Connections on Embedded Manifolds

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Abstract

In this paper we consider classical notions of differential geometry, such as Clifford and spinor bundles, together with their connections, from the point of view of their embeddings in (pseudo-)Euclidean space. It turns out that the use of an embedding allows for fairly simple expression of these notions.

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1 Introduction

Historically speaking, the notion of manifold originally meant what we now would call *a submanifold of Euclidean space*, and, more specifically, a surface in three-dimensional space. It is in this context that most notions (such as tangent vectors, curvature) were originally defined, and where they have a meaning which is fairly easy to grasp. Much more recently, it was shown by the famous theorems of Nash and Clarke (see [7] and [2]) respectively, that up to a minor continuity condition, any metric manifold can be embedded isometrically in a (pseudo-)Euclidean space of sufficiently big dimension. We can then embed the tangent and Clifford bundles in the corresponding entities of the embedding space, which makes it possible to describe connections in a quite natural way.

In this text we then consider manifolds whose Spin structure can itself be embedded in the Spin structure of the embedding space (it seems to be an open question whether any Spin manifold allows for such an embedding). Again, the spinor bundle, and the relevant connections obtain an easy form.

While this approach may not appeal to some because of the introduction of superfluous entities connected with the embedding space, the method has the advantage of being quite transparent. As an example, it turned out to be very easy to determine the eigen sections, rather than only the spectrum, of the spinor Dirac operators for the sphere and for Poincaré space, as will be shown in a forthcoming article ([3]). In the rest of this section we give the necessary conventions and notations used throughout this paper.

Orthogonal spaces and Clifford algebras. An *orthogonal space* V is a (finite dimensional) vector space with a symmetric scalar product. With $\mathbf{R}^{p,q}$ is meant the

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n-dimensional vector space over $\mathbf{R}$, where $n = p + q$, which consists of all n-tuples of real numbers $x = (x_1, \ldots, x_n)$ with the scalar product

$$\mathcal{B}(x, y) = -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{n} x_i y_i. \tag{1}$$

The *Clifford algebra over* V is denoted by $\mathcal{C}\ell(V)$. It is the free algebra generated by V modulo the relation

$$\vec{x}^2 = -\mathcal{B}(\vec{x}, \vec{x}). \tag{2}$$

This implies (by the polarisation formula) that orthogonal vectors anticommute, while parallel vectors commute.

For the Clifford algebra over the standard orthogonal space $\mathbf{R}^{p,q}$ we shall use the short notation $\mathcal{C}\ell_{p,q}$ instead of $\mathcal{C}\ell(\mathbf{R}^{p,q})$. Taking the standard basis $e_1, \ldots, e_n$ of $\mathbf{R}^{p,q}$ we see that the basis vectors anticommute. Actually equation (2) is equivalent to the relation

$$e_i e_j + e_j e_i = -2\mathcal{B}(e_i, e_j). \tag{3}$$

A basis of $\mathcal{C}\ell_{p,q}$ is given by ordered products of different basis vectors. We shall use two shorthand notations for such a product:

$$e_{t_1} \ldots e_{t_k} = e_{t_1 \ldots t_k} = e_{\{t_1, \ldots, t_k\}}, \qquad t_1 < \ldots < t_k.$$

Since all products of generators can be written (up to sign) in the form e_A, where A is a subset of $\{1, \ldots, n\}$, these e_A's form a basis of the Clifford algebra, and the dimension of $\mathcal{C}\ell_{p,q}$ over $\mathbf{R}$ is equal to 2^n. The set $\{1, \ldots, n\}$ will in the sequel be written as $\mathbf{n}$ for short.

Elements of $\mathcal{C}\ell_{p,q}$ are called Clifford numbers. Three (anti)automorphisms will be used (here $\vec{x}$ is an arbitrary vector, and a and b are arbitrary Clifford numbers):

(i) The main antiautomorphism is defined by

$$\bar{\vec{x}} = -\vec{x} \qquad \overline{(ab)} = \bar{b}\bar{a}.$$

(ii) The reversion is defined by

$$\vec{x}^* = \vec{x} \qquad (ab)^* = b^* a^*.$$

(iii) The main automorphism is defined by

$$\vec{x}' = -\vec{x} \qquad (ab)' = a'b'.$$

These (anti)morphisms will mainly be applied to products of vectors. Explicitly one obtains for a product of k vectors:

$$\begin{aligned}
\overline{(\vec{x}_1 \ldots \vec{x}_k)} &= (-\vec{x}_k) \ldots (-\vec{x}_1) \\
(\vec{x}_1 \ldots \vec{x}_k)^* &= \vec{x}_k \ldots \vec{x}_1 \\
(\vec{x}_1 \ldots \vec{x}_k)' &= (-\vec{x}_1) \ldots (-\vec{x}_k).
\end{aligned}$$

If A has k elements, then e_A is called a *k-vector*. Likewise any linear combination of k-vectors is called a k-vector, and the vector space of k-vectors is written as $\mathcal{C}\ell^k_{p,q}$; a k-vector which can be written as the product of k vectors is called a k-blade. In less than four dimensions every k-vector is a k-blade, but for $n \geq 4$, $e_1e_2 + e_3e_4$ is not a k-blade.

Obviously $\mathcal{C}\ell_{p,q}$ is the direct sum of all $\mathcal{C}\ell^k_{p,q}$ for $k \leq n$, and the projection of a Clifford number a onto $\mathcal{C}\ell^k_{p,q}$ will be written as $[a]_k$. Instead of 1-vectors the term vectors is used, and vectors are identified with elements of $\mathbf{R}^{p,q}$. Also the term bivectors is used for 2-vectors, while the elements of $\mathcal{C}\ell^n_{p,q}$ are called *pseudo-scalars*. The n-vector $e_{\mathbf{n}}$ is called the *pseudo-unit*. Notice that the product of the n basis vectors in a different order always is $e_{\mathbf{n}}$, at least up to sign: actually $e_{\mathbf{n}}$ defines an orientation of $\mathbf{R}^{p,q}$. The wedge product on the Clifford algebra is defined by

$$\vec{x} \wedge \vec{y} = \vec{x}\vec{y} + \mathcal{B}(\vec{x}, \vec{y}),$$

and can be extended using associativity. In general, for two Clifford numbers a and b, $[a]_k \wedge [b]_\ell = [ab]_{k+\ell}$. In a similar way there is a dot product defined by

$$[a]_k \cdot [b]_\ell = \begin{cases} [ab]_{|k-\ell|} & \text{if } k \text{ and } \ell \text{ are strictly positive} \\ 0 & \text{if } k\ell = 0. \end{cases}$$

Let $a = [\vec{x}_1 \dots \vec{x}_k]_k$ be a k-blade. Then a is invertible (i.e. $\bar{a}a \neq 0$; for any k-blade b we have that $\bar{b}b$ is real) if and only if $V = \mathrm{span}\{x_1, \dots, x_k\}$ is a k-dimensional vector space, such that the restriction of $\mathcal{B}$ to V is non-degenerate. The orthogonal projection of a Clifford number c onto $\mathcal{C}\ell(V)$ is given by $a^{-1}(a \cdot c) + [c]_0$. The convention $a \cdot [c]_0 = 0$ is quite unpractical for our purposes, but it is quite general.

Spin and Pin groups. For any a in the Clifford algebra such that $aa' = \pm 1$ it is clear that a'^{-1} exists, and so the the definition of the linear transformation on the Clifford algebra

$$\begin{aligned} \chi(a) : \mathcal{C}\ell_{p,q} &\rightarrow \mathcal{C}\ell_{p,q} \\ \vec{x} &\rightarrow \chi(a)\vec{x} = a\vec{x}a'^{-1} \end{aligned}$$

is well defined. The Pin group is now defined as the group of such elements which leave the underlying vector space invariant:

$$\mathrm{Pin}(p,q) = \{a \in \mathcal{C}\ell_{p,q} : aa' = \pm 1 \text{ and } \forall \vec{x} \in \mathbf{R}^{p,q} : \chi(a)\vec{x} \in \mathbf{R}^{p,q}\}.$$

χ obviously provides a group morphism of $\mathrm{Pin}(p,q)$ to a subgroup of $GL(\mathbf{R}^{p,q})$. It can be proved that, for every element a of the Pin group, $\chi(a)$ (when restricted to $\mathbf{R}^{p,q}$), is an element of the orthogonal group $O(p,q)$. Moreover $\mathrm{Pin}(p,q)$ is a double covering of $O(p,q)$, the kernel of $\chi : \mathrm{Pin}(p,q) \rightarrow O(p,q)$ being $\{-1, 1\}$, and $\mathrm{Pin}(p,q)$ is generated as a group by the vectors $\vec{x}$ for which $\vec{x}\vec{x}' = \pm 1$.

In a similar way we have the Spin group, $\mathrm{Spin}(p,q)$ which consists of the products of even numbers of unit vectors and as such gives a double covering of $SO(p,q)$.

The Lie algebra of the Pin group (which is also the Lie algebra of the Spin group, and hence isomorphic to $so(p,q)$) is the space $\mathcal{C}\ell^2_{p,q}$ of bivectors, with the Lie bracket $[a,b] = ab - ba$. Starting from the representation χ of $\mathrm{Pin}(p,q)$ on the Clifford algebra, we arrive at the derived representation $d\chi$ given by

$$d\chi(b)a = ba - ab,$$

for an arbitrary bivector b and a Clifford number a.

Submanifolds. Let M be an m-dimensional submanifold of $\mathbf{R}^{p,q}$, let a be an arbitrary point of M. There is a natural identification of an element of the tangent space T_aM with a vector of $\mathbf{R}^{p,q}$ as follows:any $X \in T_aM$ will be identified with $(a, \vec{x}) \in M \times \mathbf{R}^{p,q}$ if and only if, for any C^∞ function f on M, and any C^∞ extension F of f in a neighbourhood (in $\mathbf{R}^{p,q}$) of a we have that

$$D_X f = \partial_t F(a + t\vec{x})|_{t=0}. \tag{4}$$

In the sequel we write $(a, \vec{x})$, or simply $\vec{x}$ instead of X; any manipulation (multiplication, derivation etc.) is implied to act on the second entry. So, for example $(a, \vec{x}) + (a, \vec{y}) = (a, \vec{x} + \vec{y})$. Formally, the tangent bundle is embedded this way in $\mathbf{R}^{p,q} \times \mathbf{R}^{p,q}$; less formally of course the tangent space T_aM is considered as a subspace of $\mathbf{R}^{p,q}$.

A metric manifold (M, g) with signature (r, s) (the metric g can be Riemannian or pseudo-Riemannian) is called isometrically embedded if and only if

$$g((a, \vec{x}), (a, \vec{y})) = \mathcal{B}(\vec{x}, \vec{y}),$$

where $\mathcal{B}(\cdot, \cdot)$ is the scalar product of $\mathbf{R}^{p,q}$. For an isometrically embedded manifold, the *Clifford bundle* can be embedded in $\mathbf{R}^{p,q} \times \mathcal{C}\ell_{p,q}$ in a way quite similar to the one used for the tangent bundle. Considering T_aM as a subspace of $\mathbf{R}^{p,q}$, we see that, because g is non-degenerate, the restriction of $\mathcal{B}$ to T_aM is also non-degenerate. Hence it makes sense to construct the Clifford algebra $\mathcal{C}\ell(T_aM)$, which naturally is a subalgebra of $\mathcal{C}\ell_{p,q}$. The Clifford bundle, denoted by $\mathcal{C}\ell(M)$ is the submanifold of $\mathbf{R}^{p,q} \times \mathcal{C}\ell_{p,q}$ consisting of those elements (a, b) for which $a \in M$ and $b \in \mathcal{C}\ell(T_aM)$.

A section of the Clifford bundle is a C^∞, Clifford algebra valued function f on M, such that for each $a \in M$, $(a, f(a))$ is in the Clifford bundle $\mathcal{C}\ell(M)$. Such a section will also be called a (tangent) Clifford field. Because of the identification of the Clifford algebra with the exterior algebra, a tangent Clifford field can be identified with a differential form.

In each Clifford algebra $\mathcal{C}\ell(T_aM)$, there are two candidate pseudo-units. The choice between them defines the orientation of T_aM. We say that M is orientable if such a choice can be made in a continuous way. More formally, M is orientable if there is a Clifford field e_M, which in each point is a unit m-vector (i.e. $\overline{e_M(a)}e_M(a) = \pm 1$). If M is orientable, there are two of these fields. After choosing one, M is oriented, and e_M is called the *pseudo-unit field.* A tangent Clifford field can be characterised by the fact that

$$f = e_M^{-1}(_M \cdot f) + [f]_0.$$

Notice that a vector valued tangent Clifford field can be identified with a section of the tangent bundle.

2 Exterior derivatives and curvature

Let f be a tangent Clifford field. Since, with the embedding above, f is a function with values in a vector space (the Clifford algebra $\mathcal{C}\ell_{p,q}$) it is possible to take derivatives of

f in the classical sense. It is not necessarily true that $D_X f$ is a tangent Clifford field. Hence, for these fields, we introduce the *exterior derivative* $\nabla_X f$ by

$$\nabla_X f(a) = P_a D_X f(a), \tag{5}$$

where P_a is the projection operator onto the Clifford algebra generated by $S_a M$. Notice that

$$\nabla_X [f(a)]_k = e_M^{-1}(a)(e_M(a) \cdot [D_X f(a)]_k),$$

for $k > 0$, while $\nabla_X [f(a)]_0 = [D_X f(a)]_0$. Notice that derivation, and hence also exterior derivation preserves homogeneity: $[D_X f(a)]_k = D_X [f(a)]_k$, and also $[\nabla_X f(a)]_k = \nabla_X [f(a)]_k$. We shall shortly prove that ∇ is the classical torsion-free Koszul connection on the Clifford bundle. Before that we introduce the notion of parallel transport. Let f be a tangent field defined on a curve γ, with image going from x to y on M. If $\nabla_{\partial_t \gamma} f = 0$, then it is said that $f(x)$ is *parallel transported* to $f(y)$.

Some caution is needed as the projection operator is not distributive with respect to Clifford multiplication: in general it is not true that $P_a(\lambda\mu) = P_a(\lambda)P_a(\mu)$ (this is e.g. false when $\lambda = \mu$ is a non-isotropic vector orthogonal to $S_a M$). It is true however that

$$P_a(\lambda\mu) = P_a(\lambda)P_a(\mu) \tag{6}$$

if either λ or μ are tangent to $S_a M$.

2.1 The relation between D and ∇

The change of the pseudo-unit field gives a measure for the curvature of the manifold, i.e. the way in which M locally is different from an m-dimensional pseudo-Euclidean space. Since the pseudonorm $e_M(x)\overline{e_M(x)} = (-1)^p$ is constant, for any tangent vector $\vec{x}$, $D_{\vec{x}} e_M$ is orthogonal to e_M itself, and $\nabla_{\vec{x}} e_M = 0$. Moreover, the pseudo-unit always is an element of the Pin group.

The derivatives of the pseudo-unit are important enough to merit a separate notation. We define b_X by

$$D_X e_M = b_X e_M(x), \tag{7}$$

and, if a coordinate system is given, b_i by $D_i e_M(x) = b_i(x) e_M(x)$. Since the pseudo-unit always is an element of the Pin group, b_X and b_i are elements of the Lie algebra of the Pin group, i.e. they are bivector valued functions on M. Notice however that they are not tangent Clifford fields, unless they are zero.

Moreover $\eta = e_M^2 = \pm 1$ is constant, and so $0 = D_X \eta = b_X e_M^2 + e_M b_X e_M$, which implies that $b_X e_M = -e_M b_X$. This is only possible if b_X is the product of a vector orthogonal to e_M and a vector parallel to it, or a sum of such products. As a consequence, if $\vec{v} \in T_x M$, then $d\chi(b_X)\vec{v}$ is orthogonal to e_M. This follows in a fairly straightforward way from the rule $\vec{x}\vec{y}\vec{z} - \vec{z}\vec{x}\vec{y} = -\mathcal{B}(\vec{y}, \vec{z})\vec{x} + \mathcal{B}(\vec{x}, \vec{z})\vec{y}$.

The bivector b_X leads to an efficient description of the difference between D_X and ∇_X. We start with a tangent vector field $\vec{f}$. $D_X \vec{f}$ can be split into a part $(D_X \vec{f})_\parallel$ tangent to M and a part $(D_X \vec{f})_\perp$ orthogonal to M, both vector valued. Notice that $\vec{f} e_M + (-1)^m e_M \vec{f} = 0$, $(D_X \vec{f})_\parallel e_M + (-1)^m e_M (D_X \vec{f})_\parallel = 0$ and

$(D_X\vec{f})_\perp e_M + (-1)^m e_M (D_X\vec{f})_\perp = 2(D_X\vec{f})_\perp e_M$. Taking the derivative of the first equation gives

$$\begin{aligned} 0 &= (D_X\vec{f})e_M + \vec{f}b_X e_M + (-1)^m b_X e_M \vec{f} + (-1)^m e_M D_X \vec{f} \\ &= 2(D_X\vec{f})_\perp e_M - d\chi(b_X)f. \end{aligned}$$

Since $\nabla_X\vec{f} = D_X\vec{f} - (D_X\vec{f})_\perp$ this results in $\nabla_X\vec{f} = D_X\vec{f} - d\chi\left(\frac{b_X}{2}\right)\vec{f}$. From the product rule (6) it then follows that this relation holds for any Clifford valued tangent field, and we can use it to define ∇ for general $\mathcal{C}\ell_{p,q}$ valued functions

$$\nabla_X f = D_X f - d\chi\left(\frac{b_X}{2}\right) f. \tag{8}$$

Notice that the fact that $d\chi(b_X)\vec{f}$ is orthogonal to e_M provides an independent proof that b_X is the product of an orthogonal and a tangent vector, or a sum of such products.

2.2 The curvature tensor

The *curvature tensor* is given, for two parallel vector fields, by

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

The geometrical interpretation is the following: let f be a tangent field, and take a closed loop γ with size ϵ in the X and Y directions (notice the presence of the term in $[X,Y]$, which assures that the loop will be closed), starting in x. $f(x)$, parallel transported along γ will result in a new tangent vector, say $\Gamma f(x)$, and the difference between $f(x)$ and $\Gamma f(x)$ will be $\epsilon R(X,Y)f(x)$, up to higher order terms in ϵ. Obviously Γ is an orthogonal transformation, and as a differential of orthogonal transformations $R(X,Y)$ must be an antisymmetric transformation. We prove this more formally in terms of the derivatives of the pseudo-unit field.

Theorem 2.1. *The curvature is given by*

$$R(X,Y) = d\chi(b_X b_Y - b_Y b_X),$$

where $b_X b_Y - b_Y b_X$ is a tangent bivector.

Proof. The proof consists simply of inserting (8) into the definition of $R(X,Y)f$, and then simplifying. First calculate

$$\nabla_{[X,Y]}f = D_X D_Y f - D_Y D_X f - (1/2)d\chi(b_{[X,Y]})f,$$

where

$$\begin{aligned} b_{[X,Y]}e_M &= (D_X D_Y - D_Y D_X)e_M \\ &= D_X(b_Y e_M) - D_Y(b_X e_M) \\ &= (D_X b_Y)e_M + b_Y b_X e_M - (D_Y b_X)e_M + b_X b_Y e_M. \end{aligned}$$

Then subtract this from

$$\begin{aligned}\nabla_X \nabla_Y f &= D_X D_Y f - (1/2) d\chi(b_X) D_Y f \\ &\qquad -(1/2) D_X (d\chi(b_Y) f) + (1/4) d\chi(b_X) d\chi(b_Y) f \\ &= D_X D_Y f - (1/2) d\chi(b_X) D_Y f - (1/2) d\chi(D_X b_Y) f \\ &\qquad -(1/2) d\chi(b_Y) D_X f + (1/4) d\chi(b_X) d\chi(b_Y) f,\end{aligned}$$

and subtract the similar equation with X and Y interchanged. This gives

$$\begin{aligned}R(X,Y)f &= \frac{1}{2}(d\chi(D_Y b_X) f - d\chi(D_X b_Y) f + d\chi(b_{[X,Y]}) f \\ &\quad + \frac{1}{4}(d\chi(b_X) d\chi(b_Y) - d\chi(b_X) d\chi(b_Y)) f \\ &= -\frac{1}{4}(d\chi(b_X) d\chi(b_Y) - d\chi(b_Y) d\chi(b_X)) f \\ &= -\frac{1}{4}(d\chi(b_X b_Y - b_Y b_X)) f,\end{aligned}$$

where the last line follows from the relation $d\chi(a)d\chi(b) - d\chi(b)d\chi(a) = d\chi(ab - ba)$ which is valid for all bivectors a and b, as is easily proved by writing out $d\chi$ in multiplication form, or alternatively can be seen using the fact that $d\chi$ is a derived representation and therefore preserves the bracket. It is obvious from the definition that if f is tangent then $R(X,Y)f$ must be tangent, and so $b_X b_Y - b_Y b_X$ is itself tangent. ∎

Finally we prove that the connection defined above is the classical Levi-Civita connection (i.e. that connection which parallel transports vectors to vectors with the same norm, and which is torsion-free) on the tangent bundle, and therefore independent of the embedding chosen. Notice that for a vector field $\vec{f}$ and a curve γ, the equation $\nabla_{\partial_t \gamma} \vec{f} = 0$ implies $\partial_{\partial_t \gamma} \vec{f}$ is orthogonal to $\vec{f}$, and hence that the norm of $\vec{f}$ is constant, and we only have to prove that ∇ is torsion-free. Recall that the torsion of a connection is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{9}$$

Assume now local coordinates are given in an open set U of M. We can assume that there is a chart $\psi : V \to W$, where V and W are open in $\mathbf{R}^{p,q}$, $U = M \cap W$, $\psi(V \cap \mathbf{R}^{r,s} s) = U$, and that ψ is non-singular on V, i.e. that the chart can be extended to a neighbourhood of V in $\mathbf{R}^{p,q}$. We then have the coordinate vector fields $E_i f = \partial_i (\psi^{-1} \circ f)$, which, according to the embedding (4) can be identified in a point a with elements of $\mathbf{R}^{p,q} \times \mathbf{R}^{p,q}$ by

$$E_i(a) = (a, \partial_i \psi).$$

Proving that $T = 0$, is equivalent with proving that $T(E_i, E_j) = 0$ for any i and j. But since they are coordinate vector fields $[E_i, E_j] = 0$, and according to the definition of ∇ we have that $\nabla_i E_j = P(\partial_i \partial_j \psi)$, where P is orthogonal projection onto the tangent space. Since $\partial_i \partial_j \psi = \partial_j \partial_i \psi$, this proves that $\nabla_i E_j = \nabla_j E_i$, and so the exterior derivative is torsion-free.

3 Embedded Spin structures

For the Clifford bundle it was sufficient to have an isometric embedding in some (pseudo-)Euclidean space. If we want to introduce Spin structures however, we shall need a much stronger condition.

Definition 3.1. A set of C^∞ vector valued functions $\vec{n}_1, \ldots, \vec{n}_{n-m}$ on M is called a *global trivialisation* of the normal bundle if and only if in every point a of M and for every i and j

(i) $\vec{n}_i(a)$ is orthogonal to T_aM

(ii) $\vec{n}_i(a) \cdot \vec{n}_j(a) = \pm\delta_{ij}$.

The term global trivialisation reminds of the fact that the vector fields $\vec{n}_1, \ldots, \vec{n}_{n-m}$ defines a continuously varying basis of the orthogonal complement of T_aM in $\mathbf{R}^{p,q}$. We assume that M (or rather the embedding of M in $\mathbf{R}^{p,q}$ allows a global trivialisation of the normal bundle. We shall choose a reference point $\vec{N}$ on M, and identify the space $\mathbf{R}^{r,s}$ (recall that (r,s) is the signature of the metric on M with $T_{\vec{N}}M$. Then we can define a Spin structure on M.

Definition 3.2. A *Spin structure* Σ on M is a submanifold of $\mathbf{R}^{p,q} \times \mathrm{Spin}(p,q)$ defined by the following condition: a point $(a,\sigma) \in \mathbf{R}^{p,q} \times \mathrm{Spin}(p,q)$ is in Σ if and only if

(i) $a \in M$.

(ii) The orthogonal tranformation $\chi(\sigma)$ maps T_aM onto $T_{\vec{N}}M$, and moreover maps each $\vec{n}_i(a)$ to $\vec{n}_i(\vec{N})$.

Notice that there is a left action of $\mathrm{Spin}(r,s)$ on Σ. Indeed, if τ is an element of $\mathrm{Spin}(r,s)$, and (a,σ) is in Σ, then clearly $(a,\tau\sigma) \in \Sigma$. Moreover each fibre $\Sigma_a = \{(a,\sigma) \in \Sigma\}$ is equivalent to $\mathrm{Spin}(r,s)$, since $(a,\sigma) \in \Sigma$ and $(a,\mu) \in \Sigma$ implies that there is a $\tau \in \mathrm{Spin}(r,s)$ such that $\mu = \tau\sigma$. It turns out that Σ is a so-called pricipal $\mathrm{Spin}(r,s)$-bundle over M (see e.g. [5] for more information on this notion).

It is immediately clear that every orientable hypersurface (i.e. a submanifold where $n-m = 1$) has a Spin structure, since the orientation can be equalled to the choice of a normal vector. There is a general definition of Spin structures on manifolds, which does not make use of the embedding. Not every manifold has a Spin structure in this sense; a manifold which has is called a *Spin manifold*. It seems not to be known whether each Spin manifold allows for an isometric embedding with a global trivialisation, it is however certain that a Spin manifold can have an isometric embedding which does not allow for an embedding of the Spin structure.

As an example we have the circle S^1. With its traditional embedding in $\mathbf{R}^2$, it is a hypersurface, and hence has the obvious Spin structure. There is however a second Spin structure which cannot be realised in $\mathbf{R}^2$, but which can be realised in $\mathbf{R}^3$. We choose an orthonormal basis e_1, e_2, e_3 of $\mathbf{R}^3$, and the parametrisation of S^1 given by $\theta(\xi)$, where $\xi = \cos\theta e_1 + \sin\theta e_2$, and $-\pi < \theta \leq \pi$. Notice that $\mathrm{Spin}(1) = \{1, -1\}$. The reference point $\vec{N}$ is chosen to be e_1, so $\theta(\vec{N}) = 0$, the following picture is obtained:

(1) The Spin structure of S^1 as a hypersurface is given by $\vec{n}_1(\xi) = \xi$ (and, in $\mathbf{R}^3$, $\vec{n}_2(\xi) = e_3$). Then $\vec{n}_1(\vec{N}) = e_1$ and the Spin fibre in an arbitrary point ξ is given by

$$\Sigma_\xi = \left\{ \pm \left(\cos\frac{\theta(\xi)}{2} - e_{12}\sin\frac{\theta(\xi)}{2} \right) \right\}.$$

Notice that this Spin structure is connected: going round the circle once changes the sign of the Spin element.

(2) A second Spin structure given by $\vec{n}_1(\xi) = \xi\cos\theta + e_3\sin\theta$ and $\vec{n}_2(\xi) = -\xi\sin\theta + e_3\cos\theta$. Then $\vec{n}_1(\vec{N}) = e_1$ and $\vec{n}_2(\vec{N}) = e_3$. The Spin fibres in this case are given by

$$\Sigma_\xi = \left\{ \pm \left(\cos\frac{\theta}{2} - e_{13}\sin\frac{\theta}{2} \right) \left(\cos\frac{\theta}{2} - e_{12}\sin\frac{\theta}{2} \right) \right\}.$$

This Spin structure is not connected: it is homeomorphic to $S^1 \times \mathrm{Spin}(1)$.

Using the Spin structure we can give an important alternative characterisation of the Clifford fibres: (a, b) is an element of the Clifford fibre $\mathcal{C}\ell_a$ if and only if $\sigma b\bar{\sigma} \in \mathcal{C}\ell_{\vec{N}} = \mathcal{C}\ell_{r,s}$ for any (a, σ) in Σ_a. As the mapping $\chi(\sigma)$ is a direct orthogonal transformation on $\mathbf{R}^{p,q}$, and moreover the vectors $\vec{n}_i(a)$ have the same orientation as their image under $\chi(\sigma)$, $\vec{n}_i(\vec{N})$, this means that the mapping $\chi(\sigma)$ restricted to T_aM is a *direct* orthogonal transformation onto $T_{\vec{N}}M$, and so that $\chi(\sigma)e_M(a) = e_M(\vec{N})$.

4 Spinor bundles

Assume that M has a Spin structure Σ, with reference point $\vec{N}$. We then can define an embedded spinor bundle $\mathcal{S}$ in $\mathbf{R}^{p,q} \times \mathcal{C}\ell_{p,q}$ as the submanifold consisting of pairs (a, ψ) for which

(i) $a \in M$.

(ii) For any $(a, \sigma) \in \Sigma_a$, $\sigma\psi \in \mathcal{C}\ell_{r,s}$.

As usual, $\mathcal{S}_a = \{(a, \sigma) \in \mathcal{S}\}$ is called the *spinor fibre* in a, and *spinor sections* are defined in a way similar to that for sections of the Clifford bundles. Spinor sections will also be called *spinor fields.*

It is classical to define the spinor bundle using an irreducible representation of the Clifford algebra $\mathcal{C}\ell_{r,s}$. Let A be such a representation. In the customary definition, condition (ii) has the form $\sigma\psi \in A$. However, it is known that A is isomorphic to a minimal left ideal of $\mathcal{C}\ell_{r,s}$, so that our definition encompasses the classical one. The minimal left ideal can be written in the form $\mathcal{C}\ell_{r,s}J$, where J is a primitive idempotent. When one wants to use irreducible representations of $\mathcal{C}\ell_{r,s}$, one can restrict the attention to spinor fields (as we have defined them) which satisfy the extra condition $\psi J = \psi$.

Notice that, in a certain sense, the Clifford algebra $\mathcal{C}\ell_a$ is in the trace class of the spinor fibre. Indeed, $(a, b) \in \mathcal{C}\ell_a$ if and only if $b = \psi\bar{\phi}$ for some (a, ψ) and (a, ϕ) both in $\mathcal{S}_a$. Again we can define a connection for spinor sections. Let ψ be a spinor section and X be a tangent vector in T_aM. Then we define the connection $\not\nabla$ by

$$\not\nabla_X\psi = S_a(\partial_X\psi),$$

where S_a is orthogonal projection onto the spinor fibre $\mathcal{S}_a$. Again the bivector valued function $X \to b_X$ plays an important rôle in a more explicit expression for the connection. However, there is a certain arbitrariness in the trivialisation of the normal bundle, unless we are dealing with a hypersurface.

Theorem 4.1. *The connection $\nabla\!\!\!/$ is given by*

$$\nabla\!\!\!/_X \psi = \partial_X \psi - \frac{1}{2}(b_X + e_X)\psi.$$

Here b_X is the bivector given by (7), and e_X is a bivector in the Clifford algebra generated by the normal fibre, i.e. in the Clifford algebra generated by $\vec{n}_1, \ldots, \vec{n}_{m-n}$.

Proof. Fix a point a, and take a section σ of the Spin bundle in a neighbourhood of a. Then $\sigma\psi$ is a $\mathcal{C}\ell_{r,s}$ valued function near a, and

$$D_X(\sigma\psi) = (D_X\sigma)\psi + \sigma(D_X\psi).$$

This is also $\mathcal{C}\ell_{r,s}$ valued, and so $\sigma^{-1}D_X(\sigma\psi)$ is a local spinor section, i.e. its part orthogonal to the spinor fibre is zero:

$$0 = (\sigma^{-1}(D_X\sigma)\psi)_\perp + (D_X\psi)_\perp.$$

Now $(D_X\psi)_\perp = D_X\psi - \nabla\!\!\!/_X\psi$, and we shall prove that $(\sigma^{-1}(D_X\sigma)\psi)_\perp = (1/2)(b_X + e_X)\psi$, where e_X is fully orthogonal to the tangent space. First notice that $(\sigma^{-1}(D_X\sigma)\psi)_\perp = (\sigma^{-1}(D_X\sigma))_\perp\psi$, by (6). We shall put $\sigma^{-1}(D_X\sigma) = c_X$. This is a bivector, because σ is in the Spin group, and we split it up into

$$c_X = (c_X)_\| + (c_X)_t + \frac{1}{2}e_X,$$

where $(c_X)_\|$ is in the tangent Clifford algebra,$(c_X)_t$ is a sum of products of vectors, one parallel and one orthogonal to $\mathcal{C}\ell_a$, and e_X is fully orthogonal to $\mathcal{C}\ell_a$. We set out to prove that $(c_X)_t$ is $(1/2)b_X$. Notice that both $(c_X)_\|$ and e_X commute with the pseudo-unit field e_M, while $(c_X)_t$ anticommutes with it.

Starting point is that $\sigma e_M \sigma^{-1} = e_M(\vec{N})$ is constant. Hence

$$\begin{aligned} 0 &= (D_X\sigma)e_M\sigma^{-1} + \sigma(D_X e_M)\sigma^{-1} + \sigma e_M(D_X\sigma^{-1}) \\ &= \sigma c_X e_M \sigma^{-1} + \sigma(b_X e_M)\sigma^{-1} - \sigma e_M c_X \sigma^{-1} \\ &= \sigma\left[2(c_X)_t e_M + b_X e_M\right]\sigma^{-1}. \end{aligned}$$

This completes the proof. ∎

It can be easily seen that, while b_X is determined by e_M, c_X is completely determined by the trivialisation of the normal bundle, $\vec{n}_1, \ldots, \vec{n}_{n-m}$. Indeed, e_X is in the normal section, and for all i we have that $\sigma\vec{n}_i\sigma^{-1} = \vec{n}_i(\vec{N})$. Therefore the derivative is zero, and

$$\begin{aligned} 0 &= D_X(\sigma\vec{n}_i\sigma^{-1}) \\ &= \sigma c_X \vec{n}_i \sigma^{-1} + \sigma(D_X\vec{n}_i\sigma^{-1} - \sigma\vec{n}_i c_X \sigma^{-1}). \end{aligned}$$

Now $(c_X)_\|$ commutes with $\vec{n}_i$, and so, using the dot product notation,

$$D_X \vec{n}_i = (b_X + e_X) \cdot \vec{n}_i.$$

These equations are sufficient to determine e_X.

There is a certain arbitrariness in the trivialisation of the normal bundle, which is expressed by the extra term e_X in the expression of the connection, unless we are dealing with a hypersurface. In this case e_X must of necessity be zero, because the normal section is one-dimensional, and the expression for the connection becomes

$$\nabla\!\!\!/_X \psi = \partial_X \psi - \frac{1}{2} b_X \psi.$$

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References

[1] H. Blaine Lawson, M-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series 38, Princeton University Press, 1989.

[2] C. Clarke, *On the global isometric embedding of pseudo-Riemannian manifolds*, Proc. Roy. Soc. London, ser. A 314, (1970), 417-428.

[3] J. Cnops, *Dirac operators on Euclidean and Minkowski spheres*, in preparation.

[4] R. Delanghe, F. Sommen, V. Souček, *Clifford analysis and spinor valued functions*, Kluwer Acad. Publ., Dordrecht, 1992.

[5] J. Gilbert and M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge University Press, 1991.

[6] D. Hestenes and G. Sobczyk, *Clifford algebra to geometric calculus*, Fundamental theories in physics vol. 5, D. Reidel, Dordrecht, 1984.

[7] J. Nash, *The embedding problem for Riemannian manifolds*, Annals 63, (1956), 20-63.

[8] A. Trautman, *Spinors and the Dirac operator on hypersurfaces. I. General theory*, J. Math. Phys., 33/12, (1992), 4011-4019.

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