# Geometrical Objects on Subbundles

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**Abstract**

The reductions to a vector subbundle of a pull back vector bundle are studied. They are related to the Finsler splittings (defined earlier by one of the authors) and to geometrical objects, defined here.

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## 1 Reductions of vector bundles

Let $\xi = (E, \pi, M)$ be a vector bundle (denoted in the sequel v.b.), with the fibre $F \cong I\!R^n$, $G \subset GL_n(I\!R)$ a Lie subgroup and $\xi' = (E', \pi', M)$ be a vector subbundle (denoted in the sequel v.sb.), with the fibre $F' \cong I\!R^k \subset I\!R^n$. Let us denote as $L(\xi) = (L(E), p, M)$ the principal bundles (p.b.s) of the frames of the v.b. $\xi$ and the induced p.b. $L\xi'(\xi) = \pi'^* L(\xi) = (\pi'^* L(E) = LE'(E), p_1, E')$, which is also the p.b.s of the frames of the v.b. $\xi'(\xi) = \pi'^* \xi = (\pi'^*(E) = E'(E), \pi_1, E')$.

**Definition 1.1** If the p.b. $L(\xi)_G$ is a reduction of the group $GL_n(I\!R)$ of $L(\xi)$ to $G$, then there is a local trivial bundle $\xi_G$, associated with the p.b. $L(\xi)_G$, defined by the left action of $G$ on $F$ (it is used the left action of $GL_n(I\!R)$ on $F$ restricted to $G$). We say that the bundle $\xi_G$ is the *$G$-reduced bundle* of $\xi$. If $H$ is a subgroup of $G$ and there is a reduction of the group $G$ of $L(\xi)_G$ to $H$, in an analogous way we say that $\xi_H$ is a *$H$-reduced bundle* of $\xi_G$.

Notice that a reduction of the group $G$ of $L(\xi)_G$ to $H$ is also a reduction of the group $GL_n(I\!R)$ of $L(\xi)$ to $H$.

**Example 1.1** Consider the subgroup of the automorphisms which invariate the vector subspace $F' \cong I\!R^k$ of $I\!R^n$:

$$(1)\ G_0 = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}; A \in GL_k(I\!R),\ B \in GL_{n-k}(I\!R),\ C \in \mathrm{M}_{k,n-k}(I\!R) \right\} \subset GL_n(I\!R).$$

The p.b. $L(\xi)_{G_0}$ always exists and it consists of all the frames of $L(\xi)$ which extend frames on $\xi'$; we call in the sequel these frames as *frames on $\xi$, adapted to $\xi'$*. For the same $G_0$ as above, we can consider the p.b. $L\xi'(\xi)_{G_0}$, which also consists of frames on $L\xi'(\xi)$ which extend frames on $\xi'(\xi')$, called as frames on $\xi'(\xi)$, adapted to $\xi'(\xi')$.

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**Example 1.2** Let $G_0$ be as above, $F'' \cong I\!R^{n-k}$ a vector subspace of $F$, so that $F = F' \oplus F''$ and $H_0 \subset G_0$ the subgroup of the elements which invariate the vector subspaces $F'$ and $F''$:

(2) $$H_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ; A \in GL_k(I\!R),\ B \in GL_{n-k}(I\!R) \right\}.$$

The p.b. $L(\xi)_{H_0}$ always exists and it is a reduction of the group $G_0$ of $L(\xi)_{G_0}$ to $H_0$. It consists of frames of $L(\xi)_{G_0}$ which are adapted also to other subbundle $\xi'' = (E'', \pi'', M)$ of $\xi$. It follows that at every point $x \in M$ we have the direct sum of vector spaces $E_x = E'_x \oplus E''_x$. Such a reduction is also called *a Whitney sum* of the v.b.s $\xi'$ and $\xi''$ and it is denoted as $\xi' \oplus \xi''$. This is equivalent with a left splitting $S$ of the inclusion morphism $i : \xi' \to \xi$, when $\xi'' = \ker S$. In the case of the p.b. $L\xi'(\xi)$, a reduction of the group $G_0$ of $L\xi'(\xi)_{G_0}$ to $H_0$ is equivalent to a left splitting $S$ of the inclusion $i' = \pi'^{*} i : \pi'^{*}\xi' = \xi'(\xi') \to \pi'^{*}\xi = \xi'(\xi)$, as we have called *Finsler splitting* (see. [4]). In this case $\xi'(\xi)$ has an $H_0$-reduction as Whitney sum $\xi'(\xi') \oplus \ker S$.

It is well known that the reduction of the structural group $G$ of a p.b. $P$ to a subgroup $H \subset G$ is equivalent to the existence of a global section in a fibre bundle associated with $P$, which have the fibre $G/H$, defined by the natural action of $G$ on $G/H$ [1, pg.57, Propzition 5.6]. A direct computation leads to the following:

**Proposition 1.1** *There is a canonical identification*

(3) $$G_0/H_0 \cong \mathrm{M}_k = \left\{ \begin{pmatrix} 0 & P \\ 0 & I_{n-k} \end{pmatrix} ; P \in \mathrm{M}_{k,n-k}(I\!R) \right\},$$

*the classes being at left, such that the left action $\odot$ of the group $G_0$ on $\mathrm{M}_k$ is the adjunction, and*

(4) $$\begin{pmatrix} E & G \\ 0 & F \end{pmatrix} \odot \begin{pmatrix} 0 & P \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} 0 & (E \cdot P + G) \cdot F^{-1} \\ 0 & I_{n-k} \end{pmatrix}.$$

Given the v.sb. $\xi'$, the $G_0$-reductions of the group $GL_n(I\!R)$ of the v.b.s $L(\xi)$ and the p.b. $L\xi'(\xi)$ are uniquely defined. It follows that considering the bundles with the fibres $GL_n(I\!R)/G_0$, associated with the p.b.s of frames $L(\xi)$ and $L\xi'(\xi)$, the sections in these bundles, which correspond to the reductions of $GL_n(I\!R)$ to $G_0$, are uniquely determinated by $\xi'$. In the case of the Example 1.2, the $H_0$-reductions of the group $G_0$ of the p.b.s $L(\xi)_{G_0}$ and $L\xi'(\xi)_{G_0}$ are equivalent with sections in the bundles $F_1$ and $F_2$ which are associated with these p.b.s and have as fibres $G_0/H_0$.

## 2 Reductions of the group $G^r_{m,n}$

We use in this section some ideas from [2, Cap. IV, Secţiunea 7], but in a more general setting.

Let $\xi = (E, p, M)$ be a v.b., having the fibre $F \cong I\!R^n$, $\dim M = m$ and $G \subset GL_n(I\!R)$ a Lie subgroup. We suppose that the structural group $GL_n(I\!R)$ of $L(\xi)$ is reducible to $G$. Denote as $G^1_{m,n}$ the subgroup of $GL(m+n, I\!R)$ which consists of the matrices of the form $\begin{pmatrix} A^i_j & 0 \\ 0 & B^a_b \end{pmatrix}$, $\left(A^i_j\right) \in GL_m(I\!R)$, $(B^a_b) \in G$. Given $r \in I\!N^*$, the $r$-prolongation of the group $G^1_{n,m}$, denoted as $G^r_{n,m}$, is the set of the elements which have the form

$$a = (A^i_{j_1}, A^i_{j_1 j_2}, \dots, A^i_{j_1 j_2 \cdots j_r}; B^a_b, B^a_{bj_1}, \dots, B^a_{bj_1 \cdots j_{r-1}}), \tag{5}$$

where the components are symmetric in the indices $j_k \in \overline{1,m}$, and $\left(A^i_j\right) \in GL_m(I\!R)$, $(B^a_b) \in G$ and if we fix $j_1, j_2, \dots j_p$, then $B^a_{bj_1 \cdots j_p} \in g$, where $g$ is the Lie algebra of the Lie group $G$. The composition law of two elements of the form (5) can be done looking at the cmponents as multi-linear maps. Thus, let $a : (A, B) = ((A_1(\cdot), A_2(\cdot,\cdot), \dots, A_r(\cdot,\dots,\cdot),\ B_0(\cdot),, B_1(\cdot,\cdot), \dots, B_r(\cdot,\cdot,\dots,\cdot)))$, and $b : (C, D)$ be two elements in $G^r_{n,m}$. We denote $ba : (A', B')$, where the expresion of this composition law, using coordinates, can be found in [2, pag. 70]. Notice that if $H \subset G$ is a subgroup, then $H^r_{m,n} \subset G^r_{m,n}$ is a subgroup.

Consider the p.b. $\mathcal{O}G\xi^r$ on the base $E$, with the group $G^r_{n,m}$, defined by the structural functions

$$\varphi_{UU'}(u) = (\frac{\partial x^{i'}}{\partial x^i}(x), \dots, \frac{\partial^r x^{i'}}{\partial x^{i_1} \cdots \partial x^{i_r}}(x), g^{a'}_a(x), \dots, \frac{\partial^{r-1} g^{a'}_a}{\partial x^{j_1} \cdots \partial x^{j_{r-1}}}(x)), \tag{6}$$

where $\pi(u) = x$, and $\left\{g^{a'}_a(x)\right\}$ are structural functions of the vertical bundle $V\xi$, constant on the fibres, defined on an open cover of $E$ of the form $\{U = \pi^{-1}(V), V \subset M,\ V\ open\}$. These structural functions proceed from some structural function on $M$, so the definition is coherent and is equivalent to that used in [2]. We shall use the condition in this form in order to construct some reductions. In the case $G = GL_n(I\!R)$ we get the definitions used in [2].

Let $\xi'$ be a v.sb. of the v.b. $\xi$. We denote $\mathcal{O}G\xi'(\xi)^r = i^*\mathcal{O}G\xi^r$, where $i : E' \to E$ is the inclusion. From now to the end of the section we study the reductions of the group $G^r_{0m,n}$ of the p.b. $\mathcal{O}G_0\xi'(\xi)^r$ to the subgroup $H^r_{0m,n}$, where $G_0$ and $H_0$ are given by the formulas (1) and (2). We consider first the case $r = 1$.

**Proposition 2.1** *Let $\xi'$ be a v.sb. of the v.b. $\xi$. Then a Finsler splitting $S$ of the inclusion $i : \xi' \to \xi$ induces a $H_0$-reduction of the v.b. $\xi'(\xi)$ to a bundle $\xi'(\xi') \oplus \eta$, where $\eta = \ker S$ is isomorphic with $\xi'(\xi'')$, $\xi'' = \xi/\xi'$, such that the bundle $\xi'(\xi') \oplus \eta$ is isomorphic to the bundle $\xi'(\xi' \oplus \xi'')$. Conversely, every $H_0$-reduction of $\xi'(\xi)$ as $\xi'(\xi') \oplus \eta$ defines a Finsler splitting $S$ of the inclusion $i$, such that $\eta = \ker S$.*

**Proof.** The second statement follows from Example 1.2. In order to prove the first statement, it suffices to prove that $\eta$ is isomorphic with $\xi'(\xi'')$. Considering local coordinates, adapted to the v.b.s structures: on $M$, $E'$, $E''$ and $E$, it can be shown that $\eta$ is isomorphic with $\xi'(\xi'')$. The same reason shows that $\xi'(\xi') \oplus \eta$ is isomorphic with $\xi'(\xi') \oplus \xi'(\xi'') \cong \xi'(\xi' \oplus \xi'')$. Q.e.d.

**Theorem 2.1** *Let $\xi'$ be a v.sb. of the v.b. $\xi$ and $r \geq 1$*

*1) Every Finsler splitting of the inclusion $i : \xi' \to \xi$ defines a canonical reduction of the group $G^r_{0m,n}$ of $\mathcal{O}G_0\xi'(\xi)^r$ to $H^r_{0m,n}$, the reduced p.b. being $\mathcal{O}H_0\xi'(\xi)^r$.*

*2) Every reduction of the group $G^r_{0m,n}$ of $\mathcal{O}G_0\xi'(\xi)^r$ to $H^r_{0m,n}$ is $\mathcal{O}H_0\xi'(\xi)^r$ and it is induced by a Finsler splitting, as above.*

**Proof.** 1) Taking on $E'(E)$ a vectorial atlas, which has the structural functions from $H_0$, we obtain structural functions on the p.b. $\mathcal{O}G_0\xi'(\xi)^r$, which thake values in the subgroup $H^r_{0m,n}$. 2)Considering a reduction as in hytothesis and some structural functions on $\mathcal{O}H_0\xi'(\xi)^r$, as in the definition, it follows some structural functions on the p.b.s $\mathcal{O}H_0\xi'(\xi)^{r'}$, with $1 \leq t' \leq r$. Taking $r' = 1$ and using the second part and the proof of Propsition 2.1, we obtain a Finsler splitting $S$. Q.e.d.

## 3 Geometrical objects

Let $\xi$ be a v.b., $G \subset GL_n(\mathbb{R})$ a Lie subgroup and $\xi'$ a v.sb. of $\xi$.

**Definition 3.1** A *space of geometrical $G-$objects of order $r$* is a manifold $\Theta$ so that there is a left action of the group $G^r_{n,m}$ on $\Theta$. Consider now the fibre bundle with the fibre $\Theta$, associated with the p.b. $\mathcal{O}G\xi^r$, which correspond to this action. A section in this bundle is a *field of geometrical $G-$objects* on the v.b. $\xi$.

In an analogous way, we can consider the fibre bundle with the fibre $\Theta$ , associated with the principal bundle $\mathcal{O}G\xi^r$ A section in this bundle is a *field of geometrical $G-$objects* on the v.b. $\xi$, *restricted* to the v.sb. $\xi'$.

In the case $G = GL_n(\mathbb{R})$ we obtain the definitions used in [2] of a space of geometrical objects of order $r$ and of a field of geometrical objects on a v.b..

**Example 3.1** Take $\Theta = \mathbb{R}^k$, $G_0$ given by the formula (1) and the left action ofi $G_0$ on $\Theta$ given by $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} v = Av$. It is obvious that this action induces a left action of the group $G^1_{0m,n}$ on $\Theta$. It follows a field of geometrical $G_0$-objects of order 1 on the v.b. $\xi$, which is in fact a section in the v.b. $\xi(\xi')$ and a field of geometrical $G_0$-objects of order 1 on the v.b. $\xi$, restricted to the v.sb. $\xi'$, which is in fact a section in the v.b. $\xi'(\xi')$.

The second part of the example above, can be extended as follows:

**Proposition 3.1** *Every field of geometrical objects of order $r \geq 1$ on the v.b. $\xi'$ defines canonicaly a field of geometrical $G_0$-objects of order $r$ on the v.b. $\xi$, restricted to the v.sb. $\xi'$.*

**Example 3.2** A d-connection on the v.b. $\xi'$ induces a field of geometrical $G_0$-objects of order 2 on the v.b. $\xi$, restricted to the v.sb. $\xi'$.

Notice that a field of geometrical $G_0$-objects on the v.b. $\xi$, restricted to the v.sb. $\xi'$ is also such $H_0$-objects. So, the fields of geometrical $G_0$-objects from Proposition 3.1 and from Example 3.2 are also $H_0$-objects. A remarcable example of a field of geometrical $G_0$-objects of order $k$ on the v.b. $\xi$, restricted to the v.sb. $\xi'$, is given by the following direct consequence of the result [1, pg.57, Propzition 5.6], already stated and used in the first section:

**Proposition 3.2** *Let $G_0$ and $H_0$ be given by (1) and (2), and $\xi'$ be a v.sb. of the v.b. $\xi$. Then every reduction of the group $H^r_{0m,n}$ of the p.b. $\mathcal{O}G_0\xi'(\xi)^r$ to $H^r_{0m,n}$ is uniquely defined by a field of geometrical $G_0$-objects on the v.b. $\xi$, restricted to the v.sb. $\xi'$, of order $r$.*

**Example 3.3** Every Finsler splitting $S$ of the inclusion $i : \xi' \to \xi$ is defined by a field of geometrical $G_0$-objects on the v.b. $\xi$, restricted to the v.sb. $\xi'$, of order 1. According he above Proposition 3.2, this $G_0$-object is a section $\mathcal{S}_0$ in the bundle $\mathcal{F}_0$, which is associated with the *p.b.* $\mathrm{O}G_0\xi'(\xi)^1$ and defined by the left action of $G^1_{0m,n}$ on $G_0/H_0$ (in the form (3), determinated by the left action of $G_0$ on $G_0/H_0$ given by formula 4. This formula can be related to the change rule of the local components of a Finsler splitting, which give also the change rule of the local form of the section $\mathcal{S}_0$.

The action (4) can be extended in the general case, but this will be done elsewhere. An example of a field of geometrical $H_0$-objects on the v.b. $\xi$, restricted to the v.sb. $\xi'$, of order 2, is given by the following action of $H^2_{0m,n}$ on $F_0 = I\!R^d$, where $d = m^3 + mk^2 + m(n-k)^2 + m^2n + k^2n + (n-k)^2n$. Writting $F_0$ as $(L^i_{jl}, L^\alpha_{\beta i}, L^u_{vi}, C^i_{j\alpha}, C^i_{ju}, C^\alpha_{\beta\gamma}, C^\alpha_{\beta u}, C^u_{va}, C^u_{vw})$, we define the action of $H^2_{0m,n}$ on $F_0$, by an element $\left(A^i_j, A^i_{jk}, \begin{pmatrix} B^{\alpha'}_\alpha & 0 \\ 0 & B^{u'}_u \end{pmatrix}, \begin{pmatrix} B^{\alpha'}_{\alpha i} & 0 \\ 0 & B^{u'}_{ui} \end{pmatrix}\right)$, as

$$L^{i'}_{j'l'} = \left(A^{i'}_i L^i_{jl} - A^{i'}_{jl}\right) A^j_{j'} A^l_{l'}\,, L^{\alpha'}_{\beta' i'} = \left(A^{\alpha'}_\alpha L^\alpha_{\beta i} - A^{\alpha'}_{\beta i}\right) B^\beta_{\beta'} A^i_{i'}\,,$$

$$L^{u'}_{v'i'} = \left(B^{u'}_u L^u_{vi} - B^{u'}_{vi}\right) B^v_{v'} A^i_{i'}, C^{i'}_{j'\alpha'} = A^{i'}_i A^j_{j'} B^\alpha_{\alpha'} C^i_{j\alpha}\,,$$

$$C^{i'}_{ju'} = A^{i'}_i A^j_{j'} B^u_{u'} C^i_{ju},\ C^{\alpha'}_{\beta'\gamma'} = B^{\alpha'}_\alpha B^\beta_{\beta'} B^\gamma_{\gamma'} C^\alpha_{\beta\gamma}\,,$$

$$C^{\alpha'}_{\beta' u'} = B^{\alpha'}_\alpha B^\beta_{\beta'} B^u_{u'} C^\alpha_{\beta u}, C^{u'}_{v'\alpha'} = B^{u'}_u B^v_{v'} B^\alpha_{\alpha'} C^u_{v\alpha}\,, C^{u'}_{v'w'} = B^{u'}_u B^v_{v'} B^w_{w'} C^u_{vw}.$$

It follows that there is a local trivial bundle $\mathcal{F}_1$ with the fibre $F_1$, associated with the p.b. $OH_0\xi'(\xi)^2$.

Let $S$ be the Finsler splitting which correspomd to the reduction of the group $G_0$ of $OG_0\xi'(\xi)^1$ to $H_0$, according to Theorem 2.1.

**Definition 3.2** A *restricted d-connection* on $\xi$ (related to the v.sb. $\xi'$ and the Finsler splitting $S$) is a section in the above bundle $\mathcal{F}_1$.

If follows that a restricted d-connection is uniquely determinated by the local functions on $E'(E)$

$$(L^i_{jl}, L^\alpha_{\beta i}, L^u_{vi}, C^i_{j\alpha}, C^i_{ju}, C^\alpha_{\beta\gamma}, C^\alpha_{\beta u}, C^u_{v\alpha}, C^u_{vw}) \tag{7}$$

which have as variables $(x^i, y^\alpha)$. They are given on domains of local maps on $E'$, which belong to a vectorial atlas on $E'$, which proceed from one on $E'$. The coordinates on the fibres change following the rules $y^{\alpha'} = g^{\alpha'}_\alpha(x^i)y^\alpha + g^{\alpha'}_u(x^i)y^u$, $y^{u'} = g^{u'}_u(x^i)y^u$.The local functions (7) change according the rules

$$L^{i'}_{j'l'} = \frac{\partial x^{i'}}{\partial x^i}\left(\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^l}{\partial x^{l'}} L^i_{jl} + \frac{\partial^2 x^i}{\partial x^{j'}\partial x^{l'}}\right),\ L^{\alpha'}_{\beta' i'} = g^{\alpha'}_\alpha\left(g^\beta_{\beta'}\frac{\partial x^i}{\partial x^{i'}} L^\alpha_{\beta i} + \frac{\partial g^\alpha_{\beta'}}{\partial x^{i'}}\right),$$

$$L^{u'}_{v'i'} = g^{u'}_u\left(g^v_{v'}\frac{\partial x^i}{\partial x^{i'}} L^u_{vi} + \frac{\partial g^u_{v'}}{\partial x^{i'}}\right),\ C^{i'}_{j'\alpha'} = \frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^i} g^\alpha_{\alpha'} C^i_{j\alpha}$$

$$C^{i'}_{j'u'} = \frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^i} g^u_{u'} C^i_{ju}, C^{\alpha'}_{\beta'\gamma'} = g^{\alpha'}_\alpha g^\beta_{\beta'} g^\gamma_{\gamma'} C^\alpha_{\beta\gamma}\,,\quad C^{u'}_{v'w'} = g^{u'}_u g^v_{v'} g^w_{w'} C^u_{vw}.$$

If we compare the above formulas with those of a d-connection on the v.b. $i^*\xi$ [2, ec. (7.5), pag. 72], we obtain:

**Theorem 3.1** *Let $\xi'$ be a v.sb. of the v.b. $\xi$, $N$ a non-linear connection on $\xi$ and $S$ a Finsler splitting of the inclusion $i : \xi' \to \xi$.*

*1) Every linear d-connection on $\xi$ defines canonicaly a restricted d-conection on $\xi$ related to $\xi'$.*

*2) Every restricted d-connection on $\xi$ related to $\xi'$ defines canonicaly a d-conection on $\xi'$ and a linear Finsler $\xi'$-connection on $\xi'' = \xi/\xi'$.*

*3) A d-connection on $\xi'$ and a linear Finsler $\xi'$-connection on $\xi'' = \xi/\xi'$, defines canonicaly, using the Finsler splitting $S$, a restricted d-conection on $\xi$ related to $\xi'$.*

The proof of the theorem will be given elsewhere. For the definition of a linear Finsler $\xi'$-connection on a v.b. $\xi''$, over the same base, see [4].

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