Existence of Solutions for Quasilinear Elliptic Boundary Value Problems in Unbounded Domains.

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Abstract

Under suitable assumptions we prove, via the Leray-Schauder fixed point theorem, the existence of a solution for quasilinear elliptic boundary value problem in $C^{2,\alpha}(\bar{\Omega}) \cap W^{2,q}(\Omega), q > N$ which satisfies in addition the condition, $(1+ \mid x \mid^2)^{\frac{1}{2}} u \in C^{2,\alpha}(\bar{\Omega})$.

1 Introduction

Let G be a bounded, open and not empty subset of $\mathbb{R}^N$ with $C^{2,\alpha}$ boundary, $N \geq 2,\ 0 < \alpha < 1$ and let $\Omega := \mathbb{R}^N \backslash \bar{G}$. In this paper we consider quasilinear elliptic boundary value problems of the form,

$$(\mathcal{P}) \qquad \begin{cases} \sum a_{ij}(x,u) D_{ij} u - u &= f(x,u,Du) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

These problems have been investigated by many authors under various assumptions (see [3], [8], [10] and references mentioned there). Our aim is to establish, using the Leray-Schauder fixed point theorem, the existence of smooth solutions for $(\mathcal{P})$, under the assumptions listed below:

(A1) The function $g(x,z,p) := (1+ \mid x \mid^2)^{\frac{1}{2}} f(x,z,p)$ satisfies the conditions:

i) $\mid g(x,z,p) \mid \ \leq \ \varphi(\mid z \mid) \left(1+ \mid p \mid^2\right)$ for all $x \in \Omega$, $z \in I\!R$ and $p \in I\!R^N$.

Received by the editors July 1995.
Communicated by J. Mawhin.

ii) $| g(x,z,p) - g(x',z',p') | \leq \varphi(L)\left\{| x - x' |^{\alpha} + | z - z' | + | p - p' | \right\}$,
for all $L \geq 0$; $x, x' \in \bar{\Omega}$; $z, z' \in [-L, L]$ and $p, p' \in B_L(0)$.
where φ is a positive increasing function.

(A2) Suppose that all eventual solutions of the problem $(\mathcal{P})$ in the space $C^2(\bar{\Omega})$ tending to zero at infinite are a priori bounded in $L^{\infty}(\Omega)$.

(A3) **i)** $\nu | \xi |^2 \leq \sum a_{ij}(x,z)\xi_i\xi_j \leq \mu | \xi |^2$,
for all $x \in \Omega$, $z \in I\!R$ and $\xi \in I\!R^N$
ii) $| a_{ij}(x,z) - a_{ij}(x',z') | \leq \psi(L)\left\{| x - x' |^{\alpha} + | z - z' |\right\}$,
for all $L > 0$; $x, x' \in \Omega$ and $z, z' \in [-L, L]$.
where ν and μ are positive constants and ψ is an increasing function.

(A4) We suppose that $2\mu - (N-1)\nu < 1 + R^2$
where R is the radius of the largest ball contained in $G = \mathbb{R}^N \backslash \bar{\Omega}$.

Remarks 1.1

a. The assumption **(A2)** is satisfied if one of the following conditions holds:

(A'2) f is continuously differentiable with respect to the p and z variables. Furtheremore, for some constant $\lambda > -1$ we have,

$$\frac{\partial f}{\partial z}(x,z,0) \geq \lambda \qquad \forall\, x \in \Omega\,, \quad \forall\, z \in I\!R.$$

(A"2) There exists a constant Λ such that ,

$$zf(x,z,0) > -z^2, \qquad \text{for all } x \in \Omega \text{ and } | z | \geq \Lambda$$

b. By a further translation of the domain we assume, without loss of generality, that the ball $B_R(0)$ is contained in G. That is, $| x | \geq R \qquad \forall x \in \Omega$.

c. Our results can be generalized for general unbounded subdomains of $\mathbb{R}^N$ with smooth boundary.

The main result of this paper is stated as follows :

Theorem 1.1 *If the assumptions* **(A1)**; **(A2)**; **(A3)** *and* **(A4)** *are satisfied, then for any $q > N$, the problem $(\mathcal{P})$ has a solution u in the space $C^{2,\alpha}(\bar{\Omega}) \cap W^{2,q}(\Omega)$. Furtheremore,* $(1 + | x |^2)^{\frac{1}{2}} u \in C^{2,\alpha}(\bar{\Omega})$.

Let $p \geq \frac{N}{1-\alpha}$ be fixed. From now on we suppose that the assumptions **(A1); (A2); (A3)** and **(A4)** are satisfied.

2 A priori estimates

The purpose of this section is to establish the following theorem,

Theorem 2.1 *There exists a constant $c > 0$ such that any solution $u \in W^{2,p}(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ of the problem $(\mathcal{P})$ satisfies,*

i) $\| (1 + | x |^2)^{\frac{1}{2}} u \|_{2,\alpha,\Omega} \leq c.$

ii) $\| u \|_{2,p,\Omega'} \leq c \, \| (1 + | x |^2)^{-\frac{1}{2}} \|_{L^p(\Omega')} \qquad \forall \Omega' \subset \Omega.$

Where, here and in the following, we use the notations:

$$\| v \|_{0,0,\Omega} := \sup_{x\in\Omega} | v(x) | \quad , \quad [v]_{\alpha,\Omega} := \sup_{x,y\in\Omega, x\neq y} \frac{| v(x)-v(y) |}{| x-y |^{\alpha}}$$

$$\| v \|_{k,\alpha,\Omega} = \| v \|_{C^{k,\alpha}(\bar\Omega)} := \sum_{|s|\le k} \| D^s v \|_{0,0,\Omega} + \sum_{|s|=k} [D^s v]_{\alpha,\Omega}$$

$$\| v \|_{k,p,\Omega} = \| v \|_{W^{k,p}(\Omega)} := \left[\sum_{|s|\le k} \int_\Omega | D^s v |^p \, dx \right]^{1/p} .$$

By a standard regularity argument it is easy to verify that any solution of $(\mathcal{P})$ in the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ has the property: $(1+ | x |^2)^{\frac{1}{2}} u \in C^{2,\alpha}(\bar\Omega)$.
Before proving the theorem 2.1, we establish the following lemmas:

Lemma 2.2 *There exists a constant $c > 0$ such that any solution $u \in C^{2,\alpha}(\bar\Omega) \cap W^{2,p}(\Omega)$ of $(\mathcal{P})$ satisfies the estimate*,

$$\| Du \|_{0,\alpha,\Omega} \le c$$

Proof: The technique used here is similar to the one used in the first part of [9]. Let $\bar x \in \partial\Omega$, $q \ge 0$ and ζ be a real-valued function in $C^\infty(\mathbb{R}^N)$ with $\zeta(x) = 1$ for $| x |\le \frac{1}{2}$ and $\zeta(x) = 0$ for $| x |\ge 1$. For $r \in (0,1)$, we define the function ζ_r by setting, $\zeta_r(x) := \zeta(\frac{x-\bar x}{r})$.
In what follows c and $c(r)$ denote generic constants that depend only on ν, μ, N, q, $M :=\| u \|_{0,0,\Omega}$, and eventualy on r. By the elliptic regularity of the Laplacian together with the assumption **(A3)** we have,

$$\int_{\Omega_r} \sum | D_{ij}(\zeta_r^2 u) |^{q+2} dx \le c \int_{\Omega_r} | \sum a_{ij}(\bar x, u(\bar x)) D_{ij}(\zeta_r^2 u) |^{q+2} dx \qquad (2.1)$$

Where $\Omega_r := \Omega \cap B_r(\bar x)$.
On the other hand by [6, theorem 1], there exist a constant $r_1 < 1$ depending only on $\partial\Omega$, and two constants $c > 0$ and $\beta \in (0,1)$ depending only on ν, μ, M, r_1 and $\partial\Omega$ such that,

$$[u]_{\beta,\Omega_{r_1}} \le c \qquad (2.2)$$

According to **(A3)**, (2.1), (2.2) and the triangle inequality, we may choose $r_2 \le r_1$ small enough so that for any $r \le r_2$ we have,

$$\int_{\Omega_r} \sum | D_{ij}(\zeta_r^2 u) |^{q+2} \, dx \le c \int_{\Omega_r} | \sum a_{ij}(x, u(x)) D_{ij}(\zeta_r^2 u) |^{q+2} \, dx \qquad (2.3)$$

By differentiation we obtain,

$$D_i(\zeta_r^2 u) = \zeta_r^2 D_i u + 2\zeta_r u D_i \zeta_r \qquad (2.4)$$

$$D_{ij}(\zeta_r^2 u) = \zeta_r^2 D_{ij} u + 2\zeta_r [D_i u D_j \zeta_r + D_j u D_i \zeta_r] + 2u D_i \zeta_r D_j \zeta_r + 2\zeta_r u D_{ij} \zeta_r$$

Using (2.4), **(A1)** and **(A3)** , it is easy to verify that for any $r \le r_2$ we have,

$$\int_{\Omega_r} | \sum a_{ij}(x, u(x)) D_{ij}(\zeta_r^2 u) |^{q+2} \, dx \le c \int_{\Omega_r} (\zeta_r^2 \mid Du |^2)^{q+2} dx + c(r) \qquad (2.5)$$

and,

$$\int_{\Omega_r} \sum (\zeta_r^2 \mid D_{ij} u \mid)^{q+2} dx \leq \tag{2.6}$$
$$c\left\{\int_{\Omega_r} \sum \mid D_{ij}(\zeta_r^2 u) \mid^{q+2} dx + \int_{\Omega_r} (\zeta_r^2 \mid Du \mid^2)^{q+2} dx\right\} + c(r)$$

Combining the identities (2.4), (2.5) and (2.6) with the following interpolation inequality [9], [7] :

$$\int_{\Omega_r} (\zeta_r^2 \mid Du \mid^2)^{q+2} dx \leq c\left\{\left(\frac{\delta}{r}\right)^4 \int_{\Omega_r} (\zeta_r^2 \mid Du \mid^2)^q dx + \delta^{q+2} \int_{\Omega_r} \sum (\zeta_r^2 \mid D_{ij} u \mid)^{q+2} dx\right\}$$

where, $\delta := \parallel u - u(\bar{x}) \parallel_{0,0,\Omega_r}$
we obtain for any $r \leq r_2$ the inequality,

$$\int_{\Omega_r} \sum \mid D_{ij}(\zeta_r^2 u) \mid^{q+2} dx + \int_{\Omega_r} (\zeta_r^2 \mid Du \mid^2)^{q+2} dx \leq \tag{2.7}$$
$$c\delta^{q+2}\left\{\int_{\Omega_r} \sum \mid D_{ij}(\zeta_r^2 u) \mid^{q+2} dx + \int_{\Omega_r} (\zeta_r^2 \mid Du \mid^2)^{q+2} dx\right\}$$
$$+c.\left(\frac{\delta}{r}\right)^4 \int_{\Omega_r} (\zeta_r^2 \mid Du \mid^2)^q dx + c(r)$$

Then, from (2.2) and (2.7), we get for $\bar{r}$ small enough,

$$\int_{\Omega_{\bar{r}}} \sum \mid D_{ij}(\zeta_{\bar{r}}^2 u) \mid^{q+2} dx + \int_{\Omega_{\bar{r}}} (\zeta_{\bar{r}}^2 \mid Du \mid^2)^{q+2} dx \leq \tag{2.8}$$
$$c(\bar{r}) \int_{\Omega_{\bar{r}}} (\zeta_{\bar{r}}^2 \mid Du \mid^2)^q dx + c(\bar{r})$$

This inequality is valid for any nonnegative real q then, by induction we deduce the estimate,

$$\int_{\Omega_{\bar{r}}} (\zeta_{\bar{r}}^2 \mid Du \mid^2)^q dx \leq c(\bar{r}) \tag{2.9}$$

Combining the identities (2.4), (2.8) and (2.9) we obtain,

$$\parallel \zeta_{\bar{r}}^2 u \parallel_{W^{2,q+2}(\Omega)} \leq c(\bar{r})$$

Then, because of the arbitrariness of q in $\mathbb{R}^+$, the Sobolev imbedding theorem [1] yields,

$$\parallel \zeta_{\bar{r}}^2 u \parallel_{1,\alpha,\Omega} \leq \bar{c}$$

where $\bar{c}$ is a constant depending only on α and the parameters indicated previously. In particular,

$$\parallel u \parallel_{1,\alpha,\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x})} \leq \bar{c} \qquad \forall \bar{x} \in \partial\Omega \tag{2.10}$$

Similarly, there exist constants $r_0 < \frac{\bar{r}}{8}$ and $c_0 > 0$ depending only on $\nu, \mu, \alpha, r_0, \bar{r}, N$ and M such that for any $x_0 \in \Omega$, satisfying $dist(x_0, \partial\Omega) > \frac{\bar{r}}{3}$ we have:

$$\parallel u \parallel_{1,\alpha,B_{r_0}(x_0)} \leq c_0 \tag{2.11}$$

It then follows from (2.10) and (2.11) that,

$$\sum_i \parallel D_i u \parallel_{0,\alpha,\Omega} \leq 3r_0^{-\alpha} \max(c_0, \bar{c}).$$

Lemma 2.3 *There exists a constant $c > 0$ such that any solution u of $(\mathcal{P})$ in the space $C^{2,\alpha}(\bar{\Omega}) \cap W^{2,p}(\Omega)$ satisfies:*

$$\sup_{x\in\Omega} (1+ \mid x \mid^2)^{\frac{1}{2}} \mid u(x) \mid \leq c.$$

Proof: The desired estimate will be obtained by the construction of suitable comparaison functions in bounded subdomains Ω' of Ω. Precisely let us set,

$$w(x) := \parallel u \parallel_{0,0,\partial\Omega'} + K(1+ \mid x \mid^2)^{-\frac{1}{2}}$$

where K is a positive constant to be specified later. By differentiation we have,

$$D_i w = -Kx_i(1+ \mid x \mid^2)^{-\frac{3}{2}}$$

$$D_{ij} w = 3Kx_i x_j (1+ \mid x \mid^2)^{-\frac{5}{2}} - K\delta_{ij}(1+ \mid x \mid^2)^{-\frac{3}{2}}$$

By a direct calculation we obtain,

$$\begin{aligned} \bar{L}w &= 3K(1+ \mid x \mid^2)^{-\frac{5}{2}} \sum \bar{a}_{ij}(x)x_i x_j - K(1+ \mid x \mid^2)^{-\frac{3}{2}} \sum \bar{a}_{ii}(x) \\ &\quad -K(1+ \mid x \mid^2)^{-\frac{1}{2}} - \parallel u \parallel_{0,0,\partial\Omega'} \end{aligned}$$

Let $\lambda_1 \leq ... \leq \lambda_N$ be the eigenvalues of the matrix $A := [\bar{a}_{ij}(x)]$. Then, Since $\sum \bar{a}_{ii}(x) = \sum \lambda_i$ we have,

$$\begin{aligned} \bar{L}w &\leq K(1+ \mid x \mid^2)^{-\frac{3}{2}} \left[3\lambda_N - \sum \bar{a}_{ii}(x) \right] - K(1+ \mid x \mid^2)^{-\frac{1}{2}} \\ &\leq K(1+ \mid x \mid^2)^{-\frac{3}{2}} \left[2\lambda_N - \sum_{i=1}^{N-1} \lambda_i\right] - K(1+ \mid x \mid^2)^{-\frac{1}{2}} \\ &\leq K(1+ \mid x \mid^2)^{-\frac{3}{2}} \left[2\mu - (N-1)\nu \right] - K(1+ \mid x \mid^2)^{-\frac{1}{2}} \end{aligned}$$

But by **(A1)** we have, $\mid f(x, u(x), Du(x)) \mid \leq M_1(1+ \mid x \mid^2)^{-\frac{1}{2}}$ where, $M_1 = \varphi(\parallel u \parallel_{0,0,\Omega}) \left[1+ \parallel Du \parallel_{0,0,\Omega}^2 \right]$. Then, for having $\bar{L}(w \pm u) \leq 0$ in Ω it suffices to have,

$$2\mu - (N-1)\nu \leq \left(1 - \tfrac{M_1}{K}\right)(1+ \mid x \mid^2) \qquad \forall\, x \in \Omega$$

If we seek K in $]M_1, +\infty[$, the last condition holds if the following inequality is satisfied,

$$2\mu - (N-1)\nu \leq \left(1 - \tfrac{M_1}{K}\right)[1 + R^2]$$

by **(A4)** this inequality is equivalent to the choice,

$$K \geq K_0 := M_1 \left[1 - \frac{2\mu - (N-1)\nu}{1+R^2}\right]^{-1}$$

for this choice we have,

$$\begin{cases} \bar{L}(w \pm u) & \leq 0 \quad \text{in} \quad \Omega' \\ w \pm u & \geq 0 \quad \text{on} \quad \partial\Omega' \end{cases}$$

It then follows from the weak maximum principle that,

$$\mid u(x) \mid \leq \parallel u \parallel_{0,0,\partial\Omega'} + K(1+ \mid x \mid^2)^{-\frac{1}{2}} \qquad \forall x \in \Omega'$$

Consequently, letting $\Omega' \longrightarrow \Omega$, we obtain the desired estimate.

Proof of the theorem 2.1

Let us set $v(x) := (1+ \mid x \mid^2)^{\frac{1}{2}} u.$

By differentiation we obtain,

$$\begin{aligned} D_i v &= (1+ \mid x \mid^2)^{\frac{1}{2}} D_i u + x_i (1+ \mid x \mid^2)^{-\frac{1}{2}} u \\ D_{ij} v &= (1+ \mid x \mid^2)^{\frac{1}{2}} D_{ij} u + (1+ \mid x \mid^2)^{-\frac{1}{2}} [x_j D_i u + x_i D_j u] \\ &\quad + \delta_{ij} (1+ \mid x \mid^2)^{-\frac{1}{2}} u - x_i x_j (1+ \mid x \mid^2)^{-\frac{3}{2}} u \end{aligned} \tag{2.12}$$

Using (2.12), we obtain by a direct calculation,

$$\begin{aligned} \bar{L} v &= g(x, u, Du) + 2(1+ \mid x \mid^2)^{-\frac{1}{2}} \sum \bar{a}_{ij}(x) x_i D_j u \\ &\quad + (1+ \mid x \mid^2)^{-\frac{1}{2}} u \sum \bar{a}_{ii} - (1+ \mid x \mid^2)^{-\frac{3}{2}} u \sum \bar{a}_{ij}(x) x_i x_j \end{aligned} \tag{2.13}$$

Now, we show that,

$$\| \bar{a}_{ij} \|_{0,\alpha,\Omega} \leq c \tag{2.14}$$

$$\| g(., u, Du) \|_{0,\alpha,\Omega} \leq c \tag{2.15}$$

By the assumption **(A3)** we have,

$$\| \bar{a}_{ij} \|_{0,0,\Omega} \leq 2\mu \tag{2.16}$$

And for $x, x' \in \Omega$ we have,

$$\begin{aligned} \mid \bar{a}_{ij}(x) - \bar{a}_{ij}(x') \mid &:= \mid a_{ij}(x, u(x)) - a_{ij}(x', u(x')) \mid \\ &\leq \psi(\| u \|_{0,0,\Omega}) \, [\; \mid x - x' \mid^\alpha + \mid u(x) - u(x') \mid \;] \\ &\leq \psi(\| u \|_{0,0,\Omega}) \, [\; 1+ \| u \|_{0,\alpha,\Omega} \;] \; \mid x - x' \mid^\alpha \end{aligned} \tag{2.17}$$

Then, by virtue of the assumption **(A2)** and the lemma 2.2 , the inequalities (2.16) and (2.17) imply the estimates (2.14). In the other hand by the lemma 2.2 and the assumptions **(A1)**-**(A2)** we have,

$$\begin{aligned} \| g(., u, Du) \|_{0,0,\Omega} &\leq \varphi(\| u \|_{0,0,\Omega}) \Big[\; 1 + \| Du \|^2_{0,0,\Omega} \; \Big] \\ &\leq c \end{aligned}$$

and,

$$\begin{aligned} &\mid g(x, u(x), Du(x)) - g(x', u(x'), Du(x')) \mid \\ &\quad \leq \varphi(\| u \|_{1,0,\Omega}) \Big\{ \; \mid x - x' \mid^\alpha + \mid u(x) - u(x') \mid + \mid Du(x) - Du(x') \mid \; \Big\} \\ &\quad \leq \varphi(\| u \|_{1,0,\Omega}) \Big\{ \; 1+ \| u \|_{0,\alpha,\Omega} + \| Du \|_{0,\alpha,\Omega} \; \Big\} \mid x - x' \mid^\alpha \\ &\quad \leq c \mid x - x' \mid^\alpha . \end{aligned}$$

The estimate (2.15) is then established. So, using the estimates (2.14)-(2.15) and the identity (2.13), we deduce the estimate,

$$\| \bar{L} v \|_{0,\alpha,\Omega} \leq c \tag{2.18}$$

We apply now the Schauder estimate in unbounded domain [5], [2] to obtain,

$$\| v \|_{2,\alpha,\Omega} \;\leq\; c\{ \;\| \bar{L}v \|_{0,\alpha,\Omega} + \| v \|_{0,0,\Omega}\} \tag{2.19}$$

Hence, by virtue of the lemma 2.3, the estimates (2.18) and (2.19) imply,

$$\| v \|_{2,\alpha,\Omega} \;\leq c \tag{2.20}$$

The first assertion of the theorem 2.1 is then established. Let now Ω' be arbitrary subdomain of Ω. Using the estimate (2.20) we obtain,

$$\begin{aligned}
\| u \|^p_{2,p,\Omega'} &:= \sum_{|s|\leq 2} \int_{\Omega'} | D^s u(x) |^p \, dx \\
&\leq \sum_{|s|\leq 2} \int_{\Omega'} (1+ | x |^2)^{-\frac{p}{2}} \Big[(1+ | x |^2)^{\frac{1}{2}} \, | D^s u(x) | \;\Big]^p dx \\
&\leq c\int_{\Omega'} (1+ | x |^2)^{-\frac{p}{2}} dx
\end{aligned}$$

The theorem 2.1 is then proved

3 Proof of the main theorem

Let $\bar{E}$ and $\bar{F}$ be the closures of the sets,

$$E := \{ \; u \in C^{2,\alpha}(\bar{\Omega}) \;\; / \;\; (1+ | x |^2)^{\frac{1}{2}} u \in C^{2,\alpha}(\bar{\Omega}) \quad \text{and} \quad u = 0 \text{ on } \partial\Omega \;\; \}$$

and

$$F := \{ \; h \in C^{0,\alpha}(\bar{\Omega}) \;\; / \;\; (1+ | x |^2)^{\frac{1}{2}} h \in C^{0,\alpha}(\bar{\Omega}) \; \}$$

respectively in the Hölder spaces $C^{2,\alpha}(\bar{\Omega})$ and $C^{0,\alpha}(\bar{\Omega})$.

Let v be arbitrary and fixed in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ and define the linear operators :

$$\begin{aligned}
&L_0 := \sum_i \frac{\partial^2}{\partial x_i \partial x_i} - 1 \\
&L_1 := \sum a_{ij}(x, v(x)) D_{ij} - 1 \\
&L_t := tL_1 + (1-t)L_0 \; , \qquad t \in [0, 1]
\end{aligned}$$

Using the Schauder estimate in unbounded domains (see [5], [2]) the maximum principle and the fact that the elements of $\bar{E}$ vanish on $\partial\Omega$ and tend to zero at infinite we obtain the estimate :

$$\| u \|_{2,\alpha,\Omega} \;\leq\; c \,\| L_t u \|_{0,\alpha,\Omega} \qquad\qquad \forall u \in \bar{E} \; , \quad \forall t \in [0, 1]. \tag{3.1}$$

On the other hand it is well known that for any function $f \in F$, the linear equation $L_0 u = f$ has a unique solution in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ (see [2]). By a standard regularity argument this solution belongs in fact to the space E. Consequently, by the density of F in $\bar{F}$ and the estimate (3.1) it is easy to see that L_0 is onto from the Banach space $\bar{E}$ into $\bar{F}$. So, the method of continuity and the estimate (3.1) ensure that the linear operator L_1 is onto from $\bar{E}$ into $\bar{F}$. By a standard regularity argument it is easy to see that L_1 restricted to E is onto from E into F. In the other hand the assumption (A1) asserts that $f(., v, Dv)$ belongs to F. Then, the linear problem,

$$(P_v)\quad \begin{cases} \sum a_{ij}(x,v)D_{ij}u - u = f(x,v,Dv) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in E. Hence, the operator T which assigns for each v in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ the unique solution of (P_v) is well defined. To prove that T is completely continuous from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ into itself, let $(v_n)_n$ be a bounded sequence in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and set $u_n := Tv_n$. A similar argument as that used in the theorem 2.1 leads to the estimates:

$$\| u_n \|_{2,\alpha,\Omega} \leq c \qquad \forall n \in I\!N, \tag{3.2}$$

$$\| u_n \|_{2,p,\Omega'} \leq c \,\| (1+ | x |^2)^{-\frac{1}{2}} \|_{0,p,\Omega'} \qquad \forall n \in I\!N, \qquad \forall \Omega' \subset \Omega. \tag{3.3}$$

Using the estimates (3.2) and (3.3), it is easy to verify that the sequences of derivatives of u_n up to order 2, satisfy the assumptions of [1, theorem 2.22]. The sequence (u_n) is then precompact in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The continuity of T follows easily. According to theorem 2.1 the fixed points of the family of operators $(\sigma.T)_{\sigma\in[0,1]}$ are apriori bounded in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ by the same constant then, the Leray-Schauder fixed point theorem [4, theorem 11.3] asserts that T has a fixed point u. It is clear that u solves $(\mathcal{P})$ and satisfies, $(1+ | x |^2)^{\frac{1}{2}} u \in C^{2,\alpha}(\bar{\Omega})$. In particular, $u \in C^{2,\alpha}(\bar{\Omega}) \cap W^{2,q}(\Omega)$ for any $q > N$. The main theorem is then established.

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