Universal Properties of the Corrado Segre Embedding

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Abstract

Let $\mathcal{S}(\Pi_0, \Pi_1)$ be the *product* of the projective spaces Π_0 and Π_1, i.e. the semilinear space whose point set is the product of the point sets of Π_0 and Π_1, and whose lines are all products of the kind $\{P_0\} \times g_1$ or $g_0 \times \{P_1\}$, where P_0, P_1 are points and g_0, g_1 are lines. An *embedding* $\chi : \mathcal{S}(\Pi_0, \Pi_1) \to \Pi'$ is an injective mapping which maps the lines of $\mathcal{S}(\Pi_0, \Pi_1)$ onto (whole) lines of Π'. The classical embedding is the Segre embedding, $\gamma_0 : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$. For each embedding χ, there exist an automorphism α of $\mathcal{S}(\Pi_0, \Pi_1)$ and a linear morphism $\psi : \overline{\Pi} \to \Pi'$ (i.e. a composition of a projection with a collineation) such that $\chi = \alpha\gamma_0\psi$. (Here $\alpha\gamma_0\psi$ maps P onto $\psi(\gamma_0(\alpha(P))) =: P\alpha\gamma_0\psi$.) As a consequence, every $\mathcal{S}(\Pi_0, \Pi_1)$ which is embedded in a projective space is, up to projections, a Segre variety.

1 Introduction

Most classical varieties represent as points of a projective space some geometric objects. So such varieties are (*projective*) *embeddings*, which are somewhat canonical. For instance, take an h-flat ${}^hU$ (i.e. a subspace of dimension h) of an n-dimensional projective space Π over a commutative field F. ${}^hU$ can be associated with $\binom{n+1}{h+1}$ coordinates, the so-called *Plücker coordinates*, or *Grassmann coordinates*. They are defined up to a factor. So ${}^hU$ can be represented as a point of a $(\binom{n+1}{h+1} - 1)$-dimensional projective space. We call *Plücker map* this representation. The image of

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the Plücker map, i.e. the set of the points related to all h-flats in Π, is a *Grassmann variety*.

In 1981, H. HAVLICEK [3] investigated the embeddings of the Grassmann spaces. An embedding of the Grassmann space $\Gamma^h(\Pi)$ represents injectively the h-flats of the projective space Π as points of another projective space $\overline{\Pi}$ and maps a pencil of h-flats onto a line of $\overline{\Pi}$, where a *pencil* is the set of all h-flats which contain a given ${}^{h-1}U$ and are contained in a given ${}^{h+1}U$.

HAVLICEK proved that each such embedding is the composition of the Plücker map, described above, with a linear morphism (in German: "lineare Abbildung") of projective spaces. This linear morphism is related to a (possibly singular) semilinear transformation between the underlying vector spaces. By this property, the Plücker map is called a *universal embedding*. He actually showed a more general result: The Plücker map with domain Γ is a universal element of the covariant functor $\mathcal{F}(\Gamma, -)$, that maps a projective space Π onto the set of all linear morphisms of Γ in Π. Its action on the morphisms is defined by $\mathcal{F}(\Gamma,\psi)(\chi) := \chi\psi$.

The purpose of our work is to deal with the analogous question which concerns the product spaces and the related Corrado Segre embedding. The product of two projective spaces is defined to be a particular semilinear space (see the definition in the abstract). The word "semilinear" means that any two distinct points are joined by at most one line. The Segre embedding does not have strong universal properties like the Plücker map. In general (cf. theorem 1) an embedding of a product space $\mathcal{S}$ is the composition of three maps: (i) an automorphism of $\mathcal{S}$, (ii) the Segre embedding, and (iii) a linear morphism between projective spaces. For particular product spaces it is possible to take the first map equal to the identity map (theorem 2), so, in such cases, the Segre embedding is a universal embedding.

As a consequence of the previous results, we will establish a relationship with the notion of a regular pseudoproduct space, which has been given by N. MELONE and D. OLANDA [4]. A regular pseudoproduct space is a semilinear space which satisfies some intrinsic incidence-geometric axioms, and turns out to be isomorphic to a product space. In theorem 3, we will prove that every regular pseudoproduct space which is embedded in a projective space is, up to projections, a Segre variety.

A similar result holds for the Grassmann varieties [3][7].

b. A *semilinear space* is a pair $\Sigma = (\mathcal{U}, \mathcal{R})$, where $\mathcal{U}$ is a set, whose elements are called *points*, and $\mathcal{R} \subset 2^{\mathcal{U}}$. (In this note "$A \subset B$" just means that $x \in A$ implies $x \in B$.) The elements of $\mathcal{R}$ are *lines*. The axioms which define a semilinear space are the following: (i) $|g| \geq 2$ for every $g \in \mathcal{R}$, (ii) $\bigcup_{g\in\mathcal{R}} g = \mathcal{U}$, ($iii$) $g, h \in \mathcal{R}$, $g \neq h$ $\Rightarrow |g \cap h| \leq 1$. Two points $P, Q \in \mathcal{U}$ are *collinear*, $P \sim Q$, if a line g exists such that $P, Q \in g$ (for $P \neq Q$ we will also write $PQ := g$); otherwise, P and Q are *not collinear*, $P \not\sim Q$. An *isomorphism* between the semilinear spaces $(\mathcal{U}, \mathcal{R})$ and $(\mathcal{U}', \mathcal{R}')$ is a bijection $\alpha : \mathcal{U} \to \mathcal{U}'$ such that both α and α^{-1} map lines onto lines.

The *join* of $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{U}$ is:

$$\mathcal{M}_1 \vee \mathcal{M}_2 := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \Bigg(\bigcup_{\substack{P_i \in \mathcal{M}_i \\ P_1 \sim P_2,\, P_1 \neq P_2}} P_1P_2 \Bigg).$$

A *subspace* of Σ is a set $\mathcal{U}' \subset \mathcal{U}$ which fulfills

$$P_1, P_2 \in \mathcal{U}' \quad \Longrightarrow \quad \{P_1\} \vee \{P_2\} \subset \mathcal{U}'.$$

A subspace $\mathcal{U}'$ such that $P_1 \sim P_2$ for all $P_1, P_2 \in \mathcal{U}'$ is called a *singular subspace* of Σ.

Let $\Pi' = (\mathcal{P}', \mathcal{G}')$ be a projective space. A *linear morphism* $\psi : \Sigma \to \Pi'$ consists of: the *domain* $\mathbf{D}(\psi) \subset \mathcal{U}$; the *exceptional set* $\mathbf{A}(\psi) := \mathcal{U} \setminus \mathbf{D}(\psi)$; a mapping $\psi' : \mathbf{D}(\psi) \to \mathcal{P}'$ and the related mapping

$$\psi'' : 2^{\mathcal{U}} \longrightarrow 2^{\mathcal{P}'} : \mathcal{M} \longmapsto (\mathcal{M} \cap \mathbf{D}(\psi))\psi'.$$

We will abuse notation and write ψ to denote also the maps ψ' and ψ''. This ψ must fulfill the following axioms [3]:

$$(L1) \qquad (\{X\} \vee \{Y\})\psi = \{X\}\psi \vee \{Y\}\psi \quad \text{for } X, Y \in \mathcal{U},\ X \sim Y;$$

$$(L2) \qquad \begin{array}{c} \{X\}\psi = \{Y\}\psi,\ X, Y \in \mathcal{U},\ X \neq Y,\ X \sim Y \quad \Longrightarrow \\ \Longrightarrow \exists A \in XY \text{ such that } \{A\}\psi = \emptyset. \end{array}$$

ψ is said to be *global* when $\mathbf{D}(\psi) = \mathcal{U}$; is called *embedding* if it is global and injective. If ψ is an embedding, then $\mathrm{im}\,(\psi) := \mathcal{U}\psi$ is an *embedded semilinear space*. The *rank* of ψ is:

$$\mathrm{rk}\,\psi := \dim \Pi([\mathrm{im}\,(\psi)]).$$

Here the square brackets denote projective closure in the projective space Π', and, for any subspace U, $\Pi(U)$ is U meant as a projective space.

Proposition 1.1 *If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{U}$, then*

$$(\mathcal{M}_1 \vee \mathcal{M}_2)\psi \subset \mathcal{M}_1\psi \vee \mathcal{M}_2\psi. \square$$

We now introduce some notation, which will hold in the whole paper:
$\Pi_0 = (\mathcal{P}_0, \mathcal{G}_0)$, $\Pi_1 = (\mathcal{P}_1, \mathcal{G}_1)$ are projective spaces of finite dimensions n_0, n_1, respectively.
${}^d\check{\mathcal{U}}(\Pi)$, or simply ${}^d\check{\mathcal{U}}$, is the set of all d-flats of Π.
${}^d\check{\mathcal{U}}_i := {}^d\check{\mathcal{U}}(\Pi_i)$, $i = 0, 1$.
$\mathcal{U} := \mathcal{P}_0 \times \mathcal{P}_1$.
$\mathcal{R} := \{\{X_0\} \times g_1 | X_0 \in \mathcal{P}_0,\ g_1 \in \mathcal{G}_1\} \cup \{g_0 \times \{X_1\} | g_0 \in \mathcal{G}_0,\ X_1 \in \mathcal{P}_1\}$.
$\mathcal{S}(\Pi_0, \Pi_1) := (\mathcal{U}, \mathcal{R})$, which is a semilinear space, is the *product space* of Π_0 and Π_1.
$\chi : \mathcal{S}(\Pi_0, \Pi_1) \to \Pi'$, where $\Pi' = (\mathcal{P}', \mathcal{G}')$ is a projective space, is a linear morphism.

Proposition 1.2 $\mathrm{rk}\,\chi \leq n_0 n_1 + n_0 + n_1$.

Proof By induction on n_1. If $n_1 = 1$, then $\mathcal{U} = (\mathcal{P}_0 \times \{Q\}) \vee (\mathcal{P}_0 \times \{Q'\})$, with $Q, Q' \in \mathcal{P}_1$, $Q \neq Q'$. Prop. 1.1 gives $\mathrm{im}\,(\chi) \subset (\mathcal{P}_0 \times \{Q\})\chi \vee (\mathcal{P}_0 \times \{Q'\})\chi$. In this case, the statement follows from the representation theorem for the linear morphisms from projective spaces [3] (quoted here: prop. 1.5).

Now assume $n_1 > 1$ and take a hyperplane, say $\mathcal{H}$, in Π_1. If $(\mathcal{U}', \mathcal{R}') := \mathcal{S}(\Pi_0, \Pi(\mathcal{H}))$, then $\chi_{|\mathcal{U}'}$ is a linear morphism of $\mathcal{S}(\Pi_0, \Pi(\mathcal{H}))$ into Π'. The inductive assumption gives $\operatorname{rk} \chi_{|\mathcal{U}'} \leq n_0 n_1 + n_1 - 1$. Let $\overline{P} \in \mathcal{P}_1 \setminus \mathcal{H}$ and $\mathcal{P}_0' := (\mathcal{P}_0 \times \{\overline{P}\})\chi$. Then $\dim \Pi(\mathcal{P}_0') \leq n_0$. From $\mathcal{U} = \mathcal{U}' \vee (\mathcal{P}_0 \times \{\overline{P}\})$ it follows, by prop 1.1, $\operatorname{im}(\chi) \subset \mathcal{U}'\chi \vee \mathcal{P}_0'$. This proves our proposition. □

We say that the linear morphism χ is *regular* if $\operatorname{rk} \chi = n_0 n_1 + n_0 + n_1 = \dim \Pi'$. As an example, assume that Π_0 and Π_1 are coordinatized by a commutative field F. The *Corrado Segre embedding* $\gamma_0 : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$, where $\overline{\Pi}$ is the $(n_0 n_1 + n_0 + n_1)$-dimensional projective space coordinatized by F, is defined by

$$((x_0, x_1, \ldots, x_{n_0})F, (y_0, y_1, \ldots, y_{n_1})F)\gamma_0 := (x_i y_j)_{i=0,\ldots,n_0;\, j=0,\ldots,n_1} F.$$

The Segre embedding turns out to be a regular linear morphism.

The following proposition is contained in the proof of prop. 1.2:

Proposition 1.3 *If $\mathcal{H}$ is a hyperplane of Π_1 and $X_1 \in \mathcal{P}_1 \setminus \mathcal{H}$, then*

$$\operatorname{im}(\chi) \subset (\mathcal{P}_0 \times \mathcal{H})\chi \vee (\mathcal{P}_0 \times \{X_1\})\chi. \square$$

Throughout this paper, $\gamma : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$, where $\overline{\Pi} = (\overline{\mathcal{P}}, \overline{\mathcal{G}})$ is a projective space, will denote a regular linear morphism.

Proposition 1.4 *γ is an embedding.*

Proof Assume that γ is not global; then there exists $(X_0, X_1) \in \mathcal{U}$ such that $\{(X_0, X_1)\}\gamma = \emptyset$. If $\mathcal{H}$ is a hyperplane of Π_1 and $X_1 \notin \mathcal{H}$, then, by prop. 1.2, $\operatorname{rk} \gamma_{|\mathcal{P}_0 \times \mathcal{H}} \leq n_0 n_1 + n_1 - 1$. The non-globality assumption implies $\dim \Pi((\mathcal{P}_0 \times \{X_1\})\gamma) < n_0$. Then, prop. 1.3 gives $\operatorname{rk} \gamma < n_0 n_1 + n_0 + n_1$, a contradiction. So, γ is global.

Now assume that γ is not injective, i.e. there exist two distinct elements of $\mathcal{U}$, say (X_0, X_1) and (Y_0, Y_1), such that $(X_0, X_1)\gamma = (Y_0, Y_1)\gamma$. Since γ is global, from (L2) it follows $X_0 \neq Y_0$ and $X_1 \neq Y_1$. Now use prop. 1.3 with $\chi := \gamma$ and $\mathcal{H}$ such that $Y_1 \in \mathcal{H}$. The dimension of $[(\mathcal{P}_0 \times \mathcal{H})\gamma]$ is at most $n_0 n_1 + n_1 - 1$, the dimension of $\mathcal{P}_0 \times \{X_1\}$ is n_0 and

$$(X_0, X_1)\gamma \in (\mathcal{P}_0 \times \mathcal{H})\gamma \cap (\mathcal{P}_0 \times \{X_1\})\gamma.$$

This contradicts the hypothesis that γ is regular. □

Now we quote two known results concerning the linear morphisms from projective spaces.

Proposition 1.5 [2][3] *Every linear morphism $\psi : \Pi \to \Pi'$ is the product of a projection with center $\mathbf{A}(\psi)$ onto a complementary subspace in Π, say $\mathcal{P}^*$, and a collineation between $\mathcal{P}^*$ and $\operatorname{im}(\psi)$.* □

Proposition 1.6 [3] *Let $\Pi = (\mathcal{P}, \mathcal{G})$ be a Pappian projective space, $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ two complementary subspaces of Π, and $\Pi' = (\mathcal{P}', \mathcal{G}')$ a projective space. Let $\psi_i : \mathcal{P}_i^* \to \mathcal{P}'$, $i = 1, 2$, be linear morphisms, and assume that the following condition is satisfied:*

(V4) $\operatorname{rk}\psi_1 \geq 1$ and $\operatorname{rk}\psi_2 \geq 2$. Furthermore, there exist two lines $g_i \subset \mathbf{D}(\psi_i)$ $(i = 1, 2)$, and a projectivity $\sigma' : g_1\psi_1 \to g_2\psi_2$ such that the mapping

$$\sigma := (\psi_{1|g_1})\sigma'(\psi_{2|g_2})^{-1}$$

is a projectivity.

Then there exists a linear morphism $\psi : \mathcal{P} \to \mathcal{P}'$ *such that* $\psi_{|\mathcal{P}_i^*} = \psi_i$, $i = 1, 2$. □

2 Basic properties of the embeddings of $\mathcal{S}(\Pi_0, \Pi_1)$

Proposition 2.1 *If* $\chi : \mathcal{S}(\Pi_0, \Pi_1) \to \Pi'$ *is an embedding, then* Π' *is Pappian.*

Proof Since $\dim(\Pi') \geq 3$, Π' is Desarguesian. Let $g_i \in \mathcal{G}_i$, $i = 0, 1$, and

$$\begin{aligned} S_0 &:= \{(\{X_0\} \times g_1)\chi | X_0 \in g_0\}, \\ S_1 &:= \{(g_0 \times \{X_1\})\chi | X_1 \in g_1\}. \end{aligned}$$

For $i = 0, 1$, S_i is a regulus in Π', and every line of S_i meets every line of S_{1-i} in exactly one point. It is well-known [5] that such a configuration can occur only in Pappian projective spaces. □

A *frame* of an n-dimensional projective space is a set of $n + 2$ points, no $n + 1$ of which lie on a hyperplane. We recall that $\gamma : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$ denotes a regular linear morphism.

Proposition 2.2 *Assume that*

$$\{X_{i0}, X_{i1}, \ldots, X_{i,n_i+1}\}$$

is a frame of Π_i, $i = 0, 1$. *Let*

$$\overline{P}_{jh} := (X_{0j}, X_{1h})\gamma, \quad j = 0, 1, \ldots, n_0 + 1, \quad h = 0, 1, \ldots, n_1 + 1.$$

Then

$$\mathcal{E} := \{\overline{P}_{jh} | j = 0, 1, \ldots, n_0,\ h = 0, 1, \ldots, n_1\} \cup \{\overline{P}_{n_0+1,n_1+1}\}$$

is a frame of $\overline{\Pi}$.

Proof Let

$${}^{n_0}\overline{Y}_h := [\{\overline{P}_{0h}, \overline{P}_{1h}, \ldots, \overline{P}_{n_0\,h}\}] = (\mathcal{P}_0 \times \{X_{1h}\})\gamma \in {}^{n_0}\check{\mathcal{U}}(\overline{\Pi}), \quad h = 0, 1, \ldots, n_1.$$

We shall prove that if $Q \in \mathcal{E}$, then $[\mathcal{E} \setminus \{Q\}] \supset \operatorname{im}(\gamma)$.

If $Q = \overline{P}_{n_0+1,n_1+1}$, then $[\mathcal{E} \setminus \{Q\}] \supset {}^{n_0}\overline{Y}_h$ for $h = 0, 1, \ldots, n_1$. Furthermore,

$$\begin{aligned} (X_0, X_1)\gamma &\in [\{(X_0, X_{10}), (X_0, X_{11}), \ldots, (X_0, X_{1n_1})\}\gamma] \subset \\ &\subset [{}^{n_0}\overline{Y}_0 \cup {}^{n_0}\overline{Y}_1 \cup \ldots \cup {}^{n_0}\overline{Y}_{n_1}] \subset [\mathcal{E} \setminus \{Q\}] \end{aligned}$$

for any $(X_0, X_1) \in \mathcal{U}$.

Now assume $Q = \overline{P}_{\overline{j}\,\overline{h}} \neq \overline{P}_{n_0+1,n_1+1}$. From ${}^{n_0}\overline{Y}_h \subset [\mathcal{E} \setminus \{Q\}]$ for $h = 0, 1, \ldots, n_1$, $h \neq \overline{h}$, it follows

$$\begin{aligned}\{\overline{P}_{n_0+1,0} \quad , \quad & \overline{P}_{n_0+1,1}, \ldots, \overline{P}_{n_0+1,n_1+1}\} \setminus \{\overline{P}_{n_0+1,\overline{h}}\} \subset [\mathcal{E} \setminus \{Q\}] \\ & \Longrightarrow (\{X_{0,n_0+1}\} \times \mathcal{P}_1)\gamma \subset [\mathcal{E} \setminus \{Q\}] \\ & \Longrightarrow \overline{P}_{n_0+1,\overline{h}} \in [\mathcal{E} \setminus \{Q\}].\end{aligned}$$

Since ${}^{n_0}\overline{Y}_{\overline{h}} = [\{\overline{P}_{0\overline{h}}, \overline{P}_{1\overline{h}}, \ldots, \overline{P}_{n_0+1,\overline{h}}\} \setminus \{Q\}]$, we have ${}^{n_0}\overline{Y}_{\overline{h}} \subset [\mathcal{E} \setminus \{Q\}]$. Our assertion can now be proven as in the previous case. ◻

Proposition 2.3 If a line g of $\overline{\Pi}$ is contained in $\mathrm{im}\,(\gamma)$, then there is an $h \in \mathcal{R}$ such that $g = h\gamma$.

Proof Let $(X_0, X_1)\gamma$, $(X_0', X_1')\gamma$ and $(X_0'', X_1'')\gamma$ be three distinct points of g.

If $X_0 = X_0'$, then $g = (\{X_0\} \times X_1X_1')\gamma$ and the assertion is proven. The same argument applies to the cases $X_0 = X_0''$ and $X_0' = X_0''$. So suppose that X_0, X_0' and X_0'' are three distinct points. Take in Π_0 a hyperplane $\mathcal{H}$ such that $X_0 \notin \mathcal{H}$, $X_0' \in \mathcal{H}$. Since

$$\mathrm{im}\,(\gamma) \subset [(\{X_0\} \times \mathcal{P}_1)\gamma \cup (\mathcal{H} \times \mathcal{P}_1)\gamma],$$

the regularity of γ and prop. 1.2 give

$$(\{X_0\} \times \mathcal{P}_1)\gamma \cap [(\mathcal{H} \times \mathcal{P}_1)\gamma] = \emptyset.$$

We have $(X_0'', X_1'')\gamma \notin [(\mathcal{H} \times \mathcal{P}_1)\gamma]$ (because $g \not\subset [(\mathcal{H} \times \mathcal{P}_1)\gamma]$), whence g is the unique line of $\overline{\Pi}$ which contains $(X_0'', X_1'')\gamma$ and meets the subspaces $(\{X_0\} \times \mathcal{P}_1)\gamma$ and $[(\mathcal{H} \times \mathcal{P}_1)\gamma]$. We conclude that $g = (X_0X_0'' \times \{X_1''\})\gamma$. ◻

The following result is a corollary of prop. 2.3:

Proposition 2.4 Let U be a subspace of $\overline{\Pi}$, contained in $\mathrm{im}\,(\gamma)$. Then there exists a singular subspace $\mathcal{U}'$ of $\mathcal{S}(\Pi_0, \Pi_1)$ such that $U = \mathcal{U}'\gamma$. ◻

Prop. 2.4 could be connected with the following:

Proposition 2.5 Let $\mathcal{F}$ be the collection of all singular subspaces of $\mathcal{S}(\Pi_0, \Pi_1)$, and $\check{\mathcal{U}}_i := \bigcup_{d=-1}^{n_i} {}^d\check{\mathcal{U}}_i$ for $i = 0, 1$. Then

$$\mathcal{F} = \{\{P_0\} \times U_1 | P_0 \in \mathcal{P}_0,\ U_1 \in \check{\mathcal{U}}_1\} \cup \{U_0 \times \{P_1\} | P_1 \in \mathcal{P}_1,\ U_0 \in \check{\mathcal{U}}_0\}. ◻$$

Proposition 2.6 Let $\gamma' : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$ be a linear morphism. Assume that $\{X_{i0}, X_{i1}, \ldots, X_{i,n_i+1}\}$ is a frame of Π_i, $i = 0, 1$, and let

$$P_{jh} := (X_{0j}, X_{1h}),\ j = 0, 1, \ldots, n_0 + 1,\ h = 0, 1, \ldots, n_1 + 1.$$

If

$$P_{jh}\gamma \quad = \quad P_{jh}\gamma', \quad j = 0, 1, \ldots, n_0,\ h = 0, 1, \ldots, n_1, \quad \textit{and}$$

$$P_{n_0+1,n_1+1}\gamma \quad = \quad P_{n_0+1,n_1+1}\gamma',$$

then $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\gamma')$.

Proof By prop. 2.2, γ' is regular. Write

$${}^{n_1}\overline{Z}_j := [\{P_{j0}\gamma, P_{j1}\gamma, \ldots, P_{jn_1}\gamma\}], \quad j = 0, 1, \ldots, n_0.$$

For every j, we have ${}^{n_1}\overline{Z}_j = (\{X_{0j}\} \times \mathcal{P}_1)\gamma = (\{X_{0j}\} \times \mathcal{P}_1)\gamma'$.

The point $P_{n_0+1,n_1+1}\gamma$ belongs to exactly one n_0-flat ${}^{n_0}U$ which meets all the flats ${}^{n_1}\overline{Z}_j$, $j = 0, 1, \ldots, n_0$, since

$$\dim \Pi([{}^{n_1}\overline{Z}_0 \cup {}^{n_1}\overline{Z}_1 \cup \ldots \cup {}^{n_1}\overline{Z}_{n_0}]) = \dim \Pi([\mathrm{im}\,(\gamma)]) = n_0 n_1 + n_0 + n_1.$$

Such an n_0-flat is necessarily

$${}^{n_0}U = (\mathcal{P}_0 \times \{X_{1,n_1+1}\})\gamma = (\mathcal{P}_0 \times \{X_{1,n_1+1}\})\gamma'.$$

Now let

$${}^{n_0}\overline{Y}_h := [\{P_{0h}\gamma, P_{1h}\gamma, \ldots, P_{n_0\,h}\gamma\}], \quad h = 0, 1, \ldots, n_1 + 1.$$

By the assumptions, for $h = 0, 1, \ldots, n_1$, and the above arguments, for $h = n_1 + 1$,

$${}^{n_0}\overline{Y}_h = (\mathcal{P}_0 \times \{X_{1h}\})\gamma = (\mathcal{P}_0 \times \{X_{1h}\})\gamma' \quad \text{for } h = 0, 1, \ldots, n_1 + 1.$$

For any $(X_0, X_1) \in \mathcal{U}$, the point $(X_0, X_1)\gamma$ belongs to $(\{X_0\} \times \mathcal{P}_1)\gamma$, which is the unique n_1-flat of $\overline{\Pi}$ that contains $(X_0, X_{1,n_1+1})\gamma \in {}^{n_0}\overline{Y}_{n_1+1}$ and meets every ${}^{n_0}\overline{Y}_h$, $h = 0, 1, \ldots, n_1$. So, both $\mathrm{im}\,(\gamma)$ and $\mathrm{im}\,(\gamma')$ are the union of all the n_1-flats of $\overline{\Pi}$ which meet ${}^{n_0}\overline{Y}_0$, ${}^{n_0}\overline{Y}_1$, …, ${}^{n_0}\overline{Y}_{n_1+1}$. ▫

3 On the automorphism group of $\mathcal{S}(\Pi_0, \Pi_1)$

The automorphism group $\mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$ has been studied in [4]; here we give some further results.

Let $\mathrm{P\Gamma L}(\Pi)$ be the collineation group of Π. If Π has dimension one, then we consider every bijective transformation of the point set of Π as a collineation.

Proposition 3.1 [4] $\mathrm{P\Gamma L}(\Pi_0) \times \mathrm{P\Gamma L}(\Pi_1)$ *is a normal subgroup of* $\mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$. *If* $\mathrm{P\Gamma L}(\Pi_0) \times \mathrm{P\Gamma L}(\Pi_1)$ *does not coincide with* $\mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$, *then its index in* $\mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$ *is two.* ▫

Proposition 3.2 $\mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1) \neq \mathrm{P\Gamma L}(\Pi_0) \times \mathrm{P\Gamma L}(\Pi_1)$ *if, and only if, there exists a collineation* $\delta : \Pi_0 \to \Pi_1$. *In this case, write*

$$\Delta : \mathcal{U} \longrightarrow \mathcal{U} : (X_0, X_1) \longmapsto (X_1\delta^{-1}, X_0\delta).$$

Then $\mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$ *is the semidirect product of* $\mathrm{P\Gamma L}(\Pi_0) \times \mathrm{P\Gamma L}(\Pi_1)$ *and* $\{\mathbf{1}, \Delta\}$.

Proof The first assertion has been proven in [4] and the second one is a corollary of prop. 3.1. ▫

We now characterize the automorphisms of $\mathcal{S}(\Pi_0, \Pi_1)$ which are related to the collineations of $\overline{\Pi}$, with respect to the regular linear morphism $\gamma : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$.

Proposition 3.3 Assume: $n_0 > 1$, $\alpha \in \mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$, $P_i \in \mathcal{P}_i$, $g_i \in \mathcal{G}_i$, $i = 0, 1$, $h_0 := (\{P_0\} \times g_1)\gamma$, $h_1 := (g_0 \times \{P_1\})\gamma$, $\tilde{\alpha} := \gamma^{-1}\alpha\gamma$. *(So* $\tilde{\alpha}$ *is a bijection of* $\mathrm{im}\,(\gamma)$*). The following are equivalent:*
(i) *There is a projectivity* $\zeta : h_0 \to h_1$ *such that the mapping* $\zeta' : h_0\tilde{\alpha} \to h_1\tilde{\alpha}$*, which is defined by* $P\zeta' := P\tilde{\alpha}^{-1}\zeta\tilde{\alpha}$ *for all* $P \in h_0\tilde{\alpha}$*, is a projectivity.*
(ii) *There is a collineation* κ *of* $\overline{\Pi}$ *such that* $\kappa_{|\mathrm{im}\,(\gamma)} = \tilde{\alpha}$*.*

Proof $(i) \Rightarrow (ii)$. For $i = 0, 1$ take a frame of Π_i, say

$$\{X_{i0}, X_{i1}, \ldots, X_{i,n_i+1}\},$$

such that $X_{10} = P_1$, $X_{00} = P_0$ and g_1 contains two elements of a frame, say $X_{1h'}$ and $X_{1h''}$. Let

$$\overline{P}_{jh} := (X_{0j}, X_{1h})\gamma, \quad j = 0, 1, \ldots, n_0 + 1,\ h = 0, 1, \ldots, n_1 + 1.$$

Then, by prop. 2.2,

$$\mathcal{E} := \{\overline{P}_{jh} | j = 0, 1, \ldots, n_0;\ h = 0, 1, \ldots, n_1\} \cup \{\overline{P}_{n_0+1,n_1+1}\}$$

is a frame of $\overline{\Pi}$.

$\alpha\gamma$ is a regular linear morphism, thus $\tilde{\mathcal{E}} := \mathcal{E}\tilde{\alpha}$ is a frame of $\overline{\Pi}$. The map $\tilde{\alpha}_{|(\mathcal{P}_0 \times \{P_1\})\gamma}$ is a collineation; let κ be the unique collineation of $\overline{\Pi}$ that coincides with $\tilde{\alpha}$ on $\mathcal{E} \cup (\mathcal{P}_0 \times \{P_1\})\gamma$. The existence of κ follows from $n_0 > 1$. Since h_0 contains $\overline{P}_{0h'}, \overline{P}_{0h''} \in \mathcal{E}$, we have $h_0\tilde{\alpha} = (\{P_0\} \times g_1)\kappa$. If $X \in g_1$, then $(P_0, X)\gamma\zeta \in h_1 \subset (\mathcal{P}_0 \times \{P_1\})\gamma$. Therefore:

$$(P_0, X)\gamma\tilde{\alpha} = (P_0, X)\gamma\zeta\tilde{\alpha}(\zeta')^{-1} = (P_0, X)\gamma\zeta\kappa(\zeta')^{-1},$$

hence $\tilde{\alpha}_{|h_0} = \zeta\kappa(\zeta')^{-1}{}_{|h_0}$. Since $h_0 = \overline{P}_{0h'}\overline{P}_{0h''}$, the mappings κ and $\zeta\kappa(\zeta')^{-1}$ coincide on $\overline{P}_{0h'}$, $\overline{P}_{0h''}$, $h_0 \cap [\mathcal{E} \setminus \{\overline{P}_{0h'}, \overline{P}_{0h''}\}]$ and are related to the same automorphism of the field underlying $\overline{\Pi}$. It follows that $\kappa_{|h_0} = \tilde{\alpha}_{|h_0}$ and

$$\kappa_{|(\{P_0\} \times \mathcal{P}_1)\gamma} = \tilde{\alpha}_{|(\{P_0\} \times \mathcal{P}_1)\gamma}.$$

Now consider the regular linear morphisms $\alpha\gamma = \gamma\tilde{\alpha}$ and $\gamma\kappa$. They coincide on

$$\{(X_{0j}, X_{1h}) | j = 0, 1, \ldots, n_0;\ h = 0, 1, \ldots, n_1\} \cup \{(X_{0,n_0+1}, X_{1,n_1+1})\},$$

then (cf. prop. 2.6) $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\gamma\tilde{\alpha}) = \mathrm{im}\,(\gamma\kappa)$.

By previous arguments, if $X_0 = P_0$ or $X_1 = P_1$, then $(X_0, X_1)\gamma\tilde{\alpha} = (X_0, X_1)\gamma\kappa$. Then assume $X_0 \neq P_0$ and $X_1 \neq P_1$. By prop. 2.3, every line of $\overline{\Pi}$ which is contained in $\mathrm{im}\,(\gamma)$ is the image, through γ and also through $\gamma\tilde{\alpha}$ and $\gamma\kappa$, of an element of $\mathcal{R}$.

So there is exactly one point $P^* \in \mathrm{im}\,(\gamma)$, other than $(P_0, P_1)\gamma\tilde{\alpha}$, which is joined to the points

$$(X_0, P_1)\gamma\tilde{\alpha} = (X_0, P_1)\gamma\kappa \quad \text{and} \quad (P_0, X_1)\gamma\tilde{\alpha} = (P_0, X_1)\gamma\kappa,$$

by some line contained in $\mathrm{im}\,(\gamma)$. We conclude that $(X_0, X_1)\gamma\tilde{\alpha} = (X_0, X_1)\gamma\kappa = P^*$. Thus we proved that (i) implies (ii).

$(ii) \Rightarrow (i)$. Let ζ be a projectivity between the given lines. Then it is straightforward that also $\zeta' = (\kappa^{-1}\zeta\kappa)_{|h_0\alpha}$ is a projectivity. ▫

The hypothesis "$n_0 > 1$" has been used only to prove $(i) \Rightarrow (ii)$. It is possible to give examples which show that such a hypothesis cannot be deleted.

We conclude this section by giving a characterization of the automorphisms of $\mathcal{S}(\Pi_0, \Pi_1)$.

Proposition 3.4 *If* $\alpha : \mathcal{U} \to \mathcal{U}$ *is a bijection and* $g\alpha \in \mathcal{R}$ *for each* $g \in \mathcal{R}$*, then* $\alpha \in \mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$.

Proof We shall prove that if $g \in \mathcal{R}$, then $g\alpha^{-1} \in \mathcal{R}$. We assume, without loss of generality, that $g = \{X_0\} \times g_1$, $X_0 \in \mathcal{P}_0$, $g_1 \in \mathcal{G}_1$. In addition, let P and Q be two distinct points on g_1.

Assume $(X_0, P)\alpha^{-1} \not\sim (X_0, Q)\alpha^{-1}$ and let $(Y_0, Y_1) := (X_0, P)\alpha^{-1}$, $(Z_0, Z_1) := (X_0, Q)\alpha^{-1}$. Then $Y_0 \neq Z_0$ and $Y_1 \neq Z_1$. Take into account the following singular subspaces of $\mathcal{S}(\Pi_0, \Pi_1)$:

$$U_0 := (\{Y_0\} \times \mathcal{P}_1)\alpha, \quad U_1 := (\mathcal{P}_0 \times \{Z_1\})\alpha.$$

It holds:

$$(X_0, P) \in U_0 \setminus U_1, \quad (X_0, Q) \in U_1 \setminus U_0, \quad |U_0 \cap U_1| = 1.$$

Since the dimensions of Π_0 and Π_1 are finite, U_0 and U_1 are exactly the two maximal singular subspaces containing the singleton $U_0 \cap U_1$; this contradicts $(X_0, P) \sim (X_0, Q)$. As a consequence, $(X_0, P)\alpha^{-1} \sim (X_0, Q)\alpha^{-1}$, hence

$$g\alpha^{-1} = \{(X_0, P)\alpha^{-1}\} \vee \{(X_0, Q)\alpha^{-1}\}. ▫$$

4 Main results

We quote a result from [1] (prop. 3):

Proposition 4.1 *Assume that* Π_0 *and* Π_1 *are coordinatized by the same commutative field* F, $|F| > 3$, *and that at least one of the following is satisfied:*
(i) $\min\{n_0, n_1\} = 1$; (ii) $\mathrm{Aut}\,F \neq \{\mathbf{1}\}$.

Then there are a subset $\mathcal{M}$ *of* $\mathcal{U}$, *a projective space* $\Pi' = (\mathcal{P}', \mathcal{G}')$, *and two regular linear morphisms* $\gamma_i : \mathcal{S}(\Pi_0, \Pi_1) \to \Pi'$, $i = 0, 1$, *such that* $\mathrm{im}\,(\gamma_0) = \mathrm{im}\,(\gamma_1)$ *and that the following is true only for* $i = 0$:
(E_i) *There is a hyperplane* $\mathcal{H}_i$ *of* Π' *such that* $\mathcal{M} = \mathcal{H}_i\gamma_i^{-1}$. ▫

In the proof of prop. 4.1, assuming $n_0 \geq n_1$, a suitable non-projective collineation α' of Π_1 is considered. Then $\gamma_1 := (\mathrm{id}_{\mathcal{P}_0}, \alpha')\gamma_0$, where γ_0 is the Segre embedding.

A *universal embedding* of $\mathcal{S}(\Pi_0,\Pi_1)$ is an embedding $\rho:\mathcal{S}(\Pi_0,\Pi_1)\to\overline{\Pi}$ such that for each embedding $\chi:\mathcal{S}(\Pi_0,\Pi_1)\to\Pi'$ exactly one linear morphism $\psi:\overline{\Pi}\to\Pi'$ exists such that $\chi=\rho\psi$.

Proposition 4.2 If the assumptions of prop. 4.1 are satisfied, then there is no universal embedding of $\mathcal{S}(\Pi_0,\Pi_1)$.

Proof Assume that ρ is a universal embedding and take into account the linear morphisms γ_0 and γ_1 of prop. 4.1. The universal property of ρ gives $\operatorname{rk}\rho=n_0n_1+n_0+n_1$ (cf. also prop. 1.2). If ψ_i is the linear morphism such that $\rho\psi_i=\gamma_i$, $i=0,1$, then $\phi:=\psi_0^{-1}\psi_1$ is a collineation of Π'; hence $\mathcal{M}\gamma_1=\mathcal{M}\gamma_0\phi$. By (E_0), prop. 4.1, we obtain $\mathcal{M}\gamma_0=\mathcal{H}_0\cap\operatorname{im}(\gamma_0)$, whence $\mathcal{M}\gamma_1=\mathcal{H}_0\phi\cap\operatorname{im}(\gamma_1)$ and so (E_1) is true, in contradiction to prop. 4.1. ▫

Theorem 1 Let $\gamma:\mathcal{S}(\Pi_0,\Pi_1)\to\overline{\Pi}$, $\overline{\Pi}=(\overline{\mathcal{P}},\overline{\mathcal{G}})$, *be a regular linear morphism, and* $\chi:\mathcal{S}(\Pi_0,\Pi_1)\to\Pi'$, $\Pi'=(\mathcal{P}',\mathcal{G}')$, *an embedding;* $n_i:=\dim(\Pi_i)$, $i=0,1$; $1\le n_i<\infty$; $n_0>1$.

Then there are a collineation α' *of* Π_1 *and a linear morphism* $\psi:\overline{\Pi}\to\Pi'$ *such that* $\chi=\alpha\gamma\psi$, *where* $\alpha:=(\mathrm{id}_{\mathcal{P}_0},\alpha')$.

Proof All the projective spaces can be coordinatized by the same commutative field (cf. prop. 2.1).

Now take n_1+1 independent points of Π_1: A_0, A_1, …, A_{n_1}. Then let

$$^{n_0}\overline{Y}_h:=(\mathcal{P}_0\times\{A_h\})\gamma\in{}^{n_0}\check{\mathcal{U}}(\overline{\Pi}),\quad h=0,1,\dots,n_1.$$

We have:

$$\begin{aligned}[{}^{n_0}\overline{Y}_0 &\cup {}^{n_0}\overline{Y}_1\cup\ldots\cup{}^{n_0}\overline{Y}_{n_1}]=[(\mathcal{P}_0\times\{A_0,A_1,\dots,A_{n_1}\})\gamma]\supset[\operatorname{im}(\gamma)]=\\ &= \overline{\mathcal{P}}.\end{aligned}$$

Now let, for $t=0,1,\dots,n_1$,

$$U_t:=[{}^{n_0}\overline{Y}_0\cup{}^{n_0}\overline{Y}_1\cup\ldots\cup{}^{n_0}\overline{Y}_t]\in{}^{tn_0+t+n_0}\check{\mathcal{U}}(\overline{\Pi}).$$

Next we define, recursively on $t=0,1,\dots,n_1$, a linear morphism $\psi_t:U_t\to\mathcal{P}'$ with the property

$$(4.1)_t\qquad \psi_{t|{}^{n_0}\overline{Y}_h}=\gamma^{-1}\chi_{|{}^{n_0}\overline{Y}_h}\quad\text{for } h=0,1,\dots,t.$$

The property $(4.1)_0$ is already a definition of ψ_0. Now let $0<t\le n_1$ and assume that there is a linear morphism $\psi_{t-1}:U_{t-1}\to\mathcal{P}'$ satisfying $(4.1)_{t-1}$.

Since γ is regular, U_{t-1} and ${}^{n_0}\overline{Y}_t$ are complementary subspaces of U_t. Define

$$\psi'_t\,:\,{}^{n_0}\overline{Y}_t\longrightarrow\mathcal{P}'\,:\,P\longmapsto P\gamma^{-1}\chi.$$

We have $\operatorname{rk}\psi_{t-1},\operatorname{rk}\psi'_t\ge n_0\ge 2$. Take $h_0\in\mathcal{G}_0$ and let

$$g_1:=(h_0\times\{A_0\})\gamma;\quad g_2:=(h_0\times\{A_t\})\gamma.$$

Then $g_1 \subset \mathbf{D}(\psi_{t-1})$ and $g_2 \subset \mathbf{D}(\psi'_t)$. Define

$$\sigma' : g_1\psi_{t-1} \longrightarrow g_2\psi'_t : (X_0, A_0)\chi \longmapsto (X_0, A_t)\chi,$$

and, for $A \in A_0A_t \setminus \{A_0, A_t\}$,

$$g := (h_0 \times \{A\})\chi \in \mathcal{G}'.$$

Since $\{P\}\sigma' = (\{P\} \vee g) \cap g_2\psi'_t$ for all $P \in g_1\psi_{t-1}$, σ' is a projectivity. Now take into account

$$\sigma := (\psi_{t-1|g_1})\sigma'(\psi'_{t|g_2})^{-1}.$$

Then $(X_0, A_0)\gamma\sigma = (X_0, A_t)\gamma$ for all $X_0 \in h_0$; hence also σ is a projectivity. Thus, by prop. 1.6, there is an extension, say ψ_t, of ψ_{t-1} and ψ'_t satisfying $(4.1)_t$.

Let $\psi := \psi_{n_1}$ and, for $h = 0, 1, \ldots, n_1$ and $X_0 \in \mathcal{P}_0$:

$$\begin{array}{rcl} Q_h(X_0) & := & (X_0, A_h)\gamma; \\ T(X_0) & := & [\{Q_0(X_0), Q_1(X_0), \ldots, Q_{n_1}(X_0)\}] = (\{X_0\} \times \mathcal{P}_1)\gamma. \end{array} \tag{4.2}$$

From $Q_h(X_0)\psi = (X_0, A_h)\chi$ for all $X_0 \in \mathcal{P}_0$ and $h = 0, 1, \ldots, n_1$, it follows

$$T(X_0)\psi = (\{X_0\} \times \mathcal{P}_1)\chi. \tag{4.3}$$

The dimension of $\Pi(T(X_0)\psi)$ is n_1, then $\psi_{|T(X_0)}$ is global. Hence also $\gamma\psi$ is global.

Now assume $(X_0, X_1)\gamma\psi = (Y_0, Y_1)\gamma\psi$ and $(X_0, X_1) \neq (Y_0, Y_1)$. If $X_0 = Y_0$, then the axiom (L2) for $\gamma\psi$ implies that such a linear morphism is not global. This contradiction gives $X_0 \neq Y_0$. We have

$$\begin{array}{cc} T(X_0)\psi = (\{X_0\} \times \mathcal{P}_1)\chi, & T(Y_0)\psi = (\{Y_0\} \times \mathcal{P}_1)\chi, \\ (X_0, X_1)\gamma \in T(X_0), & (Y_0, Y_1)\gamma \in T(Y_0), \end{array}$$

hence $(\{X_0\} \times \mathcal{P}_1)\chi \cap (\{Y_0\} \times \mathcal{P}_1)\chi \neq \emptyset$, in contradiction with the assumption that χ is an embedding. So we have shown that $\gamma\psi$ is injective.

$\alpha := \chi\psi^{-1}\gamma^{-1}$ is a well-defined mapping and a bijection, since, by (4.2) and (4.3),

$$\begin{array}{rcl} P \in \mathrm{im}\,(\gamma\psi) & \Longleftrightarrow & \text{there exists } X_0 \in \mathcal{P}_0 \text{ such that } P \in T(X_0)\psi \\ & \Longleftrightarrow & P \in \mathrm{im}\,(\chi). \end{array}$$

Let $g \in \mathcal{R}$. Then $g\chi\psi^{-1} \cap \mathrm{im}\,(\gamma)$ is a line. The lines of $\overline{\Pi}$, which are contained in $\mathrm{im}\,(\gamma)$, are exactly the images of the lines of $\mathcal{S}(\Pi_0, \Pi_1)$ (cf. prop. 2.3), hence $g\alpha \in \mathcal{R}$. Therefore, by prop. 3.4, α is an automorphism of $\mathcal{S}(\Pi_0, \Pi_1)$. In addition $(\{X_0\} \times \mathcal{P}_1)\alpha \subset \{X_0\} \times \mathcal{P}_1$; then, by prop. 3.2, we conclude that α is in the form $(\mathrm{id}_{\mathcal{P}_0}, \alpha')$, α' a collineation of Π_1. □

Remarks on theorem 1.

1. Theorem 1 does not hold true for $n_0 = n_1 = 1$. For instance, if α^* is a transformation of $\Pi = \mathrm{PG}(1, \mathbf{R})$ which exchanges two points and is identical on the remaining ones, then $\chi := (\alpha^*, \alpha^*)\gamma$ cannot be represented as desired. However, every embedding of $\mathcal{S}(\Pi_0, \Pi_1)$ is a composition of the kind $\alpha\gamma\psi$, where $\alpha \in \mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$, even if $n_0 = n_1 = 1$.

2. There are examples of embeddings such that the related linear morphism ψ has a non-empty exceptional set $\mathbf{A}(\psi)$. This is possible only if $n_0, n_1 > 1$.

3. Theorem 1 cannot be extended to the case that χ is a linear morphism. We show an example, assuming $n_0, n_1 > 1$. Let g^* be a line of Π_1 and U a complementary subspace with respect to g^*. Let $\sigma : \Pi_1 \to \Pi(g^*)$ be the projection from U onto g^*. Assume that $\alpha^* : g^* \to g^*$ is a bijective transformation, and that there exists no collineation κ of Π_1 such that $\kappa_{|g^*} = \alpha^*$. Clearly, we can choose Π_0 and Π_1 so that such an α^* and a regular linear morphism $\gamma : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$ exist. Then $\chi := (\mathrm{id}_{\mathcal{P}_0}, \sigma\alpha^*)\gamma$ is a linear morphism. Now we assume that there exist $\alpha \in \mathrm{Aut}\,\mathcal{S}(\Pi_0, \Pi_1)$ and a linear morphism $\psi : \overline{\Pi} \to \overline{\Pi}$ such that

$$\chi = \alpha\gamma\psi. \tag{4.4}$$

In order to show that (4.4) leads to a contradiction we prove two propositions.

Proposition 4.3 $[\mathrm{im}\,(\chi)] \cap \mathrm{im}\,(\gamma) = (\mathcal{P}_0 \times g^*)\gamma$.

Proof Since $\mathrm{im}\,(\chi) = (\mathcal{P}_0 \times g^*)\gamma$, we have $(\mathcal{P}_0 \times g^*)\gamma \subset [\mathrm{im}\,(\chi)] \cap \mathrm{im}\,(\gamma)$.

Now suppose there is $(X_0^*, X_1^*)\gamma \in [(\mathcal{P}_0 \times g^*)\gamma]$ such that $X_1^* \notin g^*$. Let V be a subspace of Π_1 with $X_1^* \in V$ and such that g^* and V are complementary subspaces of Π_1. Then

$$\mathrm{im}\,(\gamma) \subset (\mathcal{P}_0 \times V)\gamma \vee (\mathcal{P}_0 \times g^*)\gamma, \quad \text{and} \quad [(\mathcal{P}_0 \times V)\gamma] \cap [(\mathcal{P}_0 \times g^*)\gamma] \neq \emptyset$$

contradict the regularity of γ. □

Take $P_0^* \in \mathcal{P}_0$, $P_1^* \in \mathcal{P}_1 \setminus g^*$ and let

$$\begin{array}{lll} P := (P_0^*, P_1^*), & g := \{P_0^*\} \times g^*, & \mathcal{E} := \{P\} \vee g, \\ Q := P\alpha\gamma & h := g\alpha\gamma, & \mathcal{F} := \mathcal{E}\alpha\gamma. \end{array}$$

Proposition 4.4 There exist a projection $\pi : \overline{\Pi} \to \overline{\Pi}$ and a collineation κ of $\overline{\Pi}$ such that (i) $\psi = \pi\kappa$; (ii) $\psi_{|h} = \kappa_{|h}$; (iii) $\mathcal{F}\kappa = \mathcal{E}\gamma$.

Proof By prop. 1.5, there are a map π' and an injective linear morphism $\kappa' : \mathrm{im}\,(\pi') \to \overline{\mathcal{P}}$ such that $\psi = \pi'\kappa'$. From $h\pi'\kappa' = g\gamma$ we have $h \cap \mathbf{A}(\pi') = \emptyset$. Now let V be such that V and $h\pi'$ are complementary subspaces of $\mathrm{im}\,(\pi')$. Let $W := V \vee h$. Since $\mathbf{A}(\pi') \vee V \vee h = \overline{\mathcal{P}}$, the projection $\zeta : W \to \mathrm{im}\,(\pi')$ from $\mathbf{A}(\pi')$ is a collineation. Let π be the projection from $\mathbf{A}(\pi) := \mathbf{A}(\pi')$ onto W and $\kappa'' := \zeta\kappa'$. Then $\psi = \pi'\kappa' = \pi\kappa''$. Furthermore $\mathcal{F}\pi\kappa'' = \mathcal{E}\alpha\gamma\psi = g\gamma$, hence $Q \in \mathcal{F} \setminus W$.

We have $[\mathrm{im}\,(\alpha\gamma)] = \overline{\mathcal{P}}$, whence $[\mathrm{im}\,(\alpha\gamma\pi)] = W$ and $[\mathrm{im}\,(\chi)] = [\mathrm{im}\,(\alpha\gamma\pi\kappa'')] = \mathrm{im}\,(\kappa'')$. By prop. 4.3, $P\gamma \notin \mathrm{im}\,(\kappa'')$. Since $\mathrm{rk}\,\kappa'' \geq 2$, the collineation κ'' can be extended to a $\kappa \in \mathrm{P\Gamma L}(\overline{\Pi})$ such that $Q\kappa = P\gamma$. (i) and (ii) are clearly satisfied, and

$$\mathcal{F}\kappa = \{Q\}\kappa \vee h\kappa = \{P\}\gamma \vee g\gamma = \mathcal{E}\gamma. \square$$

By prop. 4.4, the following map is a collineation of $\mathcal{E}$:

$$\overline{\alpha} : X \longmapsto X\alpha\gamma\kappa\gamma^{-1}.$$

If $X \in g$, then $X\overline{\alpha} = X\alpha\gamma\pi\kappa\gamma^{-1} = X(\mathrm{id}_{\mathcal{P}_0}, \alpha^*)$, hence α^* can be extended to a collineation of the plane $\{P_1^*\} \vee g^*$ and to a collineation of Π_1. By this contradiction, we conclude that (4.4) is false.

Theorem 2 *Assume that* Π_0 *and* Π_1 *are coordinatized by the commutative field* F *and that both the following conditions are satisfied:* (i) $n_0, n_1 > 1$*;* (ii) $\mathrm{Aut}\, F = \{\mathbf{1}\}$. *Then every regular linear morphism* $\gamma : \mathcal{S}(\Pi_0, \Pi_1) \to \overline{\Pi}$ *is a universal embedding.*

Proof Let χ be an embedding of $\mathcal{S}(\Pi_0, \Pi_1)$. By ($i$) and ($ii$), the collineation α' of theorem 1 is projective.

Now we use prop. 3.3. Let $\zeta : h_0 \to h_1$ be any projectivity. Then both $\tilde{\alpha}_{|h_1}$ and $\tilde{\alpha}^{-1}{}_{|h_0\tilde{\alpha}}$ are projectivities, because they are restrictions of the projective collineations

$$\begin{aligned} \tilde{\alpha}_{|(\mathcal{P}_0 \times \{P_1\})\gamma} &: (X_0, P_1)\gamma \longmapsto (X_0, P_1\alpha')\gamma, \\ \tilde{\alpha}^{-1}{}_{|(\{P_0\} \times \mathcal{P}_1)\gamma} &: (P_0, X_1)\gamma \longmapsto (P_0, X_1(\alpha')^{-1})\gamma, \end{aligned}$$

respectively. Thus ζ' is a projectivity and a collineation κ of $\overline{\Pi}$ exists such that $\kappa_{|\mathrm{im}\,(\gamma)} = \tilde{\alpha}$. So $\alpha\gamma = \gamma\kappa$. We conclude that $\chi = \gamma\tilde{\psi}$, with $\tilde{\psi} := \kappa\psi$. $\square$

We now take into account a special family of semilinear spaces. To this end, we need some further notions.

A semilinear space $(\mathcal{U}, \mathcal{R})$ is said to be a *projective semilinear space* if *Veblen's axiom* holds:

(V) For any $g, g' \in \mathcal{R}$, $g \cap g' = \emptyset$, $P \in \mathcal{U} \setminus (g \cup g')$, at most one line exists through P meeting both g and g'.

It should be noted that the singular subspaces of a projective semilinear space are projective spaces.

A semilinear space $(\mathcal{U}, \mathcal{R})$ is said to be *irreducible* if any line contains at least three points; if any two points in $(\mathcal{U}, \mathcal{R})$ are collinear, then $(\mathcal{U}, \mathcal{R})$ is a *linear space*; if this is not the case, $(\mathcal{U}, \mathcal{R})$ is a *proper* semilinear space.

A *regular pseudoproduct space* is a semilinear space $(\mathcal{U}, \mathcal{R})$ satisfying the following axioms [4]:

(RP1) $(\mathcal{U}, \mathcal{R})$ is proper, irreducible, projective and contains at least one subspace which properly contains a line.

(RP2) $P, Q \in \mathcal{U}$, $P \not\sim Q$ imply that precisely two points exist, P' and Q', such that $P \sim P' \sim Q$ and $P \sim Q' \sim Q$. (We obtain a quadrangle, which is called the *quadrangle with respect to (w.r.t.) P and Q*, and is denoted by $\langle P,Q,P',Q'\rangle$.)

(RP3) If $P \in \mathcal{U}$, $g \in \mathcal{R}$ and $P \not\sim Q$ for all $Q \in g$, then a unique $T_{P,g} \in \mathcal{U}$ exists such that $P \sim T_{P,g} \sim Q$ for all $Q \in g$. Furthermore, let $\langle P,Q,T_{P,g},Q'\rangle$ be the quadrangle w.r.t. P,Q; then Q' spans a line as Q ranges on g.

Proposition 4.5 [4] *A regular pseudoproduct space is isomorphic to a product space* $\mathcal{S}(\Pi_0,\Pi_1)$*, with* $\max\{\dim(\Pi_0),\dim(\Pi_1)\} \geq 2$. ▫

Assume that the projective spaces Π_0, Π_1, $\overline{\Pi}$, Π' are coordinatized by the commutative field F. We recall (prop. 1.5) that every linear morphism between projective spaces is a composition of a projection and a collineation. Hence, by theorem 1, if $\chi : \mathcal{S}(\Pi_0,\Pi_1) \to \Pi'$ is an embedding, then $\mathrm{im}\,(\chi)$ can be obtained from the Segre variety $\mathrm{im}\,(\gamma_0)$ by operating a projection and a collineation; indeed, $\mathrm{im}\,(\chi) = \mathrm{im}\,(\gamma_0)\psi$. (For the case $n_0 = n_1 = 1$ see remark 1.) So ψ is related to a semilinear transformation $\hat{\psi}$ between vector spaces. $\hat{\psi}$ is related to a field automorphism $\hat{\theta}$, and $\hat{\theta}^{-1}\hat{\psi}$ is a linear transformation. It is clear that the collineation of $\overline{\Pi}$ related to $\hat{\theta}^{-1}$ maps $\mathrm{im}\,(\gamma_0)$ onto itself. As a consequence, if ψ' is the linear morphism that is related to $\hat{\theta}^{-1}\hat{\psi}$, then $\mathrm{im}\,(\chi) = \mathrm{im}\,(\gamma_0)\psi'$. Now we can summarize:

Theorem 3 *Let $\mathcal{S}$ be a regular pseudoproduct space. Assume that $\mathcal{S}$ is embedded in the projective space* $\mathrm{PG}(n,F)$*. Then $\mathcal{S}$ is, up to projective collineations, a projection of a Segre variety. If, in addition, the embedding is regular, then $\mathcal{S}$ is projectively equivalent to a Segre variety.* ▫

In the previous theorem we use the new notion of a regular embedding of $\mathcal{S}$, which is spontaneous (cf. p. 68).

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