Equivariant weak n-equivalences

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The notion of n-type was introduced by J.H.C. Whitehead ([22, 23]) where its clear geometric meaning was presented. Following J.L. Hernandez and T. Porter ([12, 13]) we use the term *weak n-equivalence* for a map $f : X \to Y$ of path-connected spaces which induces isomorphisms $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ on homotopy groups for $k \leq n$. Certainly, weak n-equivalence of a map determines its n-connectedness but not conversely. For J.H. Baues ([2, page 364]) n-types denote the category of spaces X with $\pi_k(X) = 0$ for $k > n$. The n-type of a CW-space X is represented by P_nX, the n-th term in the Postnikov decomposition of X. Then the n-th Postnikov section $p_n : X \to P_nX$ is a weak n-equivalence. Much work has been done to classify the n-types and find equivalent conditions for a map $f : X \to Y$ to be a weak n-equivalence. J.L. Hernandez and T. Porter ([12]) showed how with this notion of weak n-equivalence and with a suitable notion of n-fibration and n-cofibration one obtains a Quillen model category structure ([20]) on the category of spaces. The case of weak n-equivalences mod a class $\mathcal{C}$ of groups (in the sense of Serre) was analyzed by C. Biasi and the second author ([3]). E. Dror ([5]) pointed out that weak equivalences of certain spaces (including nilpotent and complete spaces) can be described by means of homology groups. Then in 1977 J.H. Baues ([1]) proved the Dual Whitehead Theorem for maps of $\mathfrak{R}$-Postnikov spaces (of order $k \geq 1$), where $\mathfrak{R}$ is a commutative ring.

Given the growing interest in equivariant homotopy, it is not surprising that notions of equivariant n-types have been studied. For instance algebraic models for equivariant 2-types have been presented by I. Moerdijk and J.-A. Svensson ([18])

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whilst an equivariant theory of n-types from a Quillen model category theory viewpoint has recently been given by A.R. Garzon and J.G. Miranda ([8]). As yet however homological criteria for equivariant weak n-equivalences have not been given and one of the aims of the present paper is to develop an equivariant version of a truncated Whitehead Theorem (cf. [16, 19] for its nontruncated form) with a list of some equivalent conditions for equivariant weak n-equivalences which are basically homological. Some of these may be used to obtain a Quillen model category structure on the category of G-spaces.

We begin with the nonequivariant case by giving a list of equivalent homology conditions for weak n-equivalences of certain spaces (Proposition 1.2). Then in Theorem 1.4 such a list is obtained by means of homology with local coefficients for weak n-equivalences of any spaces. In fact we get a guide for the assumptions that we must have on the family of spaces in order to move to equivariant spaces in section 2. Bredon homology groups ([4, 14]) are used to generalize the methods presented in section 1 and some equivalent homology conditions for equivariant weak n-equivalences of G-path-connected spaces are listed in Theorem 2.1. In particular an equivariant version of the Dual Whitehead Theorem for G-path-connected nilpotent spaces developed by the first author ([9]) could be deduced. On the other hand there is also a notion of *local* covariant (resp. contravariant) coefficient system $\mathbb{L}$ on a G-space X for which the Bredon homology (resp. cohomology) groups $H_n^G(X,\mathbb{L})$ (resp. $H^n_G(X,\mathbb{L})$) for $n \geq 0$ were considered by I. Moerdijk and A.-J. Svensson ([17]) and the first author ([10]). Then we state Theorem 2.2 presenting a list of equivalent assertions describing any equivariant weak n-equivalences by means of the Bredon homology with local coefficients.

The results are for the case when the group G of the equivariant theory is discrete. Our methods might adapt to the nondiscrete case as methods and results on related areas by W.G. Dwyer and D.M. Kan ([6]) have removed the discrete conditions from many results in equivariant homotopy theory where previously it was considered necessary. The general case, say when G is a Lie group, requires extended techniques that are not yet available and will be dealt with elsewhere.

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1 Nonequivariant case and background

For a map $f : X \to Y$ of CW-spaces let $P_n f : P_n X \to P_n Y$ be the induced map of n-th Postnikov terms. Of course, in the sequel, we may assume that the map $f : X \to Y$ is an inclusion. For a connected space X and an Abelian group A let $H_n(X, A)$ (resp. $H^n(X, A)$) be its n-th homology (resp. cohomology) group with coefficients in A for $n \geq 0$. In particular, write $H_n(X) = H_n(X, Z)$ (resp. $H^n(X) = H^n(X, Z)$) where Z is the additive group of integers. Write $h_n : \pi_n(X) \to H_n(X)$ (resp. $h^n : H^n(X, A) \to \mathrm{Hom}(\pi_n(X), A)$) for the Hurewicz (resp. co-Hurewicz) map for $n \geq 0$.

Lemma 1.1. *Let $f : X \to Y$ be a map of path-connected spaces. Then the following assertions are equivalent:*

1) *the maps*

$$H_k(f,A) : H_k(X,A) \to H_k(Y,A)$$

are isomorphisms for all $k \le n$ *and the map*

$$H_{n+1}(f,A) \oplus h_{n+1} \otimes A : H_{n+1}(X,A) \oplus \pi_{n+1}(Y) \otimes A \to H_{n+1}(Y,A)$$

is an epimorphism for any Abelian group A*;*

2) *the maps*

$$H_k(f) : H_k(X) \to H_k(Y)$$

are isomorphisms for all $k \le n$ *and the map*

$$H_{n+1}(f) \oplus h_{n+1} : H_{n+1}(X) \oplus \pi_{n+1}(Y) \to H_{n+1}(Y)$$

is an epimorphism;

3) *the maps*

$$H^k(f,A) : H^k(Y,A) \to H^k(X,A)$$

are isomorphisms for all $k \le n$ *and the map* $(H^{n+1}(f,A), h^{n+1}) : H^{n+1}(Y,A) \to H^{n+1}(X,A) \times \mathrm{Hom}(\pi_{n+1}(Y),A)$ *is a monomorphism for any Abelian group* A*;*

4) *the maps* $H^k(f) : H^k(Y) \to H^k(X)$ *are isomorphisms for all* $k \le n-1$ *and the map* $H^n(f,A) : H^n(Y,A) \to H^n(X,A)$ (*resp.* $(H^{n+1}(f,A), h^{n+1}) : H^{n+1}(Y,A) \to H^{n+1}(X,A) \times \mathrm{Hom}(\pi_{n+1}(Y),A)$) *is an isomorphism* (*resp. monomorphism*) *for any Abelian group* A.

Proof. Clearly 1) $\Longrightarrow$ 2) and 3) $\Longrightarrow$ 4). For 2) $\Longrightarrow$ 1) we use first the Universal Coefficient Theorem to get the isomorphisms $H_k(f,A) : H_k(X,A) \to H_k(Y,A)$ for $k \le n$. Then, consider the following diagram

$$\begin{array}{ccccccccc}
0 & \to & H_{n+1}(X) \otimes A \oplus \pi_{n+1}(Y) \otimes A & \to & H_{n+1}(X,A) \oplus \pi_{n+1}(Y) \otimes A & \to & \mathrm{Tor}(H_n(X),A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{n+1}(Y) \otimes A & \to & H_{n+1}(Y,A) & \to & \mathrm{Tor}(H_n(Y),A) & \to & 0
\end{array}$$

and by means of the Snake Lemma one gets the epimorphism

$$H_{n+1}(X,A) \oplus \pi_{n+1}(Y) \otimes A \to H_{n+1}(Y,A).$$

To show 3) $\Longrightarrow$ 2) we may assume that the map $f : X \to Y$ is an inclusion. Then the isomorphisms $H^k(f,A) : H^k(Y,A) \to H^k(X,A)$ ensure that $H^k(Y,X;A) = 0$ for $\le n$ and any Abelian group A. By the Universal Coefficient Theorem it follows that $\mathrm{Hom}(H_k(Y,X),A) = 0$ for $k \le n$ and any Abelian groups A. Thus, for $A = H_k(Y,X)$ one gets that $H_k(Y,X) = 0$ for $k \le n$ and so the map $H_k(f) : H_k(X) \to H_k(Y)$ is an isomorphism for $k < n$. Again by the Universal Coefficient Theorem it follows that the map $\mathrm{Hom}(H_n(f),A) : \mathrm{Hom}(H_n(Y),A) \to \mathrm{Hom}(H_n(X),A)$ is an isomorphism for any Abelian group A. Hence the map $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism as well. To show the epimorphism $H_{n+1}(X) \oplus \pi_{n+1}(Y) \to H_{n+1}(Y)$ or equivalently, the monomorphism $\mathrm{Hom}(H_{n+1}(Y),A) \to \mathrm{Hom}(H_{n+1}(X),A) \times \mathrm{Hom}(\pi_{n+1}(Y),A)$ for any Abelian group A consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}(H_n(Y),A) & \to & H^{n+1}(Y,A) & \to & \mathrm{Hom}(H_{n+1}(Y),A) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Ext}(H_n(X),A) & \to & H^{n+1}(X,A)\times\mathrm{Hom}(\pi_{n+1}(Y),A) & \to & \mathrm{Hom}(H_{n+1}(X),A)\times\mathrm{Hom}(\pi_{n+1}(Y),A) & \to & 0. \end{array}$$

and apply the Snake Lemma.

The implication 2) $\Longrightarrow$ 3) is proved dually. Now we show 4) $\Longrightarrow$ 2). The isomorphisms $H^k(f) : H^k(Y) \to H^k(X)$ for $k \le n-1$ and $H^n(f,A) : H^n(Y,A) \to H^n(X,A)$ determine that $H^k(Y,X;Z) = 0$ for $k \le n-1$ and $H^n(Y,X;A) = 0$ for any Abelian group A. From the Universal Theorem it follows that $\mathrm{Ext}(H_k(Y,X),Z) = 0$ for $k \le n-2$, $\mathrm{Hom}(H_k(Y,X),Z) = 0$ for $k \le n-1$, $\mathrm{Ext}(H_{n-1}(Y,X),A) = 0$ and $\mathrm{Hom}(H_n(Y,X),A) = 0$ for any Abelian group A. Therefore, $H_n(Y,X) = 0$ and by [11] we get $H_k(Y,X) = 0$ for $k \le n-1$ as well. Then we can use the same arguments as in the implication 3) $\Longrightarrow$ 2). □

Let π be a group and G a π-group G , i.e. a group together with a homomorphism $\phi : \pi \to \mathrm{Aut}G$. Put $\Gamma^2_\pi G$ for the normal π-subgroup of G generated by all elements of the form $((\phi x)g)g^{-1}$ with $x \in \pi$, $g \in G$ and $\Gamma_n G = \Gamma_2\Gamma_{n-1}G$ for $n \ge 3$. The π-*completion* ([5]) $G^\wedge$ of the group G is defined to be the inverse limit of the following tower of epimorphisms

$$\cdots \to G/\Gamma_{n+1}G \to G/\Gamma_n G \to \cdots \to G/\Gamma_2 G \to 1.$$

We say that a π-group G is π-*complete* if the canonical map $i : G \to G^\wedge$ is an isomorphism and is π-*perfect* if $\Gamma^2_\pi G = G$. Put $\Gamma_\pi G$ for the maximal π-perfect subgroup of G. Then we can proceed to formulate the following generalization of Theorem 2.1 in [3].

Proposition 1.2. *Let $f : X \to Y$ be a map of path-connected CW-spaces such that the groups $\pi = \pi_1 X$ and $\pi_1 Y$ are complete and the groups $\Gamma_\pi\pi_k(Y,X)$ and $\Gamma_\pi\pi_k(X)$ are trivial for $k = 2,\ldots,n$. Then the following assertions are equivalent:*

1) *the map f is a weak n-equivalence;*

2) *the map*

$$f_* : [K,X] \simeq [K,P_nX] \to [K,Y] \simeq [K,P_nY]$$

of homotopy classes is a bijection for any CW-space K such that $\dim K \le n$*;*

3) *the map*

$$P_nf : P_nX \to P_nY$$

is a weak equivalence;

4) *one of the equivalent conditions listed in the previous lemma is satisfied.*

Proof. The equivalences 1) $\Longleftrightarrow$ 2) $\Longleftrightarrow$ 3) are well-known.

The implication 1)$\Longrightarrow$4) follows from [3] and now we show 4)$\Longrightarrow$1) by showing by induction that the second assertion of Lemma 1.1 determines the first one of the proposition. For $n = 1$ consider the following weak 1-equivalences $p_X : X \to K(\pi_1(X),1)$ and $p_Y : Y \to K(\pi_1(Y),1)$. Hence by the previous implication the induced maps $H_2(X) \to H_2(K(\pi_1(X),1))$ and $H_2(Y) \to H_2(K(\pi_1(Y),1))$ are epimorphisms. From the commutative diagram

$$\begin{array}{ccc}
H_2(X) \oplus \pi_2(Y) & \longrightarrow & H_2(Y) \\
\big\downarrow & & \big\downarrow \\
H_2(\pi_1(X)) & \longrightarrow & H_2(\pi_1(Y))
\end{array}$$

one gets that the map $H_2(\pi_1(X)) \to H_2(\pi_1(Y))$ is an epimorphism as well. Now, by [5, Proposition 5.1] the map $\pi_1(f)^\wedge : \pi_1(X)^\wedge \to \pi_1(Y)^\wedge$ of the completions is an isomorphism. But the groups $\pi_1(X)$ and $\pi_1(Y)$ are complete, so the result follows and $\pi_1(Y, X) = 0$.

Now assume that $\pi_i(Y, X) = 0$ for $i \leq n$. Then by the Hurewicz Theorem there is an isomorphism $\pi'_{n+1}(Y, X) \to H_{n+1}(Y, X)$, where $\pi'_{n+1}(Y, X)$ is the quotient of $\pi_{n+1}(Y, X)$ by π-action. But $H_{n+1}(Y, X) = 0$, so $\pi'_{n+1}(Y, X) = 0$ and it follows that $\Gamma^2_\pi \pi_{n+1}(Y, X) = \pi_{n+1}(Y, X)$. Therefore, $\Gamma_\pi \pi_{n+1}(Y, X) = \pi_{n+1}(Y, X)$ and by hypothesis it follows that $\pi_{n+1}(Y, X) = 0$. Consequently, the map $\pi_{n+1}(f) : \pi_{n+1}(X) \to \pi_{n+1}(Y)$ is an epimorphism. It remains to show that it is a monomorphism as well. Put $K = \ker(\pi_{n+1}(X) \to \pi_{n+1}(Y))$. Then by the same arguments as in the proof of Theorem 2.1 in [3] we get that the quotient K' of K by the π-action is trivial. So $\Gamma^2_\pi K = K$ and $K \subseteq \Gamma_\pi \pi_{n+1} X$. Finally, by hypothesis we get that K is trivial and the map $\pi_{n+1}(f) : \pi_{n+1}(X) \to \pi_{n+1}(Y)$ is a monomorphism as required.

□

For a space X let $\pi(X)$ denote its fundamental groupoid. A covariant (resp. contravariant) functor $\mathbb{L} : \pi(X) \to \mathcal{A}b$ to the category of Abelian groups $\mathcal{A}b$ is said to be a *covariant* (resp. *contravariant*) *local coefficient system* on the space X. In particular, we can consider the tensor product $\mathbb{L} \otimes \mathbb{L}'$ of (appropriate) local coefficient systems $\mathbb{L}$ and $\mathbb{L}'$ on the space X. The groupoid $\pi(X)$ yields the covariant local coefficient system $Z\pi(X)$ such that $Z\pi(X)(x) = \bigoplus_{x' \in X} Z\pi(X)(x', x)$ for any $x \in X$. Let $\boldsymbol{\pi}_n(X)$ be the covariant (or contravariant) local coefficient system on X determined by the nth homotopy group functor for $n \geq 2$. Then we get the following statements the proofs of which follow from e.g., [20, 24].

Proposition 1.3. *For a map $f : X \to Y$ of CW-spaces the following assertions are equivalent:*

1) *the map f is a weak equivalence;*

2) *the map $f_* : [K, X] \to [K, Y]$ of homotopy classes is a bijection for any CW-spaces K;*

3) *the maps $P_n f : P_n X \to P_n Y$ are weak equivalences for all $n \geq 0$;*

4) *the map $\pi(f) : \pi(X) \to \pi(Y)$ is an equivalence of fundamental groupoids and $H_n(f, Z\pi(Y)) : H_n(X, f^* Z\pi(Y)) \to H_n(Y, Z\pi(Y))$ are isomorphisms for $n \geq 0$;*

5) *the map $\pi(f) : \pi(X) \to \pi(Y)$ is an equivalence of fundamental groupoids and $H_n(f, \mathbb{L}) : H_n(X, f^* \mathbb{L}) \to H_n(Y, \mathbb{L})$ are isomorphisms for any covariant local coefficient system $\mathbb{L}$ on the space Y and $n \geq 0$;*

6) *the map $\pi(f) : \pi(X) \to \pi(Y)$ is an equivalence of fundamental groupoids and $H^n(f, \mathbb{L}) : H^n(Y, \mathbb{L}) \to H^n(X, f_* \mathbb{L})$ are isomorphisms for any covariant local coefficient system $\mathbb{L}$ on the space Y and $n \geq 0$.*

Thus, the following generalization of Proposition 1.2 may be stated.

Theorem 1.4. *For a map $f : X \to Y$ of CW-spaces the following are equivalent:*

1) *the map f is a weak n-equivalence;*

2) *the map*

$$f_* : [K, X] \simeq [K, P_n X] \to [K, Y] \simeq [K, P_n Y]$$

of homotopy classes is a bijection for any CW-space such that $\dim K \le n$*;*

3) *the map $P_n f : P_n X \to P_n Y$ is a weak equivalence;*

4) *the map $\pi(f) : \pi(X) \to \pi(Y)$ is an equivalence of fundamental groupoids, the maps $H_k(f, Z\pi(Y)) : H_k(X, f^* Z\pi(Y)) \to H_k(Y, Z\pi(Y))$ are isomorphisms for $k \le n$ and the maps $H_{n+1}(f, Z\pi(Y)) \oplus h_{n+1} : H_{n+1}(X, f^* Z\pi(Y)) \oplus \boldsymbol{\pi}_{n+1}(Y) \otimes Z\pi(Y) \to H_{n+1}(Y, Z\pi(Y))$ is an epimorphism;*

5) *the map $\pi(f) : \pi(X) \to \pi(Y)$ is an equivalence of fundamental groupoids, the maps $H_k(f, \mathbb{L}) : H_k(X, f^*\mathbb{L}) \to H_k(Y, \mathbb{L})$ are isomorphisms for any local covariant coefficient system $\mathbb{L}$ on the space Y and $k \le n$, the map $H_{n+1}(f, \mathbb{L}) \oplus h_{n+1} : H_{n+1}(X, f^*\mathbb{L}) \oplus \boldsymbol{\pi}_{n+1}(Y) \otimes \mathbb{L} \to H_{n+1}(Y, \mathbb{L})$ is an epimorphism for any covariant coefficient system $\mathbb{L}$ on the space Y;*

6) *the map $\pi(f) : \pi(X) \to \pi(Y)$ is an equivalence of fundamental groupoids, the maps $H^k(f, \mathbb{L}) : H^k(Y, \mathbb{L}) \to H^k(X, f^*\mathbb{L})$ are isomorphisms for any local contravariant coefficient system $\mathbb{L}$ on the space Y and $k \le n$, and the map $(H^{n+1}(f, \mathbb{L}), h^{n+1}) : H^{n+1}(Y, \mathbb{L}) \to H^{n+1}(X, f^*\mathbb{L}) \times \mathrm{Hom}(\boldsymbol{\pi}_{n+1}(Y), \mathbb{L})$ is a monomorphism for any covariant local coefficient system $\mathbb{L}$ on the space Y.*

Proof. Clearly 1)$\Longleftrightarrow$2)$\Longleftrightarrow$3) as in Proposition 1.2. For 1)$\Longrightarrow$4) we may assume X and Y are path-connected. Fix $x \in X$ and take $y = f(x)$. Let $\pi = \pi_1(X, x) \simeq \pi_1(Y, f(x))$ and $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be the universal coverings and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ the covering map f. The space Y is path-connected so the local coefficient system $Z\pi(Y)$ may be replaced by the group ring $Z\pi$. Then there is a map of the Cartan-Leray spectral sequences

$$\begin{array}{ccccc} E^2_{kl} & = & H_k(\pi, H_l(\widetilde{X}, p^* f^* Z\pi)) & \Longrightarrow & H_{k+l}(X, f^* Z\pi) \\ & & \big\downarrow & & \big\downarrow \\ E^2_{kl} & = & H_k(\pi, H_l(\widetilde{Y}, q^* \mathbb{Z}\pi)) & \Longrightarrow & H_{k+l}(Y, Z\pi). \end{array}$$

The map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a weak n-equivalence and the spaces $\widetilde{X}$ and $\widetilde{Y}$ are 1-connected, so by Proposition 1.2 the maps $H_k(\widetilde{X}, p^* f^* Z\pi) \to H_k(\widetilde{Y}, q^* Z\pi)$ are isomorphisms for all $k \le n$ and $H_{n+1}(\widetilde{X}, p^* f^* \mathbb{Z}\pi) \oplus \pi_{n+1}(\widetilde{Y}) \to H_{n+1}(\widetilde{Y}, q^* Z\pi)$ is an epimorphism. Hence, by the spectral mapping theorem the maps $H_i(X, f^* Z\pi) \to H_i(Y, Z\pi)$ are isomorphisms for $i \le n$ and $H_{n+1}(X, f^* Z\pi) \oplus \pi_{n+1}(Y) \otimes Z\pi \to H_{n+1}(Y, Z\pi)$ is an epimorphism.

For 4)$\Longrightarrow$1) we may again assume X and Y are connected and let $\widetilde{X}$, $\widetilde{Y}$, π, etc., be as above. Certainly, $\pi_1(X, x) \simeq \pi_1(Y, f(x))$. Since $\widetilde{X}$ and $\widetilde{Y}$ are simply connected, by Proposition 1.2 it suffices to show that the maps $H_k(\widetilde{X}) \to H_k(\widetilde{Y})$ are isomorphisms for $k \le n$ and $H_{n+1}(\widetilde{X}) \oplus \pi_{n+1}(\widetilde{Y}) \to H_{n+1}(\widetilde{Y})$ is an epimorphism. But the Leray spectral sequences for p and q degenerate giving a diagram

$$\begin{array}{ccc} H_k(X, p_*Z) & \simeq & H_k(\widetilde{X}) \\ \Big\downarrow f_* & & \Big\downarrow (\widetilde{f})_* \\ H_k(Y, q_*Z) & \simeq & H_k(\widetilde{Y}) \end{array}$$

for all $k \geq 0$, where p_* and q_* are the local coefficient systems of the homology of the fiber. By 4) the required properties of the map f_* follow and so we are finished.

In the same way one may show that 1)$\Longleftrightarrow$5) and dually the equivalence 1)$\Longleftrightarrow$6). $\square$

2 Main result

First we formulate an equivariant version of Proposition 1.2 restricting our results to the case when the group G of the equivariant theory is discrete. Let $\mathcal{O}(G)$ be the *orbit category* (the category of all transitive G-sets and all G-maps between them) of such a group. For a G-space X G.E. Bredon ([4]) and S. Illman ([14]) introduced the homology $H_n^G(X, A)$ (resp. cohomology $H^n_G(X, A)$) groups for $n \geq 0$ of X with coefficients in a functor $A : \mathcal{O}(G) \to \mathcal{A}b$ (resp. $A : \mathcal{O}(G)^{op} \to \mathcal{A}b$). Such a coefficient system A depends on the orbit category $\mathcal{O}(G)$ but not on X, and should be regarded as *constant*, from the point of view of G-spaces. Let $\boldsymbol{I}(G)$ be the *isotropy ring* of the category $\mathcal{O}(G)$ as considered in [21]. It is the free Abelian group on the set of all maps of the category $\mathcal{O}(G)$ with a ring structure imposed by composition of maps. Explicitly, we define a multiplication fg to equal the composition $f \circ g$ if maps f and g are composable and equal 0 otherwise. Then there is a one-one correspondence between (isomorphism classes) of covariant (resp. contravariant) coefficient systems and (isomorphism classes) of left (resp. right) $\boldsymbol{I}(G)$-modules. Moreover there is the following Universal Coefficient Spectral Sequence Theorem $E^2_{kl} = \mathrm{Tor}_k(H_l^G(X, \boldsymbol{I}(G)), A) \Rightarrow H^G_{k+l}(X, A)$ (resp. $E_2^{kl} = \mathrm{Ext}^k(H_l^G(X, \boldsymbol{I}(G), A) \Rightarrow H_G^{k+l}(X, A))$ derived in [21] (see also [4, 14]).

Recall that a G-map $f : X \to Y$ is said to be a *weak G-equivalence* (resp. a *weak G-n-equivalence*) if, for each subgroup $H \subseteq G$, the map f induces a weak equivalence (resp. a weak n-equivalence) $f^H : X^H \to Y^H$ between fixed point subspaces. A G-space X is said to be *G-path-connected* if the fixed point subspaces X^H are path-connected for all subgroups $H \subseteq G$.

For an Abelian group A, a G-space X determines a contravariant (resp. covariant) coefficient system $\boldsymbol{H}_n(X, A)$ (resp. $\boldsymbol{H}^n(X, A)$) such that $\boldsymbol{H}_n(X, A)(G/H) = H_n(X^H, A)$ (resp. $\boldsymbol{H}^n(X, A)(G/H) = H^n(X^H, A)$) for each subgroup $H \subseteq G$ and $n \geq 0$. For a pointed G-space X we also get the contravariant coefficient systems $\boldsymbol{\pi}_n(X)$ determined by its homotopy groups and such that $\boldsymbol{\pi}_n(X)(G/H) = \pi_n(X^H)$ for each subgroup $H \subseteq G$ and $n \geq 2$. Note that the Hurewicz map determines maps $h_n : \boldsymbol{\pi}_n(X) \otimes A \to H_n^G(X, A)$ and $h^n : H^n_G(X, A) \to \mathrm{Hom}(\boldsymbol{\pi}_n(X), A)$ for an appropriate coefficient system A and $n \geq 2$. Put $P_nf : P_nX \to P_nY$ for the induced map of equivariant nth Postnikov terms ([7]) for $n \geq 0$.

Theorem 2.1. *Let X and Y be G-path-connected CW-spaces such that the fixed point subspaces X^H and Y^H satisfy the hypotheses on spaces in Proposition* 1.2 *for* *all subgroups $H \subseteq G$. Then, for a G-map $f : X \to Y$ the following assertions are equivalent:*

1) *the map f is a weak G-n-equivalence;*
2) *the map*

$$f_* : [K, X]_G \simeq [K, P_n X]_G \to [K, Y]_G \simeq [K, P_n Y]_G$$

of G-homotopy classes is a bijection for any G-CW-space such that $\dim K \leq n$;

3) *the map*

$$P_n f : P_n X \to P_n Y$$

is a weak G-equivalence;

4) *the maps*

$$H_k^G(f, \boldsymbol{I}(G)) : H_k^G(X, \boldsymbol{I}(G)) \to H_k^G(Y, \boldsymbol{I}(G))$$

are isomorphisms for all $k \leq n$ and

$$H_{n+1}^G(f, \boldsymbol{I}(G)) \oplus h_{n+1} : H_{n+1}^G(X, \boldsymbol{I}(G)) \oplus \boldsymbol{\pi}_{n+1}(Y) \otimes \boldsymbol{I}(G) \to H_{n+1}^G(Y, \boldsymbol{I}(G))$$

is an epimorphism;

5) *the maps*

$$H_k^G(f, A) : H_k^G(X, A) \to H_k^G(Y, A)$$

are isomorphisms for all $k \leq n$ and any covariant coefficient system A and

$$H_{n+1}^G(f, A) \oplus h_{n+1} : H_{n+1}^G(X, A) \oplus \boldsymbol{\pi}_{n+1}(Y) \otimes A \to H_{n+1}^G(Y, A)$$

is an epimorphism for all covariant coefficient systems A of Abelian groups;

6) *the maps*

$$\boldsymbol{H}_k(f, A) : \boldsymbol{H}_k(X, A) \to \boldsymbol{H}_k(Y, A)$$

are isomorphisms of homology group systems for all $k \leq n$ and any Abelian group A and the map

$$\boldsymbol{H}_{n+1}(f, A) \oplus \boldsymbol{h}_{n+1} : \boldsymbol{H}_{n+1}(X, A) \oplus \boldsymbol{\pi}_{n+1}(Y) \otimes A \to \boldsymbol{H}_{n+1}(Y, A)$$

is an epimorphism of systems for all Abelian groups A;

7) *the maps*

$$H_G^k(f, A) : H_G^k(Y, A) \to H_G^k(X, A)$$

are isomorphisms for all $k \leq n$ and any contravariant coefficient system A of Abelian group and the map $(H_G^{n+1}(f, A), h^{n+1}) : H_G^{n+1}(Y, A) \to H_G^{n+1}(X, A) \times \mathrm{Hom}(\boldsymbol{\pi}_{n+1}(Y), A)$ is a monomorphism for all contravariant coefficient systems A of Abelian groups;

8) *the maps*

$$\boldsymbol{H}^k(f, A) : \boldsymbol{H}^k(Y, A) \to \boldsymbol{H}^k(X, A)$$

are isomorphisms of systems for all $k \leq n$ and any Abelian group A and the map

$$(\boldsymbol{H}^{n+1}(f, A), \boldsymbol{h}^{n+1}) : \boldsymbol{H}^{n+1}(Y, A) \to \boldsymbol{H}^{n+1}(X, A) \times \mathrm{Hom}(\boldsymbol{\pi}_{n+1}(Y), A)$$

is a monomorphism of systems for all Abelian groups A.

Proof. The equivalences 1)$\Longleftrightarrow$2) and 2)$\Longleftrightarrow$3) are well-known (e.g., [15]), however 1)$\Longleftrightarrow$6) and 1)$\Longleftrightarrow$8) follow from Proposition 1.2. Clearly 4)$\Longleftrightarrow$5) follows from the Universal Coefficient Spectral Sequence Theorem.

From the Shapiro Lemma ([19]) it follows that for a G-spaces X, a subgroup $H \subseteq G$ and an Abelian group A, there are isomorphisms $H_n(X^H, A) \simeq H_n^G(X, \mathrm{coind}(A))$ and $H^n(X^H, A) \simeq H_G^n(X, \mathrm{ind}(A))$ for all $n \geq 0$, where $\mathrm{coind}(A)(G/K) = Z[\mathcal{O}(G)(G/K, G/H))] \otimes A$ and $\mathrm{ind}(A)(G/K) = \mathrm{Hom}(Z[\mathcal{O}(G)(G/K, G/H)], A)$ for any subgroup $K \subseteq G$. Thus, the implications 5)$\Longrightarrow$6) and 7)$\Longrightarrow$8) are finished.

The opposite implications may be deduced from the map of the Bredon ([4]) homology and cohomology spectral sequences

$$\begin{array}{ccccc} E^2_{k,l} & = & \mathrm{Tor}_k(\boldsymbol{H}_l(X), A) & \Longrightarrow & H^G_{k+l}(X, A) \\ & & \downarrow & & \downarrow \\ E^2_{k,l} & = & \mathrm{Tor}_k(\boldsymbol{H}_l(Y), A) & \Longrightarrow & H^G_{k+l}(Y, A) \end{array} \quad \text{and} \quad \begin{array}{ccccc} E_2^{k,l} & = & \mathrm{Ext}^k(\boldsymbol{H}_l(X), A) & \Longrightarrow & H_G^{k+l}(X, A) \\ & & \uparrow & & \uparrow \\ E_2^{k,l} & = & \mathrm{Ext}^k(\boldsymbol{H}_l(Y), A) & \Longrightarrow & H_G^{k+l}(Y, A). \end{array}$$

□

Unfortunately, G-connectedness is a much more severe restriction on a G-space than connectedness in the nonequivariant context, since at present it is not clear how to break up a G-space into "connected components", as one would do nonequivariantly. Therefore, instead of the orbit category $\mathcal{O}(G)$ we could work over the category $\mathcal{O}(G, X)$ with one object for each component of each fixed point subspace of a G-space X.

Now we state the equivariant version of Theorem 1.4. For a G-space X let $\pi(G, X)$ denote its fundamental category. An object of $\pi(G, X)$ is a map $x : G/H \to X$ which we may identify with a pair $(G/H, x)$ for $x \in X^H$ and a map $(\alpha, [\sigma]) : (G/H, x) \to (G/H', x')$ consists of a G-map $\alpha : G/H \to G/H'$ and a homotopy class $[\sigma]$ (rel. $G/H \times \partial I$) of G-maps $\sigma : G/H \times I \to X$ with $\sigma(0) = \alpha \circ x'$ and $\sigma(1) = x$. A covariant (resp. contravariant) *local* coefficient system $\mathbb{L}$ on a G-space X is a functor $\mathbb{L} : \pi(G, X) \to \mathcal{A}b$ (resp. $\mathbb{L} : \pi(G, X)^{op} \to \mathcal{A}b$). As in the nonequivariant case we get the tensor product of (appropriate) local coefficient systems, the local covariant local coefficient system $Z\pi(G, X)$ determined by $\pi(G, X)$ and the covariant (or contravariant) local coefficient system $\boldsymbol{\pi}_n(X)$ such that $\boldsymbol{\pi}_n(X)(G/H, x) = \pi_n(X^H, x)$ for $n \geq 2$. Put $H_n^G(X, \mathbb{L})$ (resp. $H_G^n(X, \mathbb{L})$) for the Bredon homology (resp. cohomology) groups of a G-space X with coefficients in a covariant (resp. contravariant) local coefficient system $\mathbb{L}$ as considered by I. Moerdijk and J.-A. Svensson ([17]). Then the Hurewicz map yields maps $h_n : \boldsymbol{\pi}_n(X) \otimes \mathbb{L} \to H_n^G(X, \mathbb{L})$ and $h^n : H_G^n(X, \mathbb{L}) \to \mathrm{Hom}(\boldsymbol{\pi}_n(X), \mathbb{L})$ for an appropriate local coefficient system $\mathbb{L}$ on X and $n \geq 2$. By means of the results of I.Moerdijk and J.-A. Svensson ([17, 19]) one may deduce an equivariant version of Proposition 1.3 and methods used in the proof of Theorem 1.4.

Proposition 2.2. *For a map $f : X \to Y$ of G-CW-spaces the following are equivalent:*

1) *the map f is a weak G-equivalence;*

2) *the map*

$$f_* : [K, X]_G \to [K, P_nY]_G$$

of G-homotopy classes is a bijection for any G-CW-space K;

3) *the map* $P_n f : P_n X \to P_n Y$ *are weak* G*-equivalences for any* $n \geq 0$*;*

4) *the map* $\pi(G,f) : \pi(G,X) \to \pi(G,Y)$ *is an equivalence of fundamental categories, the maps* $H_n^G(f, Z\pi(G,Y)) : H_n^G(X, f^* Z\pi(G,Y)) \to H_n^G(Y, Z\pi(G,Y))$ *are isomorphisms for* $n \geq 0$*;*

5) *the map* $\pi(G,f) : \pi(G,X) \to \pi(G,Y)$ *is an equivalence of fundamental categories, the maps* $H_n^G(f, \mathbb{L}) : H_n^G(X, f^*\mathbb{L}) \to H_n^G(Y, \mathbb{L})$ *are isomorphisms for any local covariant coefficient system* $\mathbb{L}$ *on the* G*-space* Y *and* $n \geq 0$*;*

6) *the map* $\pi(G,f) : \pi(G,X) \to \pi(G,Y)$ *is an equivalence of fundamental categories, the maps* $H^n_G(f, \mathbb{L}) : H^n_G(Y, \mathbb{L}) \to H^n_G(X, f^*\mathbb{L})$ *are isomorphisms for any local contravariant coefficient system* $\mathbb{L}$ *on the* G*-space* Y *and* $n \geq 0$*.*

Now we may state a theorem which is an equivariant version of Theorem 1.4 and its proof becomes straightforward.

Theorem 2.3. *For a map* $f : X \to Y$ *of* G*-CW-spaces the following are equivalent:*

1) *the map* f *is a weak* G*-*n*-equivalence;*

2) *the map*

$$f_* : [K, X]_G \simeq [K, P_n X]_G \to [K, Y]_G \simeq [K, P_n Y]_G$$

of G*-homotopy classes is a bijection for any* G*-CW-space such that* $\dim K \leq n$*;*

3) *the map* $P_n f : P_n X \to P_n Y$ *is a weak* G*-equivalence;*

4) *the map* $\pi(G,f) : \pi(G,X) \to \pi(G,Y)$ *is an equivalence of fundamental categories, the maps* $H_k^G(f, Z\pi(G,Y)) : H_k^G(X, f^* Z\pi(G,Y)) \to H_k^G(Y, Z\pi(G,Y))$ *are isomorphisms for* $k \leq n$ *and the map* $H_{n+1}^G(f, Z\pi(G,Y)) \oplus h_{n+1} : H_{n+1}^G(X, f^* Z\pi(G,Y)) \oplus \boldsymbol{\pi}_{n+1}(Y) \otimes Z\pi(G,Y) \to H_{n+1}^G(Y, Z\pi(G,Y))$ *is an epimorphism;*

5) *the map* $\pi(G,f) : \pi(G,X) \to \pi(G,Y)$ *is an equivalence of fundamental categories, the maps* $H_k^G(f, \mathbb{L}) : H_k^G(X, f^*\mathbb{L}) \to H_k^G(Y, \mathbb{L})$ *are isomorphisms for any local covariant coefficient system* $\mathbb{L}$ *on the* G*-space* Y *and* $k \leq n$*, the map* $H_{n+1}^G(f, \mathbb{L}) \oplus h_{n+1} : H_{n+1}^G(X, f^*\mathbb{L}) \oplus \boldsymbol{\pi}_{n+1}(Y) \otimes \mathbb{L} \to H_{n+1}^G(Y, \mathbb{L})$ *is an epimorphism for any covariant coefficient system* $\mathbb{L}$ *on the* G*-space* Y*;*

6) *the map* $\pi(G,f) : \pi(G,X) \to \pi(G,Y)$ *is an equivalence of fundamental categories, the maps* $H^k_G(f, \mathbb{L}) : H^k_G(Y, \mathbb{L}) \to H^k_G(X, f^*\mathbb{L})$ *are isomorphisms for any local contravariant coefficient system* $\mathbb{L}$ *on the* G*-space* Y *and* $k \leq n$*, and the map* $(H^{n+1}_G(f, \mathbb{L}), h^{n+1}) : H^{n+1}_G(Y, \mathbb{L}) \to H^{n+1}_G(X, f^*\mathbb{L}) \times \operatorname{Hom}(\boldsymbol{\pi}_{n+1}(Y), \mathbb{L})$ *is a monomorphism for any covariant local coefficient system* $\mathbb{L}$ *on the* G*-space* Y*.*

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