Split Semi-Biplanes in Antiregular Generalized Quadrangles

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Abstract

There are a number of important substructures associated with sets of points of antiregular quadrangles. Inspired by a construction of P. Wild, we associate with any four distinct collinear points p, q, r and s of an antiregular quadrangle an incidence structure which is the union of the two biaffine planes associated with $\{p, r\}$ and $\{q, s\}$. We investigate when this incidence structure is a semi-biplane.

1 Introduction

The definitions of both semi-biplanes and antiregular generalized quadrangles involve 0-2 conditions. That this is more than just a coincidence is shown in [4], where we investigate a first construction of semi-biplanes from anti-regular quadrangles.

It turns out that associated with any pair of non-collinear points in an antiregular generalized quadrangle is an incidence structure which is the union of semi-biplanes. All these semi-biplanes look very much like homology semi-biplanes, that is, projective planes that have been folded up using a homology involution. In [6] we prove that in the classical case and the topological case the resulting semi-biplanes are all homology semi-biplanes.

The definition of the semi-biplane associated with two non-collinear points still makes sense if we use a pair of collinear points instead. The resulting incidence structure is a biaffine plane, that is, a projective plane minus an incident point-line pair.

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The next step, which we are concentrating on in this paper, is to split up the pair of collinear points into two pairs of collinear points. The resulting incidence structure is, of course, the union of two biaffine planes, but not only that. In many instances it turns out that in this incidence structure the two biaffine planes are meshed together in just the right way to form a new kind of semi-biplane.

The paper is organized as follows. Section 2 gives a brief introduction to generalized quadrangles and the notation and conventions that we will be using in the following. In Section 3 we give an overview over the different kinds of incidence structures associated with sets of points of an antiregular generalized quadrangle. In Section 4 we define an incidence structure associated with four collinear points and give necessary and sufficient conditions for this incidence structure to be a semi-biplane, a divisible semi-biplanes or not a semi-biplane at all. In Sections 5 and 6 we apply the results of the previous section in a detailed discussion of the classical and the topolocial case, respectively.

2 Generalized quadrangles

We begin by reviewing some well-known facts about the incidence structures we will be dealing with. For proofs and more details the reader is referred to [2], [4] and [9].

The incidence structures we are interested in are of the form $\mathfrak{I} = (\mathcal{V}_1, \mathcal{V}_2, I)$. The elements of $\mathcal{V}_1$ are always called *points*. The elements of $\mathcal{V}_2$ are called *lines* if $\mathfrak{I}$ is a generalized quadrangle, a biaffine, affine or projective plane. They are called *circles* if $\mathfrak{I}$ is a Laguerre plane and *blocks* if $\mathfrak{I}$ is a semi-biplane. The relation I specifies when a point and an element of $\mathcal{V}_2$ are incident. As usual, we will often consider a point as the set of elements in $\mathcal{V}_2$ incident with it and an element of $\mathcal{V}_2$ as the set of points incident with it. Let $\{i, j\} = \{1, 2\}$. Two elements x and y of $\mathcal{V}_i$ are called *parallel* if $x = y$ or if there is no element of $\mathcal{V}_j$ incident with both x and y. If one of the two parallelisms is an equivalence relation, then its equivalence classes are called *parallel classes*. The incidence structure $\mathfrak{I}$ is *divisible* if parallelism is an equivalence relation of both $\mathcal{V}_1$ and $\mathcal{V}_2$. A pair $(v, w) \in \mathcal{V}_1 \times \mathcal{V}_2$ is called a *flag*, if v is incident with w, that is, if $v\ I\ w$. The set of flags is denoted $\mathcal{F}$. An element of $\mathcal{V}_1 \times \mathcal{V}_2 \setminus \mathcal{F}$ is called an *anti-flag.*

A *generalized quadrangle* is an incidence structure $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L}, I)$ with point set $\mathcal{P}$ and line set $\mathcal{L}$ satisfying the following axioms:

(Q1) Any two distinct points have at most one joining line.

(Q2) For every anti-flag $(p, k) \in \mathcal{P} \times \mathcal{L} \setminus \mathcal{F}$ there exists exactly one flag $(q, l) \in \mathcal{F}$ such that $(p, l) \in \mathcal{F}$ and $(q, k) \in \mathcal{F}$.

(Q3) Every point is incident with at least three lines, and every line is incident with at least three points.

One of the classical examples of a generalized quadrangle is constructed as follows: Let Q be a non-singular quadric of projective index 1 in $PG(4, \mathbb{K})$, where $\mathbb{K}$ is a field. If $\mathcal{P}$ and $\mathcal{L}$ are the sets points and lines of Q, respectively, then $Q(4, \mathbb{K}) := (\mathcal{P}, \mathcal{L}, \in)$ is a generalized quadrangle (cf. [2]).

Axiom Q2 yields two mappings $\pi : \mathcal{P} \times \mathcal{L} \setminus \mathcal{F} \to \mathcal{P}$ and $\lambda : \mathcal{P} \times \mathcal{L} \setminus \mathcal{F} \to \mathcal{L}$ with $\pi(p,k) = q$ and $\lambda(p,k) = l$. Two points p and q are said to be *collinear*, denoted $p \sim q$, if they can be joined by a line. For two distinct collinear points p and q the unique joining line is denoted $p \vee q$. The set of points collinear with a given point p is denoted $p^\perp$. For three points p, q and r the intersection $p^\perp \cap q^\perp \cap r^\perp$ is called the *centre* of p, q and r. A generalized quadrangle where no centre is empty is called a *centric* generalized quadrangle. A set of three pairwise non-collinear points is called a *triad.* If $(p,l) \in \mathcal{F}$, let l_p be the punctured line $l \setminus \{p\}$. The set of all lines through p that have all been punctured in p is denoted $||_p$.

In this note we will be dealing exclusively with *antiregular* quadrangles, that is, generalized quadrangles in which the centre of every triad of points contains either 0 or 2 points. A classical example of an antiregular quadrangle is the generalized quadrangle $Q(4,\mathbb{K})$, where $\mathbb{K}$ is a field not of characteristic 2.

3 Some substructures of antiregular quadrangles

A multitude of other important incidence structures, such as circle planes, projective planes, biaffine planes and semi-biplanes, occur as subgeometries of generalized quadrangles. Generalized quadrangles can therefore be regarded as especially tightly packed bundles of such geometries. This is one of the most attractive features of generalized quadrangles. It gives rise to many characterizations of the different kinds of generalized quadrangles in terms of special properties of these subgeometries. See [2], [7] and [8] for examples of such characterizations. Our main aim is to describe and investigate a new way in which semi-biplanes, which are defined below, are embedded as subgeometries in antiregular quadrangles.

We start with a description of the known subgeometries of an antiregular quadrangle $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L}, I)$ associated with points.

Let $p \in \mathcal{P}$ and let

$$\mathfrak{GQ}_p := (\mathcal{P}_p, \mathcal{L}_p, \sim) := (p^\perp \setminus \{p\}, \mathcal{P} \setminus p^\perp, \sim).$$

Parallelism is an equivalence relation on the point set $\mathcal{P}_p$ with the elements of $||_p$ as parallel classes. The incidence structure $\mathfrak{GQ}_p$, called the *derivation of* $\mathfrak{GQ}$ *at* p, is a *Laguerre plane* (cf. [9, 3.1]), and as such satisfies the following axioms:

(L1) Three pairwise non-parallel points are contained in a unique circle (remember that in the Laguerre plane setting elements of $\mathcal{L}_p$ are usually referred to as circles rather than lines).

(L2) Given a circle c, and two point p and q, such that $p \in c$ and q is not parallel to p, then there is a unique circle that contains both points and touches c at p, that is, coincides with c or intersects c only in p.

(L3) Given a circle c and a point p there is precisely one point parallel to p and contained in c.

(L4) There is a circle with at least three points and not all points are on the same circle.

We remark that for finite Laguerre planes of odd order and for locally compact finite-dimensional connected Laguerre planes this construction can be reversed (cf. [9]). For general information on Laguerre planes we refer to [1] and [10].

The Laguerre plane $\mathfrak{GQ}_p$ is a substructure of $\mathfrak{GQ}$ in a broader sense since $\mathcal{L}_p \not\subset \mathcal{L}$ and incidence in $\mathfrak{GQ}_p$ is only indirectly inherited from incidence in $\mathfrak{GQ}$. In the case of $\mathfrak{GQ} = Q(4, \mathbb{K})$, where $\mathbb{K}$ is a field not of characteristic 2, the Laguerre plane $\mathfrak{GQ}_p$ is isomorphic to the geometry of plane sections of a quadratic cone with its vertex v removed in $PG(3, \mathbb{K})$. The parallel classes of points are the projective lines through v contained in the cone that have all been punctured in the point p. The four axioms are easily checked in this case. We note that even if $\mathbb{K}$ is of characteristic 2, such a geometry of plane sections in $PG(3, \mathbb{K})$ is a Laguerre plane, although $Q(4, \mathbb{K})$ is no longer an antiregular quadrangle.

Let $p, q \in \mathcal{P}$ be distinct and let

$$\mathfrak{GQ}_{p,q} := (\mathcal{P}_{p,q}, \mathcal{L}_{p,q}, \sim) := \Big(p^\perp \setminus (p^\perp \cap q^\perp), q^\perp \setminus (p^\perp \cap q^\perp), \sim\Big).$$

Let p and q be collinear points with connecting line l. Then $p^\perp \cap q^\perp$ is the connecting line of the two points and $\mathfrak{GQ}_{p,q}$ is a *biaffine plane*, that is, a projective plane with one incident point-line pair removed, or equivalently, an affine plane with one parallel class of lines removed (this parallel class of lines is the set of parallel classes of points in the biaffine plane). As a subgeometry of the Laguerre plane $\mathfrak{GQ}_p$, this affine plane is usually referred to as the *derived affine plane of* $\mathfrak{GQ}_p$ *at the point* q. If $\mathfrak{GQ} = Q(4, \mathbb{K})$, this affine plane is isomorphic to the Desarguesian affine plane over $\mathbb{K}$. Of course, the biaffine plane $\mathfrak{GQ}_{p,q}$ is divisible with the elements of $||_p \setminus \{l_p\}$ being the parallel classes of $\mathcal{P}_{p,q}$ and the elements of $||_q \setminus \{l_q\}$ being the parallel classes of $\mathcal{L}_{p,q}$.

Let p and q be non-collinear points. It has been noticed only recently (cf. [4] and [6]) that in this case $\mathfrak{GQ}_{p,q}$ is the union of semi-biplanes. See also [3] and [5] for plane models of such semi-biplanes.

A *semi-biplane* $\mathfrak{S} = (\mathcal{P}, \mathcal{B})$ is an incidence structure satisfying the following axioms:

(S1) Any two distinct points are incident with 0 or 2 blocks.

(S2) Any two distinct blocks are incident with 0 or 2 points.

(S3) The incidence graph of $\mathfrak{S}$ is connected.

(S4) Every point is incident with at least three blocks and every block is incident with at least three points.

In the following we describe a second way in which semi-biplanes occur as substructures of antiregular quadrangles.

4 Semi-biplanes associated with four collinear points

Let p, q, r and s be distinct collinear points of $\mathfrak{GQ}$ with connecting line l and let $\mathcal{P}_p^l := p^\perp \setminus l$. Furthermore, let

$$\mathfrak{GQ}_{p,q,r,s} := (\mathcal{P}_{p,q,r,s}, \mathcal{L}_{p,q,r,s}, \sim) := (\mathcal{P}_p^l \cup \mathcal{P}_q^l, \mathcal{P}_r^l \cup \mathcal{P}_s^l, \sim).$$

We are going to show that in many antiregular quadrangles the four points can be chosen such that $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane. This construction of semi-biplanes was inspired by one of P. Wild's constructions (cf. [11], [12] and [13]). The odd order part of his main result [13, Theorem 11], interpreted in the quadrangle setting, is Corollary 5.3 below.

Notice that for $x \in \{p, q\}$ and $y \in \{r, s\}$ the biaffine plane $\mathfrak{GQ}_{x,y} = (\mathcal{P}^l_x, \mathcal{P}^l_y, \sim)$ is a subgeometry of $\mathfrak{GQ}_{p,q,r,s}$. Given $x \in \{p, q\}$, the point sets of $\mathfrak{GQ}_{x,r}$ and $\mathfrak{GQ}_{x,s}$ coincide and points are parallel in $\mathfrak{GQ}_{x,r}$ if and only if they are parallel in $\mathfrak{GQ}_{x,s}$. The dual statement is true for lines. Notice also that the four sets $\mathcal{P}^l_p$, $\mathcal{P}^l_q$, $\mathcal{P}^l_r$ and $\mathcal{P}^l_s$ are mutually disjoint. This means that $\mathfrak{GQ}_{p,q,r,s}$ is the 'disjoint union' of the two biaffine planes $\mathfrak{GQ}_{v,w}$ and $\mathfrak{GQ}_{x,y}$ whenever $\{v, x\} = \{p, q\}$ and $\{w, y\} = \{r, s\}$ and that it can be represented as such a union in two different ways. This observation suggests to call a semi-biplane isomorphic to some $\mathfrak{GQ}_{p,q,r,s}$ a *split semi-biplane.*

Theorem 4.1 *Let $\mathfrak{GQ}$ be an antiregular quadrangle and let p, q, r and s be four distinct collinear points with connecting line l. Then $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane if and only if the following two conditions are satisfied.*

(C1) If $x \in \mathcal{P}^l_p$ and $y \in \mathcal{P}^l_q$ are such that x and y are not collinear as points of $\mathfrak{GQ}$, then $|x^\perp \cap y^\perp \cap r^\perp| + |x^\perp \cap y^\perp \cap s^\perp| \leq 2$.

(C2) If $x \in \mathcal{P}^l_r$ and $y \in \mathcal{P}^l_s$ are such that x and y are not collinear as points of $\mathfrak{GQ}$, then $|x^\perp \cap y^\perp \cap p^\perp| + |x^\perp \cap y^\perp \cap q^\perp| \leq 2$.

Proof. We first consider Axiom S1, that is, we have to show that any two points can be joined by exactly two or by no blocks. Let $x, y \in \mathcal{P}_{p,q,r,s}$ be distinct points. We distinguish four cases.

Let $x, y \in \mathcal{P}^l_p$. Then the number of blocks containing the two points equals the sum of the number of lines in the biaffine plane $\mathfrak{GQ}_{p,r}$ and the number of lines in the biaffine plane $\mathfrak{GQ}_{p,s}$ connecting the two points. From the remark preceding the theorem it follows that these two numbers are either both equal to 0 or both equal to 1. Hence the number of blocks containing both points is either 0 or 2.

If both points are contained in $\mathcal{P}^l_q$, a similar argument yields the same conclusion.

Let $x \in \mathcal{P}^l_p$ and $y \in \mathcal{P}^l_q$. Assume that x and y are collinear points in $\mathfrak{GQ}$ and that k is the connecting line. Then k does not intersect l. By Axiom Q2, there is exactly one element of $\mathcal{P}^l_r$ contained in k. This element is distinct from x and y and is the only element in this set collinear with both x and y. Similarly, there is exactly one element in $\mathcal{P}^l_s$ collinear with both points. This gives a total of 2 blocks containing the two points.

Let $x \in \mathcal{P}^l_p$ and $y \in \mathcal{P}^l_q$. Assume that x and y are points in $\mathfrak{GQ}$ that are not collinear. Then both $\{x, y, r\}$ and $\{x, y, s\}$ are triads. Hence, by antiregularity, their centres contain either 0 or 2 points, which means that the sum $|x^\perp \cap y^\perp \cap r^\perp| + |x^\perp \cap y^\perp \cap s^\perp|$ is equal to 0, 2 or 4. Thus Axiom S1 is satisfied if and only if Condition C1 holds. Note that Condition C1 was only needed in this last case. With almost the same arguments one verifies that Axiom S2, which is the dual of Axiom S1, is satisfied if and only if Condition C2 holds.

The biaffine planes $\mathfrak{GQ}_{x,y}$, $x \in \{p, q\}$, $y \in \{r, s\}$ are subgeometries of $\mathfrak{GQ}_{p,q,r,s}$. The incidence graphs of biaffine planes are automatically connected. This means that in the incidence graph of $\mathfrak{GQ}_{p,q,r,s}$ every element of $\mathcal{P}_{p,q,r,s}$ is connected with every element of $\mathcal{L}_{p,q,r,s}$. We conclude that axiom S3 is satisfied.

Every point of a biaffine plane is contained in at least two lines. This means that every point in $\mathfrak{GQ}_{p,q,r,s}$ is contained in at least 4 blocks. Similarly, every block in $\mathfrak{GQ}_{p,q,r,s}$ contains at least 4 points, which shows that Axiom S4 is satisfied. We conclude that $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane if and only if the two conditions in the theorem are satisfied. ■

Notice that the sums in Conditions C1 and C2 give an idea of how closely the two biaffine plane $\mathfrak{GQ}_{p,r}$ and $\mathfrak{GQ}_{q,s}$ are meshed together to form the incidence structure $\mathfrak{GQ}_{p,q,r,s}$. As we just pointed out in the proof, the possible values for all the sums are 0, 2 and 4. We call $\mathfrak{GQ}_{p,q,r,s}$ *k-mixed*, $k \in \{0, 2, 4\}$, if all the sums in the theorem are equal to k. It seems natural to pay special attention to these three special kinds of incidence structures.

Theorem 4.2 *Let $\mathfrak{GQ}$ be an antiregular quadrangle and let p, q, r and s be four distinct collinear points with connecting line l.*

The incidence structure $\mathfrak{GQ}_{p,q,r,s}$ is never 0-mixed.

The incidence structure $\mathfrak{GQ}_{p,q,r,s}$ is 2-mixed if and only if it is a divisible semi-biplane. If it is a divisible semi-biplane, then the set of parallel classes of points is $(||_p \cup ||_q) \setminus \{l_p, l_q\}$ and the set of parallel classes of blocks is $(||_r \cup ||_s) \setminus \{l_r, l_s\}$.

If $\mathfrak{GQ}_{p,q,r,s}$ is 4-mixed, then it is not a semi-biplane.

Proof. We show that $\mathfrak{GQ}_{p,q,r,s}$ is never 0-mixed. Let $u \in \mathcal{P}^l_r$, and let m and n be distinct lines through u that do not intersect l. Set $x = \pi(p, m)$ and $y = \pi(q, n)$. Then x and y are not collinear, $x \in \mathcal{P}^l_p$ and $y \in \mathcal{P}^l_q$. Furthermore, the point u is contained in $x^\perp \cap y^\perp \cap r^\perp$. Hence $\mathfrak{GQ}_{p,q,r,s}$ is not 0-mixed.

Let $x \in \mathcal{P}^l_p$ and $y \in \mathcal{P}^l_q$. Suppose $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane. If x and y are collinear in $\mathfrak{GQ}$, then x is not parallel to y. If x and y are not collinear in $\mathfrak{GQ}$, then x is parallel to y if and only if $|x^\perp \cap y^\perp \cap r^\perp| + |x^\perp \cap y^\perp \cap s^\perp| = 0$. Let z denote the unique point on the line $q \vee y$ collinear with x. Then $z \in \mathcal{P}^l_q$ and z is parallel to y but not parallel to x. Thus, if there are points x and y such that $|x^\perp \cap y^\perp \cap r^\perp| + |x^\perp \cap y^\perp \cap s^\perp| = 0$, then $\mathfrak{GQ}_{p,q,r,s}$ is not divisible. Conversely, if $|x^\perp \cap y^\perp \cap r^\perp| + |x^\perp \cap y^\perp \cap s^\perp| = 2$ for all possible x and y, then no point in $\mathcal{P}^l_p$ is parallel to a point in $\mathcal{P}^l_q$. But then the punctured lines through p or q are the parallel classes of points in $\mathfrak{GQ}_{p,q,r,s}$. The dual arguments work for blocks.

The last part is an immediate consequence of Theorem 4.1. ■

The following result shows that some antiregular quadrangles do not contain the kind of semi-biplanes under discussion.

Corollary 4.3 *Let $\mathfrak{GQ}$ be a centric antiregular quadrangle and let p, q, r and s be four distinct collinear points with connecting line l. Then $\mathfrak{GQ}_{p,q,r,s}$ is 4-mixed and therefore not a semi-biplane.*

Proof. Let $x \in \mathcal{P}^l_p$ and $y \in \mathcal{P}^l_q$ such that x is not collinear with y. Then both $\{x, y, r\}$ and $\{x, y, s\}$ are triads. Therefore, the centres of both triads contain two points each and $|x^\perp \cap y^\perp \cap r^\perp| + |x^\perp \cap y^\perp \cap s^\perp| = 4$. By Theorem 4.1, the derived structure $\mathfrak{GQ}_{p,q,r,s}$ is not a semi-biplane. ■

The antiregular quadrangle $Q(4,\mathbb{C})$ is an example of a centric antiregular quadrangle. In fact $Q(4,\mathbb{K})$, with $\mathbb{K}$ a field not of characteristic 2, is centric if and only if $\mathbb{K}$ is a quadratically closed field. We will see later on that all semi-biplanes of the form $\mathfrak{GQ}_{p,q,r,s}$ in $\mathfrak{GQ} = Q(4,\mathbb{R})$ are 2-mixed.

Next we consider a special situation which we will come across frequently in the following. Let x and y be two non-collinear points and let l be a line that is not incident with any point of $x^\perp \cap y^\perp$. In particular, this implies that neither x nor y is incident with l. The *square-projection* $\mu_{x,y,l} : x^\perp \cap y^\perp \to l : t \mapsto \pi(t,l)$ is one of the most important tools in the study of antiregular quadrangles (cf. [4] and [9]). A pair $\{p,q\}$ of collinear points with joining line l is called *splitting* if $l \setminus \{p,q\}$ is the disjoint union of two sets A and B such that for all $x \in \mathcal{P}^l_p$, $y \in \mathcal{P}^l_q$, with $x \nsim y$, the image of the square-projection $\mu_{x,y,l}$ is either $A \cup \{p,q\}$ or $B \cup \{p,q\}$. This means that if $A \cup \{p,q\}$ is the image, then the preimage of every point in A contains two points, the preimages of p and q contain one point each and the preimage of every point in B is empty. If $\{p,q\}$ is a splitting pair, then A and B are called the *components* of $\{p,q\}$. If only one of the components occurs in the image for all possible choices of x and y, then we call $\{p,q\}$ *one-sided*.

The first part of the following theorem shows that one of the components of a one-sided splitting pair is necessarily empty, that is, the whole of l is the image of all square-projection under consideration. For example, if $\mathfrak{GQ}$ is a centric antiregular quadrangle, then all relevant pairs $\{p,q\}$ are one-sided splitting with $A = l \setminus \{p,q\}$ and $B = \emptyset$. We call $\{p,q\}$ *two-sided* if both A and B are non-empty. Let $\{p,q\}$ and $\{r,s\}$ be two splitting pairs of points such that all four points are contained in the same line l. We say that the two pairs are *intertwined* if p and q are contained in different components of $\{r,s\}$ and r and s are contained in different components of $\{p,q\}$. As we shall see later, in an 'ideal world' like $Q(4,\mathbb{R})$ all pairs of collinear points are two-sided splitting and, given four distinct collinear points, they can be divided up into two intertwined pairs. In a situation like this the following result applies.

Theorem 4.4 *Let $\mathfrak{GQ}$ be an antiregular quadrangle and let p,q,r and s be four distinct collinear points with connecting line l.*

Let $\{p,q\}$ be a one-sided splitting pair. Then one of its components is $l \setminus \{p,q\}$ and the other component is empty.

Let both $\{p,q\}$ and $\{r,s\}$ be two-sided splitting pairs. Then $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane if and only if the two pairs are intertwined. If $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane, then it is divisible, or equivalently, 2-mixed.

Proof. Let $\{p,q\}$ be a one-sided splitting pair and let A and B be its two components. Let A be in the image of any of the relevant square-projections. Assume that B is non-empty and $t \in B$. Let $u \in \mathcal{P}^l_t$, let m be a line distinct from l through p and let n be a line distinct from l through q. Neither of the two lines contains u. If $\lambda(u,m) \neq \lambda(u,n)$, then $\pi(u,m) \in \mathcal{P}^l_p, \pi(u,n) \in \mathcal{P}^l_q$ and $\pi(u,m) \nsim \pi(u,n)$. Moreover, $u \in \pi(u,m)^\perp \cap \pi(u,n)^\perp$ and $\mu_{\pi(u,m),\pi(u,n),l}(u) = r$. This is a contradiction. If $\lambda(u,m) = \lambda(u,n)$, let m' be a line through p different from l and m. Then $\lambda(u,m') \neq \lambda(u,n)$ and we arrive at a contradiction as before. Hence B is empty.

Let both $\{p,q\}$ and $\{r,s\}$ be two-sided splitting pairs. If the two pairs are intertwined, then all sums in Theorem 4.1 are equal to 2. Hence, by Theorem 4.2, the derived structure $\mathfrak{GQ}_{p,q,r,s}$ is a divisible semi-biplane.

If the two pairs are not intertwined, we may assume, without loss of generality, that p and q are contained in the same component of $l \setminus \{r,s\}$. Since $\{p,q\}$ is two-sided splitting, its two components are non-empty. The above considerations show that there exist $x \in \mathcal{P}_p^l$ and $y \in \mathcal{P}_q^l$, $x \nsim y$ such that $|x^\perp \cap y^\perp \cap r^\perp| + |x^\perp \cap y^\perp \cap s^\perp| = 4$ and, by Theorem 4.1, the derived structure $\mathfrak{GQ}_{p,q,r,s}$ is not a semi-biplane. ∎

It would be interesting to investigate whether some of the classical antiregular quadrangles can be characterized by the fact that they contain 'many' splitting pairs and whether the existence of 'many' splitting pairs in an antiregular quadrangle implies that all relevant pairs are splitting.

5 The classical case

Let $\mathfrak{GQ} = Q(4,\mathbb{K})$, where $\mathbb{K}$ is a field not of characteristic 2 and let p,q,r and s be four collinear points with connecting line l. In the Laguerre plane $\mathfrak{GQ}_p$ the three points q,r and s are contained in the parallel class l_p. As we already noted above, $\mathfrak{GQ}_p$ can be considered as the geometry of plane sections of a quadratic cone K with its vertex v removed. Let c be the base of K (a non-degenerate conic) and let $K' := K \setminus \{v\}$. We identify c with the projective line $\mathbb{K} \cup \{\infty\}$ in the natural way such that the point of intersection of c with l gets identified with ∞. Following this, we can identify K' with $(\mathbb{K} \cup \{\infty\}) \times \mathbb{K}$ '=' $\mathbb{K}^2 \cup (\{\infty\} \times \mathbb{K})$ via a stereographic projection through the point q such that the following holds:

1. the parallel class l_p gets identified with $\{\infty\} \times \mathbb{K}$;
2. all other parallel classes get identified with the verticals in $\mathbb{K}^2$;
3. the points q,r and s get identified with $(\infty,0),(\infty,a_r)$ and (∞,a_s), respectively;
4. the circles of the Laguerre plane get identified with the sets $\bar{c}_{a,b,c} := c_{a,b,c} \cup \{(\infty,a)\}$ where $c_{a,b,c} := \{(x, ax^2+bx+c) \mid x \in \mathbb{K}\}$, $a,b,c \in \mathbb{K}$.

More details about this identification can be found in [1].

Let $C(a) := \{c_{a,b,c} \mid b,c \in \mathbb{K}\}$, $a \in \mathbb{K}$. Notice that $C(0)$ consists of all non-vertical lines in $\mathbb{K}^2$ and that $C(a)$, $a \neq 0$ is the set of translates in $\mathbb{K}^2$ of the parabola $c_{a,0,0}$. Now $\mathfrak{GQ}_{p,q,r,s}$ '=' $(\mathbb{K}^2 \cup C(0), C(a_r) \cup C(a_s), I)$, where a point x is incident with a block b, that is $x\ I\ b$, if and only if $x \in b$ for $x \in \mathbb{K}^2$ and x touches b for $x \in C(0)$, that is, $|x \cap b| = 1$.

Let $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ and let $k \in \mathbb{K}^*$. Since $\operatorname{char}\mathbb{K} \neq 2$, there are either exactly two or no solutions of the equation $x^2 = k$. If there is a solution, then k is called a *square*. A *non-square* is an element of $\mathbb{K}^*$ that is not a square. Also remember that the set S of squares is a subgroup of $\mathbb{K}^*$ and that therefore the product of two squares is a square, the product of a square and a non-square is a non-square and that the inverse of a (non-)square is a (non-)square. Let NS denote the set of non-squares.

Theorem 5.1 *Let $\mathfrak{GQ} = Q(4, \mathbb{K})$, $\operatorname{char} \mathbb{K} \neq 2$, let p, q, r and s be distinct collinear points with connecting line l and let $(\mathbb{K}^2 \cup C(0), C(a_r) \cup C(a_s), I)$ be a representation of $\mathfrak{GQ}_{p,q,r,s}$ as above. Then:*

1. *$\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane if and only if $a_r a_s$ is a non-square.*
2. *$\mathfrak{GQ}_{p,q,r,s}$ is a divisible semi-biplane if and only if $a_r a_s$ is a non-square and S is a subgroup of index 2 in $\mathbb{K}^*$.*

Proof. The following maps extend to automorphisms of the Laguerre plane $\mathfrak{GQ}_p$ that fix the parallel class l_p point-wise:

$$\begin{aligned} \mathbb{K} \to \mathbb{K} &: (x, y) \to (x, y + ax + b), \\ \mathbb{K} \to \mathbb{K} &: (x, y) \to (x + a, y), \end{aligned}$$

where $a, b \in \mathbb{K}$.

We check when exactly Conditions C1 and C2 in Theorem 4.1 are satisfied. Let $x \in \mathbb{K}^2$ and let $y \in C(0)$ such that $x \notin y$. With the above automorphisms, it suffices to consider the cases where $x = (0, k)$, $k \in \mathbb{K}^*$ and y is the horizontal line $c_{0,0,0}$. We conclude that Condition C1 is satisfied if and only if for all $k \in \mathbb{K}^*$ at most one of the two values k/a_r and k/a_s is a square. If this is the case, we can set $k = 1/a_r$ to conclude that $a_r a_s$ is a non-square. On the other hand, let $a_r a_s$ be a non-square and let $k \in \mathbb{K}^*$. If both k/a_r and k/a_s were squares, then their product $k^2/(a_r a_s)$ and therefore also $a_r a_s$ would be squares, which is a contradiction. This shows that Condition C1 and the condition that $a_r a_s$ is a non-square are equivalent.

Let $x \in C(a_r)$ and $y \in C(a_s)$ such that $|x \cap y| \neq 1$. With the above automorphisms, it suffices to consider the cases where $x = c_{a_r,0,0}$ and $y = c_{a_s,0,k}$, $k \in \mathbb{K}^*$. Clearly, x intersects y in (necessarily two) points of $\mathbb{K}^2$ if and only if $k/(a_r - a_s)$ is a square. The elements of $C(0)$ incident with $c_{a,0,k}$, $a \in \mathbb{K}^*$ are $c_{0,2at,-at^2+k}$, $t \in \mathbb{K}$ (these are just the tangents of the parabola $c_{a,0,k}$). Hence x and y intersect in (necessarily two) blocks of $C(0)$ if and only if the following system of equations in the variables t and u has solutions:

$$\begin{aligned} 2a_r t &= 2a_s u \\ -a_r t^2 &= -a_s u^2 + k. \end{aligned}$$

This is the case if and only if $(a_r/a_s)(k/(a_r - a_s))$ is a square. So, Condition C2 is satisfied if and only if for all $k \in \mathbb{K}^*$ at most one of the values $k/(a_r - a_s)$ and $(a_r a_s)(k/(a_r - a_s))$ is a square. As above it is easy to show that this condition is equivalent to $a_r a_s$ being a non-square. This completes the proof of the first part of the theorem.

Let $\mathfrak{GQ}_{p,q,r,s}$ be a divisible semi-biplane. Then, as a consequence of Theorem 4.2 and the above considerations, for all $k \in \mathbb{K}^*$ either ka_r is a square and ka_s is a non-square, or the other way around. Without loss of generality, let a_r be a square, let a_s be a non-square. We conclude that $Sa_s \subset NS$ and $NSa_s \subset S$. Hence $Sa_s^2 = S \subset NSa_s$ and therefore $S = NSa_s$, which means that S is a subgroup of index 2 in $\mathbb{K}^*$.

Let $\mathfrak{GQ}_{p,q,r,s}$ be a semi-biplane and let S be a subgroup of index 2 in $\mathbb{K}^*$. Then the product of two non-squares is a square and we may assume, without loss of generality, that a_r is a square and that a_s is a non-square. Then it is clear that for all $k \in \mathbb{K}^*$ exactly one of the two values ka_r and ka_s is a square, which in turn translates, via the above considerations, into the fact that all the sums in Theorem 4.1 are equal to 2 and that as a consequence of this, $\mathfrak{GQ}_{p,q,r,s}$ is a divisible semi-biplane.

This completes the proof of the second part of the theorem. ∎

The squares in this result and the square-projections that we used to define splitting pairs have more than just the word 'square' in common. In fact, let p, q and l be as in Theorem 5.1, let $x \in \mathcal{P}_p^l$ and $y \in \mathcal{P}_q^l$ be non-collinear points and let $\mathbb{K}$ be identified with $l_p = \{\infty\} \times \mathbb{K}$ in the natural way. Then we can show with the same arguments as in the above proof that the image of the square-projection $\mu_{p,q,l}$ is one of the cosets Sa, $a \in \mathbb{K}^*$ to which $\{p, q\}$ has been joined and that all these cosets occur in an image of one of these projections. This implies, in particular, that $\{p, q\}$ is a splitting pair if and only if S is a subgroup of index 2 in $\mathbb{K}^*$. We summarize some immediate consequences of this remark and the results of the previous sections.

Theorem 5.2 *Let $\mathfrak{GQ} = Q(4, \mathbb{K})$, $\operatorname{char} \mathbb{K} \neq 2$, let p, q, r and s be distinct collinear points with connecting line l and let $\mathfrak{GQ}_{p,q,r,s}$ be a semi-biplane. Then the following are equivalent:*

1. *There exists a splitting pair of collinear points in $\mathfrak{GQ} = Q(4, \mathbb{K})$.*
2. *The set of squares S is a subgroup of index 2 in $\mathbb{K}^*$.*
3. *$\mathfrak{GQ}_{p,q,r,s}$ is divisible.*

If one of the above is satisfied, then all pairs of collinear points are two-sided splitting and $\{p, q\}$ and $\{r, s\}$ are intertwined pairs.

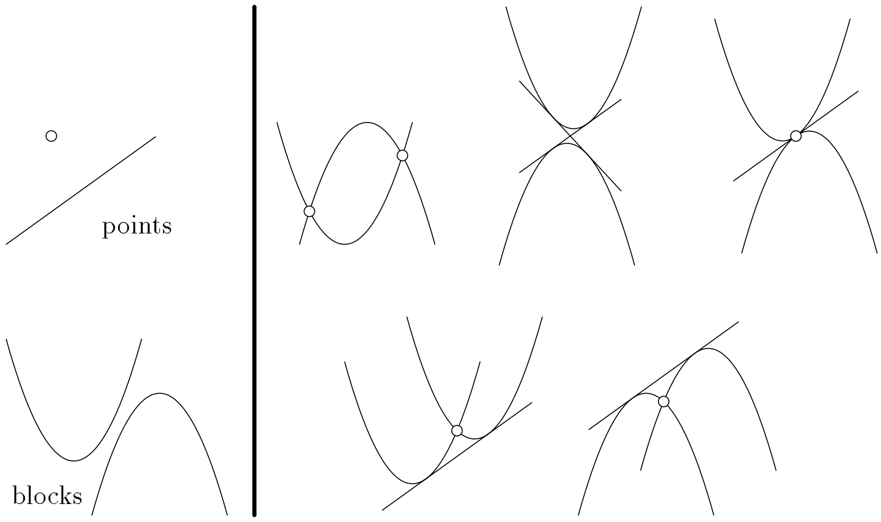
The 'finite' part of the following result corresponds to that part of [13, Theorem 11] that deals with the odd order case.

Corollary 5.3 *Let $\mathfrak{GQ} = Q(4, \mathbb{K})$, where $\mathbb{K}$ is a finite field of odd order or $\mathbb{K} = \mathbb{R}$ and let p, q, r and s be distinct collinear points with connecting line l. Then $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane if and only if $a_r a_s$ is a non-square. If $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane, then it is divisible.*

Proof. By the last theorem it suffices to remark that in all cases under discussion the set of squares is a subgroup of index 2 in $\mathbb{K}^*$. ∎

Many more interesting properties of the finite semi-biplanes constructed in this manner are discussed in [11], [12] and [13]. We mention, in particular, that Conditions C1 and C2 in Theorem 4.1 correspond to [13, Condition 4, p. 124]. Furthermore, the isomorphism problem in the finite case is dealt with in [12]. Among other things it is shown that two semi-biplanes constructed as above from one of the finite anti-regular quadrangles need not be isomorphic.

Let us have a closer look at the case $\mathfrak{GQ} = Q(4, \mathbb{R})$. As a consequence of Theorem 5.1, the incidence structure $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane if and only if a_r and a_s do have opposite signs. Let $a_s = -a_r = 1$. The pictures below, which correspond to the different cases treated in Theorem 4.1, illustrate the various ways in which two points in this semi-biplane are joined by blocks.



As in the finite case (see [12]), it is possible to prove in the real case that semi-biplanes associated with different sets of four points may be non-isomorphic.

6 Compact antiregular quadrangles

A *compact antiregular quadrangle* $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L}, I)$ is an antiregular quadrangle such that $\mathcal{P}$ and $\mathcal{L}$ are compact topological spaces and such that the two maps π and λ are continuous. We only mention a few properties of compact antiregular quadrangles. For the proofs and more details the reader is referred to [9].

There are two different kinds of compact antiregular quadrangles: those whose points and lines are homeomorphic to $\mathbb{S}_1$ and those whose points and lines are homeomorphic to $\mathbb{S}_2$. They are called the compact antiregular quadrangles with parameter 1 and 2, respectively. The antiregular quadrangles $Q(4, \mathbb{R})$ and $Q(4, \mathbb{C})$ are the two classical examples of these two kinds of antiregular quadrangles. It helps to think of a compact antiregular quadrangle with parameter i, $i = 1, 2$, as a distortion of the classical geometry with the same parameter, since most corresponding geometrical objects associated with the two antiregular quadrangles have been shown to be homeomorphic topological spaces.

Theorem 6.1 *Let $\mathfrak{GQ}$ be a compact antiregular quadrangle and let p, q, r and s be distinct collinear points with connecting line l.*

If $\mathfrak{GQ}$ has parameter 1, then $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane if and only if p and q are contained in different connected components of $l \setminus \{r, s\}$. If $\mathfrak{GQ}_{p,q,r,s}$ is a semi-biplane, then it is divisible.

If $\mathfrak{GQ}$ has parameter 2, then $\mathfrak{GQ}_{p,q,r,s}$ is not a semi-biplane.

Proof. Let $\mathfrak{GQ}$ have parameter 1. Then, by [4, Proposition 4.5], every pair of collinear points $\{p, q\}$ is splitting and the components of $\{p, q\}$ are just the connected components of $l \setminus \{p, q\}$. We apply Theorem 4.4 to wrap up the proof of the first part of this result.

Let $\mathfrak{GQ}$ have parameter 2. Then, by [9, Proposition 2.14], $\mathfrak{GQ}$ is a centric antiregular quadrangle. We apply Corollary 4.3 to conclude that $\mathfrak{GQ}_{p,q,r,s}$ is not a semi-biplane. ∎

Notice that in a compact antiregular quadrangle with parameter 1 four distinct points can always be divided up into two intertwined pairs of splitting points.

Also, in the parameter 1 case, both the point space and the block space of $\mathfrak{GQ}_{p,q,r,s}$ are disjoint unions of two topological spaces homeomorphic to $\mathbb{R}^2$, and every point and block is the disjoint union of two topological spaces homeomorphic to $\mathbb{R}$. This follows from [9, 2.3].

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