The Helmholtz Conditions for the Difference Equations Systems

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Abstract

This paper present the analogous of the Helmholtz conditions for first and second order difference equations systems. We obtain variational implicit algoritms used in numerical analysis and numerical integration schemes for Hamilton systems. We give some representative examples.

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1 Introduction

The inverse problem of the calculus of variations is a subject which has been studied over several decades and an important research field. For first and second order ordinary differential equations there are numerous contributions and we can mention the following papers [3], [4], [8].

The purpose of the present paper is to extend this analysis and to derive the Helmholtz conditions for difference equations systems too. We derive the first and second order variational relations for Lagrangians that depend on a set of points $(q_k)_{k\in Z}$ on a differentiable submanifold $\Gamma \subset Q\times B$. In these conditions we formulate the d'Alembert–Lagrange principle which leads us for $L(k) = tr(q_k J q_{k+1}^T)$, with $q_k \in O(n)$, to the discrete Euler equations for the rigid body [9].

For a function system $\{F_i(k)\}_{i=\overline{1,n}}$ with $F_i(k) = F_i(q_{k-1}, q_k, q_{k+1})$ we deduce the discrete Helmholtz conditions.

These conditions are satisfied for $\{e_i(k)\}_{i=\overline{1,n}}$ where $e_i(k)$ come from the variational principle for $L(k)$. We are now able to write the discrete Hamilton equations used in numerical algorithms for Hamiltonian systems.

In the final section we give the Helmholtz conditions for some examples of difference equations systems.

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2 The first and second variation formulae

Let Q be a n-dimensional differentiable manifold and B a m -dimensional differentiable manifold.

Let $(q_k)_{k\in Z}$ be a set of points from Q and $S : Q \times Q \to B$ a differentiable function denote $S(k) = S(q_k, q_{k+1})$.

We consider $L : Q \times B \to R$ where

$$L(k) = L(q_k, S(k))$$

The functional

$$\mathcal{A}(q) = \sum_{k\in Z} L(k) \tag{1}$$

is called the action of L.

For $q_k \in Q$ let $\lambda : Q \times I \to Q$ where $I = (-a, a) \in R$, denote with $\lambda(q_k, \varepsilon) = q_k(\varepsilon)$ and let be

$$\eta_k(\varepsilon) = \frac{\partial\lambda(q_k, \varepsilon)}{\partial\varepsilon} \qquad \xi_k(\varepsilon) = \frac{\partial^2\lambda(q_k, \varepsilon)}{\partial\varepsilon^2}$$

$$\eta_k = \eta_k(\varepsilon)\big|_{\varepsilon=0} \qquad \xi_k = \xi_k(\varepsilon)\big|_{\varepsilon=0}$$

Proposition 1. *The first variation of the functional $\mathcal{A}(q)$ with respect to λ is*

$$\delta\mathcal{A}(q, \eta) = \sum_{k\in Z} e_i(k)\eta_k^i \tag{2}$$

The second variation of $\mathcal{A}(q)$ with respect to λ is

$$\delta^2\mathcal{A}(q, \eta, \xi) = \sum_{k\in Z} e_i(k)\xi_k^i + \sum_{k\in Z}\Big[Hess^{(1)}L(k)(\eta_k, \eta_k) + \\ + 2Hess^{(2)}L(k)(\eta_k, \eta_{k+1})\Big] \tag{3}$$

where

$$e_i(k) = \frac{\partial L(k)}{\partial q_k^i} + \frac{\partial L(k)}{\partial S^\alpha(k)}\frac{\partial S^\alpha(k)}{\partial q_k^i} + \frac{\partial L(k-1)}{\partial S^\alpha(k-1)}\frac{\partial S^\alpha(k-1)}{\partial q_k^i} \tag{4}$$

$$Hess^{(1)}L(k)(\eta_k, \eta_k) = \frac{\partial e_i(k)}{\partial q_k^j}\eta_k^i\eta_k^j \tag{5}$$

$$Hess^{(2)}L(k)(\eta_k, \eta_{k+1}) = \frac{\partial}{\partial q_k^i}\left(\frac{\partial L(k)}{\partial S^\alpha(k)}\frac{\partial S^\alpha(k)}{\partial q_{k+1}^j}\right)\eta_k^i\eta_{k+1}^j \tag{6}$$

We can prove this proposition if we compute the first and second derivative with respect to ε for

$$\mathcal{A}(q(\varepsilon)) = \sum_{k\in Z} L(q_k(\varepsilon), S(q_k(\varepsilon), q_{k+1}(\varepsilon)))$$

For $L(k) = L(q_k, q_{k+1})$ we obtain

$$e_i(k) = \frac{\partial L(k)}{\partial q_k^i} + \frac{\partial L(k-1)}{\partial q_k^i} \tag{7}$$

$$Hess^{(1)} L(k)(\eta_k, \eta_k) = \left(\frac{\partial^2 L(k)}{\partial q_k^i \partial q_k^j} + \frac{\partial^2 L(k-1)}{\partial q_k^i \partial q_k^j} \right) \eta_k^i \eta_k^j \tag{8}$$

$$Hess^{(2)} L(k)(\eta_k, \eta_{k+1}) = \frac{\partial^2 L(k)}{\partial q_k^i \partial q_{k+1}^j} \eta_k^i \eta_{k+1}^j \tag{9}$$

For $L : Q \times R^n \to R$ with $S(q_k, q_{k+1}) = q_k^1 = q_{k+1}^i - q_k^i$ results [1], [2], [5], [7]

$$e_i(k) = \frac{\partial L(k)}{\partial q_k^i} - \frac{\partial L(k)}{\partial q_k^{1i}} + \frac{\partial L(k-1)}{\partial q_{k-1}^{1i}} \tag{10}$$

$$Hess^{(1)} L(k)(\eta_k, \eta_k) = \left(\frac{\partial^2 L(k)}{\partial q_k^i \partial q_k^j} + \frac{\partial^2 L(k-1)}{\partial q_{k-1}^{1i} \partial q_{k-1}^{1j}} \right) \eta_k^i \eta_k^j \tag{11}$$

$$Hess^{(2)} L(k)(\eta_k, \eta_k^1) = \left(\frac{\partial^2 L(k)}{\partial q_k^i \partial q_k^{1j}} - \frac{\partial^2 L(k)}{\partial q_{k-1}^{1i} \partial q_k^{1j}} \right) \eta_k^i \eta_k^{1j} \tag{12}$$

We define the one-forms β associated to $L(k)$

$$\beta(k) = \frac{\partial L(k-1)}{\partial S^\alpha(k-1)} \frac{\partial S^\alpha(k-1)}{\partial q_k^i} dq_k^i$$

and the submanifold $\Gamma \subset Q \times B$

$$\Gamma = \left\{ (q_k, S(q_k, q_{k+1})), e_i(k) = 0 \qquad i = \overline{1, n} \right\}$$

Proposition 2. *The submanifold* Γ *is an isotopic submanifold with respect to the two–form*

$$\Omega(k, k+1) = d\beta(k+1) - d\beta(k)$$

From this proposition follows that the Lagrangian L is a generating function of Γ in the domain where $Hess^{(2)} L(k)(\eta_k, \eta_{k+1})$ is nondegenerate.

3 The Lagrange–d'Alembert principle

We consider a set of functions $f_a : Q \times B \to R$, $a = \overline{1, p}$, where $f_a(k) = f_a(q_k, S(q_k, q_{k+1}))$ with

$$rang \left\| \frac{\partial f_a(k)}{\partial S^{\alpha(k)}} \right\| = p < m$$

and the restrictions $\mathcal{R} \subset Q \times B$ given by

$$\mathcal{R} = \left\{ (q_k, S(q_k, q_{k+1})) \quad , \quad f_a(k) = 0 \quad , \quad a = \overline{1, p} \right\}$$

We define the virtual variation for $\mathcal{R}$, $\lambda(q_k,\varepsilon)=q_k(\varepsilon)$ that satisfy

$$\frac{\partial f_a(k)}{\partial S^\alpha(k)}\frac{\partial S^\alpha(k)}{\partial q_k^i}\eta_k^i=0 \qquad a=\overline{1,p} \tag{13}$$

The Lagrange–d'Alembert principle for the system $(Q,L,\mathcal{R})$ system is:
The elements $q_k\in Q$ represents the motion of the $(Q,L,\mathcal{R})$ system if

$$e_i(k)\eta_k^i=0 \tag{14}$$

for all virtual variations η_k for $\mathcal{R}$.

From §3. (1) and (2) we deduce that the equations of the system described above $(Q,L,\mathcal{R})$ satisfies

$$e_i(k)=\mu^a\frac{\partial f_a(k)}{\partial S^\alpha(k)}\frac{\partial S^\alpha(k)}{\partial q_k^i} \qquad i=\overline{1,n} \tag{15}$$

$$f_a(k)=0$$

Using the Lagrange–d'Alembert principle for $L:O(n)\times O(n)\to R$ with

$$L(k)=tr(q_kJq_{k+1}^T) \tag{16}$$

where J is a symmetric positive matrix, we obtain Arnold equations [9]

$$M_{k+1}=\omega_kM_k\omega_k^{-1} \tag{17}$$

$$M_k=\omega_k^TJ-J\omega_k$$

where $\omega_k=q_k^Tq_{k-1}$ [5].

If $q_k=q(t_k)$ with $t_k=t_0+k\varepsilon$, §3. (5) leads to the Euler equations of the rigid body

4 The Helmholtz conditions for discrete second order equations

The equations system that describe the motion $e_i(k)=0$ is caracterized by the functions $F_i(k)=F_i(q_{k-1},q_k,q_{k+1})$, $i=\overline{1,n}$.

The inverse problem consist of finding a set of conditions for $\{F_i(k)\}_{i=\overline{1,n}}$ in order to get $F_i(k)=e_i(k)$ where $e_i(k)$ is the first variation of the Lagrangian $L(k)$. These conditions constitute the discrete variant of the Helmholtz conditions for differential equations (that means the continuous case).

For $\lambda:Q\times I\to Q$ with $\lambda(q_k,\varepsilon)=q_k(\varepsilon)$, $\varepsilon\in(-a,a)=I$ we define the Fréchet derivative of $F_i(k)$

$$DF_i(k)(\eta(k))=\frac{d}{d\varepsilon}F_i(q_{k-1}(\varepsilon),q_k(\varepsilon),q_{k+1}(\varepsilon))\bigg|_{\varepsilon=0} \tag{18}$$

where $\eta(k)=(\eta_{k-1},\eta_k,\eta_{k+1})$. From §3 (1) we obtain

$$(19)\qquad DF_i(k)(\eta(k)) = \frac{\partial F_i(k)}{\partial q^j_{k-1}}\eta^j_{k-1} + \frac{\partial F_i(k)}{\partial q^j_k}\eta^j_k + \frac{\partial F_i(k)}{\partial q^j_{k+1}}\eta^j_{k+1} \qquad i = \overline{1,n}$$

The adjoint function of $DF_i(k)\eta(k)$ is:

$$(20)\qquad D^*F_i(k)(\eta(k)) = \frac{\partial F_j(k-1)}{\partial q^i_k}\eta^j_{k-1} + \frac{\partial F_j(k)}{\partial q^i_k}\eta^j_k + \frac{\partial F_i(k+1)}{\partial q^i_k}\eta^j_{k+1}$$

From §4. (2) and (3) we have

$$\tilde{\eta}^i_k DF_i(k)(\eta(k)) - \eta^i_k D^*F_i(k)(\tilde{\eta}(k)) = \Delta(\eta(k+1), \tilde{\eta}(k+1)) - \Delta(\eta(k), \tilde{\eta}(k))$$

where

$$\Delta(\eta(k), \tilde{\eta}(k)) = \frac{\partial F_i(k-1)}{\partial q^j_k}\eta^i_{k-1}\tilde{\eta}^j_k - \frac{\partial F_i(k)}{\partial q^j_{k-1}}\eta^i_k\tilde{\eta}^j_{k-1}$$

The system is called selfadjoint if $DF_i(k) = D^*F_i(k)$, $i = \overline{1,n}$.

Proposition 3. *The functions* $\{F_i(k)\}_{i=\overline{1,n}}$ *are selfadjoint iff* a)

$$(21)\qquad \frac{\partial F_i(k)}{\partial q^j_k} = \frac{\partial F_j(k)}{\partial q^i_k};$$

b)

$$\frac{\partial F_i(k)}{\partial q^j_{k+1}} = \frac{\partial F_j(k+1)}{\partial q^i_k}$$

These conditions are the Helmholtz conditions for discrete systems.

Proposition 4. *The functions* $e_i(k)$, $i = \overline{1,n}$, *given by §2. (4) satisfied §4. (4).*

Proposition 5. *A solution of the system §4. (4) is*

$$(22)\qquad F_i(k) = \frac{\partial \varphi(k)}{\partial q^i(k)} + \frac{\partial \varphi(k-1)}{\partial q^i(k)}$$

with $\varphi(k) = \varphi(q_{k-1}, q_k)$.

Assuming a functions set $\{F_i(k)\}_{i=\overline{1,n}}$ and $\tilde{F}_i(k) = c^j_i(k)F_j(k)$ with $det(c^j_i(k)) \neq 0, \forall k \in Z$.

Proposition 6. *The function system* $\left\{\tilde{F}_i(k)\right\}_{i=\overline{1,n}}$ *satisfies the selfadjoint conditions iff*

$$(23)\qquad \left(\frac{\partial c^h_i(k)}{\partial q^j_k} - \frac{\partial c^h_j(k)}{\partial q^i_k}\right) F_h(k) + \left(c^h_i(k)\delta^l_j - c^h_j(k)\delta^l_i\right) \frac{\partial F_h(k)}{\partial q^l_k} = 0$$

$$\left(\frac{\partial c^h_i(k)}{\partial q^j_{k+1}} - \frac{\partial c^h_j(k+1)}{\partial q^i_k}\right) F_h(k+1) + c^h_i(k)\frac{\partial F_h(k)}{\partial q^j_{k+1}} - c^h_j(k+1)\frac{\partial F_h(k+1)}{\partial q^i_k} = 0$$

The matrix $(c^i_j(k))$ is called an integrant factor. In order to determine this integrant factor it must be consider the special cases with functions of the following type $c^i_j(k, q_k, q_{k+1})$.

5 Hamilton equations

Let $S : Q \times Q \to R^n$, $L : Q \times R^n \to R$ and $L(k) = L(q_k, S(q_k, q_{k+1}))$. We introduce

$$p_\alpha(k) = \frac{\partial L(k)}{\partial S^\alpha(k)} \qquad \alpha = \overline{1,m} \tag{24}$$

$L(k)$ is regular with respect to $S(k)$ if

$$det\left(\frac{\partial^2 L(k)}{\partial S^\alpha(k) \partial S^\beta(k)}\right) \neq 0 \qquad \forall k \in Z \tag{25}$$

Under that assumptions we can see that

$$S^\alpha(k) = l^\alpha(q_k, p(k))$$

If

$$H(k) = H(p(k), q(k)) = p_\alpha(k) l^\alpha(q_k, p(k)) - L(q_k, l(q_k, p(k))) \tag{26}$$

we construct his action

$$\mathcal{H}(k) = \sum_{k \in Z} [p_\alpha(k) l^\alpha(q_k, p(k)) - L(q_k, l(q_k, p(k)))] \tag{27}$$

Proposition 7. *The first variation of* $\mathcal{H}(k)$ *is*

$$\delta \mathcal{H}(k)(\xi, \eta) = \sum_{k \in Z} \left[h_i(k) \eta^i_k + m^\alpha(k) \xi_\alpha(k)\right] \tag{28}$$

where

$$m^\alpha(k) = l^\alpha(k) - p_\beta(k) \frac{\partial e^\beta(k)}{\partial p_\alpha(k)} \tag{29}$$

$$h_i(k) = p_\alpha(k-1) \frac{\partial e^\alpha(k)}{\partial q^i_k} - p_\alpha(k) \frac{\partial e^\alpha(k+1)}{\partial q^i_k}$$

Let $\Gamma^* \subset Q \times R^m \times R^{m*}$ given by

$$\Gamma^* = \{(q_k, S(k), p(k)), h_i(k) = 0, m^\alpha(k) = 0\} \tag{30}$$

Using §5. (3) and (6) we see that Γ^* is characterized by a set of discrete equations – the discrete Hamilton equations

$$\frac{\partial H(k)}{\partial q^i_k} = p_\alpha(k) \frac{\partial S^\alpha(k)}{\partial q^i_k} + p_\alpha(k-1) \frac{\partial S^\alpha(k-1)}{\partial q^i_k} \tag{31}$$

$$\frac{\partial H(k)}{\partial p_\alpha(k)} = S^\alpha(k)$$

If $S(k) = q^1_k \in R^n$, §5. (8) can be written in the following form [5]

$$p_i(k-1) - p_i(k) = \frac{\partial H(k)}{\partial q^i_k} \tag{32}$$

$$q^i_{k+1} - q^i_k = \frac{\partial H(k)}{\partial p^i(k)}$$

Such equations are used in numerical algorithms for solving first order differential equations systems.

6 The Helmholtz conditions for first order difference equations

The equations §5. (9) are in this case

$$F(k) = F^i(k)\xi_i(k) \qquad G(k) = G_i(k)\eta^i_k \tag{33}$$

where

$$F^i(k) = F^i(q_k, p_k, q_{k+1}) \qquad G_i(k) = G_i(p_{k-1}, q_k, p_k) \tag{34}$$

We define the action of the pair $(F(k), G(k))$

$$\mathcal{A}(F,G)(\eta,\xi) = \sum_{k\in Z}\left[F^i(k)\xi_i(k) + G_i(k)\eta^i_k\right] \tag{35}$$

The first variation for §6. (3) is

$$\delta\mathcal{A}(F,G)(\eta,\xi,\bar\eta,\bar\xi) = \sum_{k\in bfZ}\left[DF^i(k)\xi_i(k) + DG_i(k)\eta^i_k + F^i(k)\bar\xi_i(k) + G_i(k)\eta^i_k\right] \tag{36}$$

where

$$DF^i(k) = \left(\frac{\partial F^i(k)}{\partial q^j_k} + \frac{\partial F^i(k-1)}{\partial q^j_k}\right)\eta^j_k + \frac{\partial F^i(k)}{\partial p_j(k)}\xi_j(k)$$

$$DG_i(k) = \frac{\partial G_i(k)}{\partial q^j_k}\eta^j_k + \left(\frac{\partial G_i(k)}{\partial p_j(k)} + \frac{\partial G_i(k+1)}{\partial p_j(k)}\right)\xi_j(k) \tag{37}$$

$$\bar\xi_i(k) = \left.\frac{\partial^2\theta_i(p_k,\varepsilon)}{\partial\varepsilon\partial\varepsilon}\right|_{\varepsilon=0} \qquad \bar\eta^i_k = \left.\frac{\partial^2\lambda^i(q_k,\varepsilon)}{\partial\varepsilon\partial\varepsilon}\right|_{\varepsilon=0}$$

with $\theta(p(k),0) = p(k)$, $\lambda(q_k,0) = q_k$.

We call the adjoint pair of $(F(k), G(k))$ the functions pair $(F^*(k), G^*(k))$ given by

$$F^*(k) = F^j(k)\xi_j(k) - (G_j(k) + G_j(k+1))\eta^j_k \tag{38}$$

$$G^*(k) = -(F^j(k) + F^j(k-1))\xi_j(k) + G_j(k)\eta^j_k$$

The first variation for $F^*(k)$ with respect to $p(k)$ and for $G^*(k)$ with respect to q_k are

$$\delta\mathcal{A}_p(F^*)(\xi,\bar\xi,\bar\eta) = \sum_{k\in Z}\left[D^*F^i(k)\xi_i(k) + F^*(k)(\bar\xi,\bar\eta)\right] \tag{39}$$

$$\delta\mathcal{A}_q(G^*)(\eta,\bar\xi,\bar\eta) = \sum_{k\in Z}\left[D^*G^i(k)\eta^i_k + G^*(k)(\bar\xi,\bar\eta)\right]$$

where

$$D^*F^i(k) = \frac{\partial F^j(k)}{\partial p_i(k)}\xi_j(k) + \left(\frac{\partial G_j(k)}{\partial p_i(k)} + \frac{\partial G_j(k+1)}{\partial p_i(k)}\right)\eta^j_k \tag{40}$$

$$D^*G_i(k) = -\left(\frac{\partial F^j(k)}{\partial q^i_k} + \frac{\partial F^j(k-1)}{\partial q^i_k}\right)\xi_j(k) + \frac{\partial G_j(k)}{\partial q^i_k}\eta^j_k$$

The pair $(F(k), G(k))$ is selfadjoint if

$$D^*F^i(k) = DF^i(k) \qquad D^*G_i(k) = DG_i(k) \qquad i = \overline{1,n}, k \in Z \tag{41}$$

The relations §6. (5) and (8) implies $(F(k), G(k))$ to be self adjoint iff

$$\frac{\partial F^j(k)}{\partial p_i(k)} = \frac{\partial F^i(k)}{\partial p_j(k)} \qquad \frac{\partial G_j(k)}{\partial q_k^i} = \frac{\partial G_i(k)}{\partial q_k^j} \tag{42}$$

$$\frac{\partial F^i(k)}{\partial q_k^j} + \frac{\partial F^i(k-1)}{\partial q_k^j} + \frac{\partial G_j(k)}{\partial p_i(k)} + \frac{\partial G_j(k+1)}{\partial p_i(k)} = 0$$

$i, j = \overline{1,n}, \forall k$.

Proposition 8. *The functions system*

$$F^i(k) = q_{k+1}^i - q_k^i - h\frac{\partial H(k)}{\partial p_i(k)} \tag{43}$$

$$G_i(k) = p_i(k) - p_i(k-1) + h\frac{\partial H(k)}{\partial q_k^i}$$

is selfadjoint.

This system represents the Euler implicit algorithm for numerical integration of an Hamiltonian system.

Identically with the functions §6. (1) we can deduce that the functions system

$$F^i(k) = F^i(q_k, p_k, q_{k+1}), \qquad G_i(k) = G_i(q_k, p_k, p_{k+1}) \tag{44}$$

is selfadjoint iff

$$\frac{\partial F^i(k)}{\partial p_j(k)} = \frac{\partial F^j(k)}{\partial p_i(k)}, \qquad \frac{\partial G_i(k)}{\partial q_k^j} = \frac{\partial G_j(k)}{\partial q_k^i}, \tag{45}$$

$$\frac{\partial F^i(k)}{\partial q_k^j} + \frac{\partial F^i(k-1)}{\partial q_k^j} + \frac{\partial G_j(k)}{\partial p_i(k)} + \frac{\partial G_j(k-1)}{\partial p_i(k)} = 0$$

$i, j = \overline{1,n}, \forall k$.

Proposition 9. *The system* a).

$$F^i(k) = q_{k+1}^i - f^i(k) \tag{46}$$

b).

$$G_i(k) = p_i(k+1) - g_i(k)$$

where $f^i(k) = f^i(q_k)$ *is selfadjoint iff*

$$g_i(k) = \left(2\delta_i^j - \frac{\partial f^j(k)}{\partial q_k^i}\right) p_j(k) + \frac{\partial \psi(k)}{\partial q_k^j} \tag{47}$$

where $\psi(k) = \psi(q_k)$.

It can be profed using §6. (12). The system §6. $(14)_{b).}$ is called the adjoint of §6. $(14)_{a).}$.

We can give some examples. For the logistic function

$$F(k) = q_{k+1} - aq_k^n(1 - q_k^r) \tag{48}$$

the adjoint function is

$$G(k) = p_{k+1} - [2 - naq_k^{n-1} - a(n+r)q_k^{n+r-1}]p_k + \frac{\partial \psi(k)}{\partial q_k} \tag{49}$$

For the functions system

$$F^i(k) = q^i_{k+1} - a^i_j q^j_k \tag{50}$$

the adjoint functions are

$$G_i(k) = p_i(k+1) - (2\delta^j_i - a^j_i)p_j(k) + \frac{\partial \psi}{\partial q^i_k} \tag{51}$$

Examples

1. Let be the function $\mathcal{F}(k) = aq_{k-1} + bq_{k+1} + f(q_k)$. The selfadjoint condition §3. (7) leads to $a = b$. Let be $p_k = q_k - q_{k-1}$. We replace it in $\mathcal{F}(k)$ and obtain

$$F(k) = q_k - q_{k-1} - p_k \tag{52}$$

$$G(k) = bp_{k+1} - ap_k + (a+b)q_k + F(q_k)$$

The selfadjoint condition §6. (12) for the new functions system leads us to $a = b$. These two conditions in this case are equivalent.

2. Let be the functions system

$$\mathcal{F}_i(k) = A_{ij}q^j_{k+1} - f_i(q_k, q_{k-1}) \qquad i = \overline{1,n} \tag{53}$$

with $A_{ij} = A_{ji}$, $det(A_{ij}) \neq 0$. The system §6. (14) is selfadjoint iff

$$f_i(k) = -A_{ij}q^j_{k-1} + \frac{\partial \psi(k)}{\partial q^j_k} \tag{54}$$

From §6. (14) and (15) we obtain

$$\mathcal{F}_i(k) = A_{ij}q^j_{k+1} + A_{ij}q^j_{k-1} - \frac{\partial \psi(k)}{\partial q^i_k} \tag{55}$$

Let be $p_i(k) = A_{ij}(q^j_{k-1} - q^j_k)$. We replace it in $F_i(k)$ and obtain the functions system

$$F^i(k) = q^i_k - q^i_{k-1} + \tilde{A}^{ij}p_j(k) \tag{56}$$

$$G_i(k) = p_i(k+1) - p_i(k) - 2A_{ij}q^j_k + \frac{\partial \psi(k)}{\partial q^i_k}$$

The last functions system is selfadjoint.

Conclusions

All the principal results of this paper have been discussed throughout the text.

We studied the general inverse problem of a difference equation system. We established the selfadjoint conditions and deduced the discrete Helmholtz conditions. Finally we wrote the discrete Hamilton equations used in numerical schemes and gave some examples.

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