n_2 \ldots n_{s-2}}$ there is $j_{n_1 n_2 \ldots n_{s-2} n_{s-1}}$ such that, for $n_s \geq j_{n_1 n_2 \ldots n_{s-2} n_{s-1}}$, $H_{n_1 n_2 \ldots n_{s-2} n_{s-1} n_s} \cap \langle D \rangle$ is barrelled and dense in $\langle D \rangle$. Then, $Z_{n_1 n_2 \ldots n_{s-2} n_{s-1} n_s} \cap \langle D \rangle$ is also barrelled and dense in $\langle D \rangle$ for $n_1 \geq j_0$,

$n_2 \geq j_{n_1}$, $n_3 \geq j_{n_1 n_2}$, ..., $n_s \geq j_{n_1 n_2 \ldots n_{s-2} n_{s-1}}$, and $D \subset \langle B_{n_1 n_2 \ldots n_{s-1} n_s} \rangle = Z_{n_1 n_2 \ldots n_{s-1} n_s}$. Hence we have that D is contained in the intersection

$$\cap\{Z_{n_1 n_2 \ldots n_s} : n_1 \geq j_0, n_2 \geq j_{n_1}, \ldots, n_s \geq j_{n_1 n_2 \ldots n_{s-1}}\}$$

which is a subset of H_{j_0}. This contradicts the selection of g_{j_0}. □

In [16] Valdivia has given the following definition: *A space E is said to be a Γ_r-space if every quasicomplete subspace of $E^*(\sigma(E^*, E))$ intersecting $E'(\sigma(E', E))$ in a dense subspace contains E'.*
In the next theorem, if $W = \{E_{n_1, n_2, \ldots, n_k}, n_r \in \mathbb{N}, 1 \leq r \leq k \leq s\}$ is an s-net in E, we will denote by W_s the family $\{E_{m_1 m_2 \ldots m_s}, m_r \in \mathbb{N}, 1 \leq r \leq s\}$.

Theorem 3. *Let f be a mapping with closed graph from $L_p(\mu, X)$ in a space E. Let W be an s-net in E and suppose that every $L \in W_s$ has a finer topology $\Im_L$ "finer than the induced by E" such that $L(\Im_L)$ is a Γ_r-space. Then there is some $L \in W_s$ such that f is a continuous mapping from $L_p(\mu, X)$ in $L(\Im_L)$.*

Proof. We have that $f^{-1}(W)$ is an s-net in $L_p(\mu, X)$ and therefore, by Theorem 2, there is some $L \in W_s$ such that $f^{-1}(L)$ is barrelled and dense in $L_p(\mu, X)$. Then by [16], Theorems 1 and 14, we have that $f/f^{-1}(L)$ has a continuous extension h from $L_p(\mu, X)$ into $L(\Im_L)$ and, since f has closed graph, we have that $f = h$. □

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