Bubbletons in 3-dimensional space forms

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Abstract

We construct constant mean curvature (CMC) bubbleton surfaces in the three-dimensional space forms $\boldsymbol{R}^3$, S^3 and H^3 using the DPW method. We show that bubbletons in S^3 and H^3 have properties analogous to the properties of bubbletons in $\boldsymbol{R}^3$. In particular, we give explicit parametrizations in all three space forms, and we show that the parallel CMC surface is congruent to the original bubbleton in all three space forms. Furthermore, we prove that the construction here via the DPW method gives the same surfaces as Bianchi's Bäcklund transformation (as in [20]) for the case of the round cylinders.

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1 Introduction

We consider a smooth surface f immersed in one of the three-dimensional space forms $\boldsymbol{R}^3, S^3$ or H^3 that are the unique complete simply-connected three-dimensional Riemann manifolds of constant sectional curvature 0, 1 and -1, respectively. Defining H to be the mean curvature of f, we say that f is a constant mean curvature (CMC) surface if H is constant on f. Soap films have the property of attaining the least area with respect to the fixed volumes they bound, and are examples of CMC surfaces. Mathematically, H being constant implies that compact portions of the surface f are critical values for boundary-preserving, volume-preserving variations.

The sphere is a simple example of a closed CMC surface. For a long time, there were no known closed CMC immersions besides the sphere, and Hopf asked if there are no closed CMC surfaces different from the standard sphere. It was plausible to believe there were no other surfaces, because:

(1) Hopf showed that the only genus-zero closed CMC surfaces in $\boldsymbol{R}^3$ are spheres.

(2) Alexandrov showed that the only embedded closed CMC surfaces in $\boldsymbol{R}^3$ are spheres.

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However, Wente [23] showed existence of immersed CMC tori in 1984 and this led to renewed interest in the field. Then Pinkall, Sterling, and Bobenko found all CMC tori [16], [1], [2].

The bubbletons are CMC surfaces made from Bäcklund transformations (in Bianchi's sense) of round cylinders. They are shaped like cylinders with attached bubbles, thus they are called bubbletons [20], [22]. The parallel constant positive Gaussian curvature surfaces of bubbletons are well known, and as they were first found by Sievert [18], they are called Sievert surfaces. Bubbletons in $\boldsymbol{R}^3$ have been closely examined by Kilian, Sterling and Wente [12], [20], [22].

In the present paper, analogous to Delaunay surfaces in $\boldsymbol{R}^3$ we define Delaunay surfaces in S^3 and H^3 (see Definition 2 in Section 4.1). Using loop group techniques applied to harmonic maps (via the DPW method), we represent these Delaunay surfaces in space forms. We then define bubbletons based on Delaunay surfaces in space forms by a simple type dressing action, like those of Terng and Uhlenbeck [21], on loop groups (see Definition 3 in Section 5.1). Then we solve the period problems for these Delaunay bubbletons and additionally find explicit immersion formulas for those bubbletons in space forms based on round cylinders. In the case of $\boldsymbol{R}^3$, this was originally done in [12], [20], [22]. Furthermore we prove that the cylinder bubbletons produced here (by the DPW method) are the same as those produced in [20] in the case of $\boldsymbol{R}^3$, and that the parallel CMC surface of a round cylinder bubbleton is congruent to the starting bubbleton, in any of the three space forms. Recently, Mahler [14] interpreted the simple type dressing as the Bianchi Bäcklund transformation in the case of $\boldsymbol{R}^3$.

So what is new in this paper is the following:

(1) We show existence of round cylinder bubbletons and Delaunay bubbletons in all three space forms.

(2) We give explicit parametrizations for round cylinder bubbletons in all three space forms.

(3) We show the equivalence of Bianchi-Bäcklund transformation and simple type dressing for round cylinders in $\boldsymbol{R}^3$.

The first two of these three items are new results for the cases of S^3 and H^3, and the third item is a new result in $\boldsymbol{R}^3$.

In order to apply the DPW method, we first note that classical surface theory can be rewritten in modern fashion using quaternions. If we write quaternions using 2×2 matrices and identify the 3-dimensional Euclidean space with the space of imaginary quaternions, the classical surface theory can be described using 2×2 matrices. For CMC surfaces, the Gauss-Codazzi equations are then the compatibility conditions for a particular system of equations of Lax pair type and allow a one-parameter family of deformations preserving H and the metric that changes only the Hopf differential. The parameter for this family is called the spectral parameter. Existence of this spectral parameter means that we are working with an integrable equation. The solutions of this Lax pair that we use are in $SU(2) = 2 \times 2$ special unitary matrices, and we call these solutions the extended frames for the CMC surfaces, which are inserted into a CMC immersion formula called the Sym-Bobenko formula. In fact, these solutions in $SU(2)$ depend on the spectral parameter and thus lie in the loop group $\Lambda SU(2)$ (which we define in Section 3). The spectral parameter then becomes the loop parameter in the unit circle of the complex plane.

The DPW method [5] was created by Dorfmeister and Pedit and Wu for making CMC surfaces in $\boldsymbol{R}^3$, and is advantageous for dealing with the asymptotic behavior and period problems of CMC surfaces. The DPW method uses loop group theory involving the loop groups $\Lambda SL(2,\boldsymbol{C})$, $\Lambda SU(2)$ and $\Lambda_+ SL(2,\boldsymbol{C})$ (defined in Section 3) and is related to methods of integrable systems. The DPW method also (equivalently) makes extended frames corresponding to harmonic maps from Riemann surfaces to the unit sphere S^2. Using holomorphic 1-forms, the DPW method constructs holomorphic maps to $\Lambda SL(2,\boldsymbol{C})$ and extended frames corresponding to harmonic maps. More concretely, one first chooses a $\Lambda sl(2,\boldsymbol{C})$-matrix-valued holomorphic 1-form called a holomorphic potential. Next one solves a linear first-order (homogeneous) ordinary differential equation whose coefficient is the above holomorphic potential. The solution of this equation is ! in $\Lambda SL(2,\boldsymbol{C})$ when the initial condition is chosen in $\Lambda SL(2,\boldsymbol{C})$. We then decompose $\Lambda SL(2,\boldsymbol{C})$ to $\Lambda SU(2)\times\Lambda SL_+(2,\boldsymbol{C})$ via Iwasawa splitting, producing a $\Lambda SU(2)$ element from a $\Lambda SL(2,C)$ element. The $\Lambda SU(2)$ element is an extended frame of a CMC surface. Finally, the Sym-Bobenko formula takes the extended frame and produces a CMC immersion.

The paper is organized as follows: In Section 2 we give basic notations and results for all space forms CMC surfaces, using 2×2 matrices. In Section 3 we explain the DPW method. In Section 4 we construct simple examples (cylinders and Delaunay surfaces) by the DPW construction. In Section 5 we give the construction and explicit parametrization of CMC bubbletons, using the simple examples from Section 4, and we show equivalence of the simple type dressing and Bianchi's Bäcklund transformation. Finally, at the end of Section 5, we show that the parallel CMC surface of a bubbleton is congruent to that bubbleton.

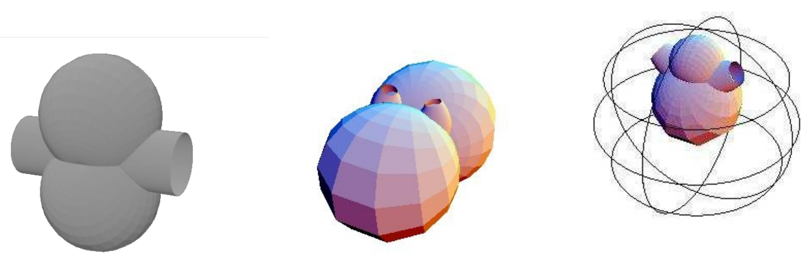

Figure 1: CMC bubbletons in $\boldsymbol{R}^3$, S^3 and H^3. The $\boldsymbol{R}^3$ bubbleton was first described in [22].

2 Lax pairs for CMC surfaces in space forms

2.1 Surfaces in space forms.

$\boldsymbol{R}^3$ (resp. S^3, H^3) is the unique complete simply-connected three-dimensional Riemannian manifold with constant sectional curvature 0 (resp. 1, -1). $\boldsymbol{R}^3$ is the standard flat Euclidean three-space. S^3 is the unit three-sphere in $\boldsymbol{R}^4$ with the metric induced by $\boldsymbol{R}^4$. To define H^3 we shall use the Lorentz space $\boldsymbol{R}^{3,1}$:

$$H^3 = \{(t,x,y,z) \in R^{3,1} \,|\, x^2+y^2+z^2-t^2 = -1 \,,\, t>0\}$$

with the metric induced by $\boldsymbol{R}^{3,1}$, where $\boldsymbol{R}^{3,1}$ is the four-dimensional Lorentz space

$$\{(t,x,y,z) \,|\, t,x,y,z \in \boldsymbol{R}\}$$

with the Lorentz metric

$$\langle (t_1,x_1,y_1,z_1),(t_2,x_2,y_2,z_2)\rangle = x_1x_2+y_1y_2+z_1z_2-t_1t_2 \;.$$

To visualize surfaces in S^3 and H^3, we use specific projections. In the case of S^3, we stereographically project S^3 from its north pole to the space $\boldsymbol{R}^3 \cup \{\infty\}$. In the case of H^3, we use the Poincare model, which is stereographic projection of the Minkowski model in Lorentz space from the point $(0,0,0,-1)$ to the 3-ball $\{(0,x,y,z) \in R^{3,1} \,|\, x^2+y^2+z^2<1\} \cong \{p=(x,y,z) \in \boldsymbol{R}^3 \,|\, |p|<1\}$.

Let Σ be a simply connected surface with conformal coordinate $z = x+iy$ defined on Σ, and let $f : \Sigma \to \mathcal{M}^3$ be a CMC conformal immersion, where $\mathcal{M}^3$ is either $\boldsymbol{R}^3$ or S^3 or H^3. We write $f = f(z,\bar z)$ as a function of both z and $\bar z$ to emphasize that f is not holomorphic in z.

Each of the three space forms lies isometrically in a vector space V: $\mathcal{M}^3 = \boldsymbol{R}^3 \subset V = \boldsymbol{R}^3$, $\mathcal{M}^3 = S^3 \subset V = \boldsymbol{R}^4$, or $\mathcal{M}^3 = H^3 \subset V = \boldsymbol{R}^{3,1}$. Let $\langle\cdot,\cdot\rangle$ be the inner product associated to V, which is the Euclidean inner product in the first two cases, and the Lorentz inner product in the third case. Then the space form metric for each of $\boldsymbol{R}^3$, S^3 and H^3 is the one induced from the metric of the associated vector space V. Since $\mathcal{M}^3 \subset V$ is an embedding, we may also view f as a C^∞ map into V,

$$f : \Sigma \to \mathcal{M}^3 \subseteq V \;\;, \;\; \text{where } V \text{ is } \boldsymbol{R}^3 \text{ or } \boldsymbol{R}^4 \text{ or } \boldsymbol{R}^{3,1} \;.$$

The derivatives $f_x = \partial_x f$ and $f_y = \partial_y f$ are vectors in the tangent space $T_{f(z,\bar z)}V$ of V at $f(z,\bar z)$. Because V is a vector space, f_x and f_y can be viewed as lying in V itself. We will also use $f_z = (1/2)(f_x - if_y)$ and $f_{\bar z} = (1/2)(f_x + if_y)$, defined in the complex extension $V_{\boldsymbol{C}} = \{v_1+iv_2 | v_1,v_2 \in V\}$ of V. The inner product of V extends to a bilinear form $\langle v_1+iv_2, v_1+iv_2\rangle = \langle v_1,v_1\rangle + 2i\langle v_1,v_2\rangle + \langle v_2,v_2\rangle$ (which we also denote by $\langle\cdot,\cdot\rangle$ although it is not actually a true inner product on $V_{\boldsymbol{C}}$). Note that f is conformal if and only if

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0 \;,\;\; \langle f_z, f_{\bar z}\rangle = 2e^{2\mu} \;, \tag{2.1}$$

where the right-most equation defines the function $\mu : \Sigma \to \boldsymbol{R}$.

In each space form, a unit normal vector $N = N(z,\bar z) \in T_{f(z,\bar z)}V \equiv V$ of f is defined by the properties

(1) $\langle N, N\rangle = 1$,

(2) $N \in T_{f(z,\bar{z})}\mathcal{M}^3$, and

(3) $\langle N, f_z\rangle = \langle N, f_{\bar{z}}\rangle = 0$.

The mean curvature of f is then given by

$$H = \frac{1}{2e^{2\mu}}\langle f_{z\bar{z}}, N\rangle \ . \tag{2.2}$$

We also define the Hopf differential

$$\mathcal{Q} = \langle f_{zz}, N\rangle dz^2 \ . \tag{2.3}$$

2.2 The vector spaces V in terms of quaternions

Define the matrices

$$\sigma_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \ , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

We can think of $\boldsymbol{H} = \mathrm{span}_{\boldsymbol{R}}\{i\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3\}$ as the quaternions because it has the quaternionic algebraic structure.

The $\boldsymbol{R}^3$ case. When $\mathcal{M}^3 = V = \boldsymbol{R}^3$, we associate $\mathcal{M}^3$ with the imaginary quaternions $\mathrm{Im}\,\boldsymbol{H} = \mathrm{span}_{\boldsymbol{R}}\{i\sigma_1, i\sigma_2, i\sigma_3\} \subseteq \boldsymbol{H}$ by the map

$$(x_1, x_2, x_3) \to x_1\frac{i}{2}\sigma_1 + x_2\frac{i}{2}\sigma_2 + x_3\frac{i}{2}\sigma_3 \ . \tag{2.4}$$

Then for $X, Y \in \mathrm{Im}\,\boldsymbol{H}$, the inner product inherited from $\boldsymbol{R}^3$ is

$$\langle X, Y\rangle = -2\cdot\mathrm{trace}(XY) = 2\cdot\mathrm{trace}(XY^*) \ , \tag{2.5}$$

where $Y^* = \bar{Y}^t$. Also, any oriented orthonormal basis $\{X, Y, Z\}$ of vectors of $\mathcal{M}^3 \equiv \mathrm{Im}\,\boldsymbol{H}$ satisfies

$$X = F\left(\frac{i}{2}\sigma_1\right)F^{-1} \ , \quad Y = F\left(\frac{i}{2}\sigma_2\right)F^{-1} \ , \quad Z = F\left(\frac{i}{2}\sigma_3\right)F^{-1} \tag{2.6}$$

for some $F \in \mathrm{SU}(2)$, and this F is unique up to sign. In other words, rotations S of $\boldsymbol{R}^3$ fixing the origin are represented in the quaternionic representation $\mathrm{Im}\,\boldsymbol{H}$ of $\boldsymbol{R}^3$ by matrices $F \in \mathrm{SU}(2)$. And the image of F under $\mathrm{Im}\,\boldsymbol{H} \to SO(3)$ is the rotation S.

The S^3 case. When $\mathcal{M}^3 = S^3$ and $V = \boldsymbol{R}^4$, we associate V with $\boldsymbol{H}$ by the map

$$(x_1, x_2, x_3, x_4) \to x_1 i\sigma_0 + x_2 i\sigma_1 + x_3 i\sigma_2 + x_4 i\sigma_3 \ , \tag{2.7}$$

so points $(x_1, x_2, x_3, x_4) \in V = \boldsymbol{R}^4$ are matrices of the form

$$X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \ , \tag{2.8}$$

where $a = x_1 + ix_4$ and $b = x_3 + ix_2$. That is, they are matrices X that satisfy

$$X = \sigma_2 \bar{X} \sigma_2 \ . \tag{2.9}$$

The inner product on $\boldsymbol{H}$ inherited from V is

$$\langle X, Y \rangle = (1/2) \cdot \mathrm{trace}(XY^*) \ , \tag{2.10}$$

where $Y^* = \bar{Y}^t$. Note that this inner product is the same as in (2.5), up to a factor of 4, and this factor of 4 appears only because we include a factor of 1/2 in (2.4) but not in (2.7).

The H^3 case. When $\mathcal{M}^3 = H^3$ and $V = \boldsymbol{R}^{3,1}$, we can associate V with the set of self-adjoint 2×2 matrices $\{X \in Mat(2, \boldsymbol{C}) \,|\, X^* = X\}$ by the map

$$(x_0, x_1, x_2, x_3) \in R^{3,1} \to X = x_0 i\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \ . \tag{2.11}$$

One can check that $\sigma_2 X^t \sigma_2 = X^{-1} \det X$ and that the inner product inherited from V is

$$\langle X, Y \rangle = (-1/2)\mathrm{trace}(X\sigma_2 Y^t \sigma_2) \ ,$$

for self-adjoint matrices X, Y. Thus $\langle X, X \rangle = -\det X$.

2.3 The Lax Pair in the space forms

Let f be a conformal immersion, as in Section 2.1. Because f is a surface in $\mathcal{M}^3 = \boldsymbol{R}^3$ (resp. S^3, H^3), μ and H and $\mathcal{Q}$ satisfy the following Gauss and Codazzi equations for $\boldsymbol{R}^3$ (resp. S^3, or H^3), and the F_1 (resp. F_1, F_2, or F_1), which correspond to the moving frame of a surface f by the map (2.4) (resp. (2.7), (2.11)), satisfies the following Lax pair equations.

$$4\mu_{z\bar{z}} - \mathcal{Q}\bar{\mathcal{Q}}e^{-2\mu} + 4H_k e^{2\mu} = 0 \ , \quad \mathcal{Q}_{\bar{z}} = 2H_z e^{2\mu} \ , \tag{2.12}$$

and

$$F_{k,z} = F_k U_k \ , \quad F_{k,\bar{z}} = F_k V_k \tag{2.13}$$

with

$$U_k = \frac{1}{2}\begin{pmatrix} \mu_z & -2H_k e^{\mu}\lambda^{-1} \\ \mathcal{Q}e^{-\mu}\lambda^{-1} & -\mu_z \end{pmatrix} \ , \quad V_k = \frac{1}{2}\begin{pmatrix} -\mu_{\bar{z}} & -\bar{\mathcal{Q}}e^{-\mu}\lambda \\ 2H_{k+1}e^{\mu}\lambda & \mu_{\bar{z}} \end{pmatrix} \ , \tag{2.14}$$

where H_k is H (resp. $H - (-1)^k i$, $H - (-1)^k$) in the case of $\boldsymbol{R}^3$ (resp. S^3, H^3), with $k \in \{1, 2\}$. (k will be 2 only in the S^3 case.)

For H constant, we see that the Gauss and Codazzi equations for $\mathcal{M}^3$ remain satisfied when $\mathcal{Q}$ is replaced by $\lambda^{-2}\mathcal{Q}$ for any $\lambda \in S^1 = \{p \in \boldsymbol{C} \,|\, |p| = 1\}$. Hence, up to rigid motions, there is a unique surface f_λ with metric determined by μ and with mean curvature H and Hopf differential $\lambda^{-2}\mathcal{Q}$. (We use the notation f_λ to state that f depends on λ; it does not denote the derivative $\partial_\lambda f$.) The surfaces f_λ for $\lambda \in S^1$ form a one-parameter family called the *associate family* of f. The parameter λ is called the *spectral parameter* and is essential to the DPW method. From [15], we have the following facts. When the ambient space is $\boldsymbol{R}^3$, the parallel surfaces of f_λ are

$$f_{\lambda,t} = f_\lambda + tN \ , \quad t \in \boldsymbol{R}.$$

When the ambient space is S^3, the parallel surfaces of f_λ are

$$f_{\lambda,t} = \cos(t) f_\lambda + \sin(t) N \ , \ \ t \in \boldsymbol{R}.$$

When the ambient space is H^3, the parallel surfaces of f_λ are

$$f_{\lambda,t} = \cosh(t) f_\lambda + \sinh(t) N \ , \ \ t \in \boldsymbol{R}.$$

There are special values of t for which the parallel surfaces $f_{\lambda,t}$ also have CMC surfaces. In the case of $\boldsymbol{R}^3$ (resp. S^3, H^3), this is true when the parallel surface is $f^*_\lambda = f_{\lambda,1/(2H)}$ (resp. $f^*_\lambda = f_{\lambda,\mathrm{arccot}(H)}, f^*_\lambda = f_{\lambda,\mathrm{arccoth}(H)}$).

The $\boldsymbol{R}^3$ case. In the case of $\boldsymbol{R}^3$, by applying a homothety if necessary, we may assume $H = 1/2$. We have the following theorem, proven in [2] and [13] using different notations, with the notations here matching those of [5], [6]. We also include information on the parallel surfaces f^*_λ here.

Theorem 2.1. *Let u and Q solve the Gauss-Codazzi equations*

$$4u_{z\bar{z}} - Q\bar{Q}e^{-2u} + e^{2u} = 0 \ , \quad Q_{\bar{z}} = 0 \ , \tag{2.15}$$

and let $F(z, \bar{z}, \lambda)$ be a solution of the system

$$F_z = FU \ , \quad F_{\bar{z}} = FV \tag{2.16}$$

with

$$U = \frac{1}{2}\begin{pmatrix} u_z & -e^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix} \ , \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u}\lambda \\ e^u\lambda & u_{\bar{z}} \end{pmatrix} \tag{2.17}$$

such that $F(z, \bar{z}, \lambda) \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and $F(z, \bar{z}, \lambda)$ is complex analytic in λ. Define

$$f_\lambda = \left[\frac{-i}{2}F\sigma_3F^{-1} - i\lambda(\partial_\lambda F)\cdot F^{-1}\right]\Bigg|_{\lambda=1} \ , \ \ N = \frac{-i}{2}F\sigma_3F^{-1} \ . \tag{2.18}$$

$$f^*_\lambda = f - 2N \ , \ \ N^* = -N \ . \tag{2.19}$$

Then f_λ and f^*_λ are of the form

$$r\cdot\frac{i}{2}\sigma_1 + s\cdot\frac{i}{2}\sigma_2 + t\cdot\frac{i}{2}\sigma_3 \ \text{ and } \ r^*\cdot\frac{i}{2}\sigma_1 + s^*\cdot\frac{i}{2}\sigma_2 + t^*\cdot\frac{i}{2}\sigma_3 \ , \tag{2.20}$$

where r, s, t, r^*, s^* and t^* are real-valued, and (r, s, t) and (r^*, s^*, t^*) are both conformal parametrizations of CMC with $H = 1/2$ surfaces in $\boldsymbol{R}^3$, parametrized by z. f_λ and f^*_λ are parallel surfaces. Also, μ and $\mathcal{Q}$ satisfy $e^{2\mu} = e^{2u}$ and $\mathcal{Q} = Q$, where μ and $\mathcal{Q}$ are defined as in (2.1) and (2.3). Furthermore, with μ^* and $\mathcal{Q}^*$ defined by $2e^{2\mu^*} = \langle f^*_{\lambda,z}, f^*_{\lambda,\bar{z}}\rangle$ and $\mathcal{Q}^* = \langle f^*_{\lambda,zz}, N^*\rangle$, we have $e^{2\mu^*} = e^{-2u}|Q|^2$ and $\mathcal{Q}^* = Q$.

Conversely, for every conformal CMC immersion with $H = 1/2$ into $\boldsymbol{R}^3$, there exists a system (2.16)-(2.17) and solution F producing the immersion via (2.18).

The S^3 case. Consider a conformal CMC immersion $f : \Sigma \to \mathcal{M}^3 = S^3 \subset V = \boldsymbol{R}^4$, with $\langle \cdot, \cdot \rangle$ as in (2.10). Similar to the $\boldsymbol{R}^3$ case, we have the following theorem, proven in [2] and [13] using different notations.

Theorem 2.2. *Let u and Q solve* (2.15) *and let $F(z, \bar{z}, \lambda)$ be a solution of the system* (2.16)-(2.17) *such that $F(z, \bar{z}, \lambda) \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and $F(z, \bar{z}, \lambda)$ is complex analytic in λ. Define $F_1 = F(z, \bar{z}, \lambda = e^{i\gamma_1})$ and $F_2 = F(z, \bar{z}, \lambda = e^{i\gamma_2})$ for some fixed $\gamma_1, \gamma_2 \in \boldsymbol{R}$, and set*

$$f_\lambda = F_1 A F_2^{-1} \ \ , \ \ N = iF_1 A \sigma_3 F_2^{-1}, \ \ \text{where} \ \ A = \begin{pmatrix} e^{\frac{i(\gamma_1-\gamma_2)}{2}} & 0 \\ 0 & e^{\frac{i(\gamma_2-\gamma_1)}{2}} \end{pmatrix}, \tag{2.21}$$

$$f_\lambda^* = \cos(\gamma_2 - \gamma_1) f + \sin(\gamma_2 - \gamma_1) N \ \ , \ \ N^* = \sin(\gamma_2 - \gamma_1) f - \cos(\gamma_2 - \gamma_1) N \ \ . \tag{2.22}$$

Then f_λ and f_λ^* are conformal CMC immersions with $H = \cot(\gamma_2 - \gamma_1)$ into S^3. f_λ and f_λ^* are parallel surfaces. Also, μ and $\mathcal{Q}$ satisfy $e^{2\mu} = \sin^2(\gamma_2 - \gamma_1) \cdot e^{2u}/4$ and $\mathcal{Q} = \sin(\gamma_2 - \gamma_1) \cdot Q$, where μ and $\mathcal{Q}$ are defined as in (2.1) *and* (2.3). *Furthermore, with μ^* and $\mathcal{Q}^*$ defined by $2e^{2\mu^*} = \langle f_{\lambda,z}^*, f_{\lambda,\bar{z}}^* \rangle$ and $\mathcal{Q}^* = \langle f_{\lambda,zz}^*, N^* \rangle$, we have $e^{2\mu^*} = \sin^2(\gamma_2 - \gamma_1) \cdot |Q|^2 e^{-2u}/4$ and $\mathcal{Q}^* = \sin(\gamma_2 - \gamma_1) \cdot Q$.*

Conversely, for every conformal CMC immersion with $H = \cot(\gamma_2 - \gamma_1)$ into S^3, there exists a system (2.16)-(2.17) *and solution F producing the immersion via* (2.21).

The H^3 case. Let $f : \Sigma \to \mathcal{M}^3 = H^3 \subset V = \boldsymbol{R}^{3,1}$ be a conformal CMC immersion with $H > 1$ in H^3. Again, similar to the $\boldsymbol{R}^3$ and S^3 cases, we have the following theorem, proven in [2] and [13] with different notations.

Theorem 2.3. *Let u and Q solve* (2.15) *and let $F = F(z, \bar{z}, \lambda = e^{q/2} e^{i\psi})$, for some fixed $q, \psi \in \boldsymbol{R}$ (with $q \neq 0$), be a solution of the system* (2.16)-(2.17) *such that $F(z, \bar{z}, \lambda) \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and $F(z, \bar{z}, \lambda)$ is complex analytic in λ. We set*

$$f_\lambda = FAF^* \ \ , \ \ N = FA\sigma_3 F^* \ , \text{where} \ \ A = \begin{pmatrix} e^{q/2} & 0 \\ 0 & e^{-q/2} \end{pmatrix}, F^* = \overline{F}^t. \tag{2.23}$$

$$f_\lambda^* = \cosh(-q) f + \sinh(-q) N \ \ , \ \ N^* = \sinh(-q) f - \cosh(-q) N \ \ . \tag{2.24}$$

Then f_λ and f_λ^* are CMC conformal immersions with $H = \coth(-q) > 1$ into H^3. f_λ and f_λ^* are parallel surfaces. Also, μ and $\mathcal{Q}$ satisfy $e^{2\mu} = \sinh^2(-q) \cdot e^{2u}/4$ and $\mathcal{Q} = \sinh(-q) \cdot Q$, where μ and $\mathcal{Q}$ are defined as in (2.1) *and* (2.3). *Furthermore, with μ^* and $\mathcal{Q}^*$ defined by $2e^{2\mu^*} = \langle f_{\lambda,z}^*, f_{\lambda,\bar{z}}^* \rangle$ and $\mathcal{Q}^* = \langle f_{\lambda,zz}^*, N^* \rangle$, we have $e^{2\mu^*} = \sinh^2(-q) \cdot |Q|^2 e^{-2u}/4$ and $\mathcal{Q}^* = \sinh(-q) \cdot Q$.*

Conversely, for every CMC conformal immersion with $H = \coth(-q)$ into H^3, there exists a system (2.16)-(2.17) *and solution F producing the immersion via* (2.23).

Remark. The parallel surface f_λ^* can have singular points. At points where the Hopf differential $\mathcal{Q}$ of the original CMC surface f_λ is zero, then the metric of f_λ^* is degenerate.

3 The DPW recipe

We saw in Section 2 that finding CMC surfaces with $H \neq 0$ in $\boldsymbol{R}^3$ and CMC surfaces in S^3 and CMC surfaces with $H > 1$ in H^3 is equivalent to finding integrable Lax pairs of the form (2.16)-(2.17) and their solutions F. Then the surfaces are found by using the Sym-Bobenko type formulas (2.18), (2.21) and (2.23). So to prove that the DPW recipe finds all of these types of surfaces, it is sufficient to prove that the DPW recipe produces all integrable Lax pairs of the form (2.16)-(2.17) and all their solutions F. Here we describe how these Lax pairs and solutions F are found by the DPW method in [5].

3.1 The loop groups and Iwasawa splitting

Let $C_r := \{\lambda \in \boldsymbol{C} \mid \ |\lambda| = r\}$ be the circle of radius r with $r \in (0,1]$, and let $D_r := \{\lambda \in \boldsymbol{C} \mid \ |\lambda| < r\}$ be the open disk of radius r. We denote the closure of D_r by $\overline{D_r} := \{\lambda \in \boldsymbol{C} \mid \ |\lambda| \leq r\}$.

Definition 1. For any $r \in (0,1] \subset \boldsymbol{R}$, we define the following loop algebra and loop groups:

(1) The twisted $sl(2,\boldsymbol{C})$ r-loop algebra is

$$\Lambda_r sl(2,\boldsymbol{C}) = \left\{ A : C_r \xrightarrow{C^\infty} sl(2,\boldsymbol{C}) \;\middle|\; A(-\lambda) = \sigma_3 A(\lambda)\sigma_3 \right\} .$$

(2) The twisted $\mathrm{SL}(2,\boldsymbol{C})$ r-loop group is

$$\Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) = \left\{ \phi : C_r \xrightarrow{C^\infty} \mathrm{SL}(2,\boldsymbol{C}) \;\middle|\; \phi(-\lambda) = \sigma_3 \phi(\lambda)\sigma_3 \right\} .$$

(3) The twisted $\mathrm{SU}(2)$ r-loop group is

$$\begin{aligned}\Lambda_r \,\mathrm{SU}(2) = \big\{ F \in \Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) \;\big|\, & F(\lambda)^{-1} = F(\bar{\lambda}^{-1})^*, \text{ and} \\ & F(\lambda) \text{ extends holomorphically to } \ D_{1/r} \setminus \overline{D_r} \ \big\} .\end{aligned}$$

When $r = 1$, we abbreviate $\Lambda_1 \,\mathrm{SU}(2)$ to $\Lambda \,\mathrm{SU}(2)$, and in this case the condition that F extends holomorphically to $D_{1/r} \setminus \overline{D_r}$ is vacuous.

(4) The twisted plus r-loop group with $\boldsymbol{R}^+$ is

$$\begin{aligned}\Lambda_{+,r} SL(2,\boldsymbol{C}) = \{ B \in \Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) \mid \ & B(\lambda) \text{ extends holomorphically to } D_r, \\ & \text{and } B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ with } \rho > 0 \ \} .\end{aligned}$$

When $r = 1$, we abbreviate $\Lambda_{+,1} SL(2,\boldsymbol{C})$ to $\Lambda_+ SL(2,\boldsymbol{C})$. Here we defined $\Lambda_{+,r} SL(2,\boldsymbol{C})$ such that $\Lambda_{+,r} SL(2,\boldsymbol{C}) \cap \Lambda_r \,\mathrm{SU}(2) = \mathrm{id}$.

We will give $\Lambda_r SL(2,\boldsymbol{C})$ an H^s-topology for a fixed $s > 1/2$ and take its completion. Then, $\Lambda_r SL(2,\boldsymbol{C})$ is a complex Banach Lie group and its elements have Fourier expansions in the loop parameter λ. We quote the following result from [5].

Lemma 3.1. (Iwasawa decomposition) *For any* $r \in (0, 1]$*, we have the following globally defined real-analytic diffeomorphism from* $\Lambda_r\,\mathrm{SL}(2, \boldsymbol{C})$ *to* $\Lambda_r\,\mathrm{SU}(2) \times \Lambda_{+,r}\,\mathrm{SL}(2, \boldsymbol{C})$ *: For any* $\phi \in \Lambda_r SL(2, \boldsymbol{C})$*, there exist unique* $F \in \Lambda_r\,\mathrm{SU}(2)$ *and* $B \in \Lambda_{+,r} SL(2, \boldsymbol{C})$ *so that*

$$\phi = FB \ .$$

We call this r-Iwasawa splitting *of* ϕ*. When* $r = 1$*, we call it simply* Iwasawa splitting*. Because the diffeomorphism is real-analytic, if* ϕ *depends real-analytically* (*resp. smoothly*) *on some parameter* z*, then* F *and* B *do as well.*

3.2 The DPW method

We now describe the DPW method. Let

$$\xi = A(z, \lambda) dz \ , \quad A(z, \lambda) \in \Lambda sl(2, \boldsymbol{C}) \ \ , \ \ \lambda \in \boldsymbol{C} \setminus \{0\} \ , \tag{3.1}$$

where $A := A(z, \lambda)$ is holomorphic in both z and λ for $z \in \Sigma$. Furthermore, we assume that A has a pole of order at most 1 at $\lambda = 0$, and the upper-right and lower-left entries of A have poles of order exactly 1 at $\lambda = 0$. We call ξ a *holomorphic potential*.

Let ϕ be the solution to

$$d\phi = \phi\xi \ , \qquad \phi(z_*) = \mathrm{id}$$

for some base point $z_* \in \Sigma$. Then ϕ is holomorphic in $z \in \Sigma$ and $\lambda \in \boldsymbol{C}^*$, and

$$\phi \in \Lambda SL(2, \boldsymbol{C}) \ .$$

By Lemma 3.1 above, we can perform an Iwasawa splitting, and write the result as

$$\phi = FB \ . \tag{3.2}$$

From [9], we have the following proposition.

Proposition 3.2. *Up to a conformal change of the coordinate* z*,* F *is a solution to a Lax pair of the form* (2.16)-(2.17)*, and then the Sym-Bobenko formula* (2.18) *or* (2.21) *or* (2.23) *produces a conformal CMC immersion in the corresponding space form* $\boldsymbol{R}^3$*,* S^3 *or* H^3*.*

Also from [9], conversely the conformal CMC immersion has a holomorphic potential, as the next proposition shows.

Proposition 3.3. *For any solution* $F \in \Lambda\,\mathrm{SU}(2)$*, defined for all* $z \in \Sigma$ *and all* $\lambda \in \boldsymbol{C} \setminus \{0\}$ *with* $F(z_*) = \mathrm{id}$*, to a Lax pair of type* (2.16)-(2.17)*, there exists a holomorphic potential* $\xi = Adz$ *with* A *as in* (3.1) *and a solution* $\phi \in \Lambda\,\mathrm{SL}(2, \boldsymbol{C})$ *of* $d\phi = \phi\xi$ *so that* ϕ *Iwasawa splits into* $\phi = FB$ *for some* $B \in \Lambda_+ SL(2, \boldsymbol{C})$*.*

3.3 Dressing

Let Σ be a simply connected surface and let ϕ be a solution to $d\phi = \phi\xi$ with $\phi(z_*) = \mathrm{id}$ on Σ, where ξ is defined as in Equation (3.1). If we define

$$\hat{\phi} = h_+ \cdot \phi \ ,$$

for $h_+ = h_+(\lambda) \in \Lambda_{+,r}\,\mathrm{SL}(2, \boldsymbol{C})$ depending only on λ, then this multiplication on the left by h_+ is called a dressing.

Note that $\hat{\phi}$ satisfies $d\hat{\phi} = \hat{\phi}\xi$, because h_+ is independent of z. Hence the dressing h_+ does not change the potential ξ, and changes only the resulting surface. To see how the surface is changed by h_+, one must Iwasawa split h_+F into $h_+F = \tilde{F}\tilde{B}$, and then $\tilde{F} \in \Lambda SU(2)$ is the frame for the changed surface. This change in the frame from F to $\tilde{F}$ is nontrivial to understand in general, hence the change in the surface is also nontrivial to understand.

3.4 Period problems

Let Σ be a connected Riemann surface with universal cover $\tilde{\Sigma}$ and let Δ denote the group of deck transformations. Let ξ be a holomorphic potential as in Equation (3.1) and ϕ be a solution of $d\phi = \phi\xi$. We assume $\tau^*\xi = \xi$ for any $\tau \in \Delta$. For each $\tau \in \Delta$, we define the monodromy matrix M_τ of ϕ by $M_\tau(\lambda) = (\phi \circ \tau) \cdot \phi^{-1}$.

From [8], we have the following theorem.

Theorem 3.4. *Let M_τ be the monodromy matrix of a solution ϕ, with respect to some deck transformation $\tau \in \Delta$ of $\tilde{\Sigma}$. Then M_τ is unitarizable via dressing for some $r \in (0, 1]$ if and only if, for all $\lambda \in S^1$,*

$$\mathrm{trace}(M_\tau) \in (-2, 2) \ \ \textit{or} \ \ M_\tau = \pm id\,. \tag{3.3}$$

We introduce the following theorem to solve the period problems in $\boldsymbol{R}^3$, S^3 or H^3, respectively, as in [13].

Theorem 3.5. *Let M_τ be as in Theorem 3.4. Assume $M_\tau \in \Lambda_r SU(2)$, so M_τ is also the monodromy matrix of F about τ, where F is as in* (3.2)*. Let f be one of the Sym-Bobenko formulas* (2.18) *or* (2.21) *or* (2.23) *for F, respectively. Then*

- *$\boldsymbol{R}^3$ case: $f \circ \tau = f$ holds if and only if*

$$M_\tau|_{\lambda=1} = \pm id \quad \textit{and} \quad \partial_\lambda M_\tau|_{\lambda=1} = 0\,, \tag{3.4}$$

- *S^3 case: $f \circ \tau = f$ holds if and only if*

$$M_\tau|_{\lambda=e^{i\gamma_1}} = M_\tau|_{\lambda=e^{i\gamma_2}} = \pm\,\mathrm{id}\,, \tag{3.5}$$

- *H^3 case: $f \circ \tau = f$ holds if and only if*

$$M_\tau|_{\lambda=e^{q/2}e^{i\psi}} = \pm id\,. \tag{3.6}$$

4 Surfaces of Revolution

4.1 Delaunay surfaces via DPW

Delaunay surfaces are periodic surfaces of revolution in $\boldsymbol{R}^3$ and are described via DPW in detail in [12]. The generalization of Delaunay surfaces, which are rotational W-hypersurfaces of σ_1-type in H^{n+1} and S^{n+1}, are studied in [19]. We also give a description here. First we give the definition of Delaunay surfaces in space forms.

Definition 2. Let $f : \Sigma = S^2 \setminus p_1, p_2 \to \boldsymbol{R}^3$ (resp. S^3, H^3) be a CMC immersion. Then f is a Delaunay surface in $\boldsymbol{R}^3$ (resp. S^3, H^3) if f is a surface of revolution in $\boldsymbol{R}^3$ (resp. S^3, H^3), i.e. a surface of revolution about a fixed geodesic line in $\boldsymbol{R}^3$ (resp. S^3, H^3).

Using stereographic projection and a Moebius transformation, we may assume $\Sigma = \boldsymbol{C}^* = \boldsymbol{C}\setminus\{0\}$. Define

$$\xi = D\frac{dz}{z} \ , \qquad \text{where } D = \begin{pmatrix} l & s\lambda^{-1} + t\lambda \\ s\lambda + t\lambda^{-1} & -l \end{pmatrix} \ , \tag{4.1}$$

with $l, s, t \in \boldsymbol{R}$.

One solution of $d\phi = \phi\xi$ is

$$\phi = \exp\left(\ln z \cdot D\right) \ . \tag{4.2}$$

This ϕ can be split (this is not r-Iwasawa splitting) in the following way:

$$\phi = F_1 B_1 \ , \quad F_1 = \exp\left(i\theta D\right) \ , \quad B_1 = \exp\left(\ln\rho \cdot D\right) \ ,$$

where $z = \rho e^{i\theta}$, with $\rho = |z|$ and $\theta = \arg(z)$. We note that $F_1 \in \Lambda_r \,\mathrm{SU}(2)$.

Since $D^2 = X^2 \,\mathrm{id}$, where $X = \sqrt{l^2 + (s+t)^2 + st(\lambda - \lambda^{-1})^2}$, we see that

$$F_1 = \begin{pmatrix} \cos(\theta X) + ilX^{-1}\sin(\theta X) & iX^{-1}\sin(\theta X)(s\lambda^{-1} + t\lambda) \\ iX^{-1}\sin(\theta X)(s\lambda + t\lambda^{-1}) & \cos(\theta X) - ilX^{-1}\sin(\theta X) \end{pmatrix} \ , \tag{4.3}$$

$$B_1 = \begin{pmatrix} \cosh(\ln\rho \cdot X) + lX^{-1}\sinh(\ln\rho \cdot X) & X^{-1}\sinh(\ln\rho \cdot X)(s\lambda^{-1} + t\lambda) \\ X^{-1}\sinh(\ln\rho \cdot X)(s\lambda + t\lambda^{-1}) & \cosh(\ln\rho \cdot X) - lX^{-1}\sinh(\ln\rho \cdot X) \end{pmatrix} \ .$$

We can now r-Iwasawa split B_1, i.e. $B_1 = F_2 \cdot B$, where $F_2 \in \Lambda_r SU(2)$ and $B \in \Lambda_{+,r} SL(2, \boldsymbol{C})$. We define $F = F_1 \cdot F_2$. Thus $\phi = FB$ is the r-Iwasawa splitting of ϕ (for any choice of $r \in (0, 1]$).

Because F_2 and B depend only on $|z| = \rho$ and F_1 depends only on θ, we have that, under the rotation of the domain

$$z \to R_{\theta_0}(z) = e^{i\theta_0} z \ , \ \theta_0 \in \boldsymbol{R} \ ,$$

the following transformations occur:

$$F \to M_{\theta_0} F \quad \text{and} \quad B \to B \ , \ \text{ where } \ M_{\theta_0} = \exp(i\theta_0 D) \ .$$

We note that $M_{\theta_0} \in \Lambda_r \,\mathrm{SU}(2)$, and that M_{θ_0} is of the same explicit form as F_1 in (4.3), but evaluated at $\theta = \theta_0$. When $\theta = 2\pi$, we have

$$M_\tau = M_{2\pi} \ . \tag{4.4}$$

Clearly M_τ is the monodromy matrix of the generating counterclockwise deck transformation $\tau \in \Delta$ of the universal cover of $\boldsymbol{C} \setminus \{0\}$.

Now we consider the closing conditions in each of the three space forms:

- When $\mathcal{M}^3 = \boldsymbol{R}^3$, $M_{2\pi}$ must satisfy (3.4) for $\lambda = 1$, so that the surface will close about the deck transformation τ. This is satisfied if

$$l^2 + (s+t)^2 = 1/4 \,, \tag{4.5}$$

so we impose this condition when $\mathcal{M}^3 = \boldsymbol{R}^3$.

- When $\mathcal{M}^3 = S^3$, $M_{2\pi}$ must satisfy (3.5), so that the surface will close about τ. With $\lambda_1 = e^{i\gamma}$ and $\lambda_2 = e^{-i\gamma}$, (3.5) is satisfied if

$$l^2 + (s+t)^2 - 4st\sin^2(\gamma) = 1/4 \,, \tag{4.6}$$

so we impose this when $\mathcal{M}^3 = S^3$.

- When $\mathcal{M}^3 = H^3$, $M_{2\pi}$ must satisfy (3.6), so that the surface will close about τ. With $\lambda = q/2 \in \boldsymbol{R}^+$, (3.6) is satisfied if

$$l^2 + (s+t)^2 + 4st\sinh^2(\frac{q}{2}) = 1/4 \,, \tag{4.7}$$

so we impose this when $\mathcal{M}^3 = H^3$.

With these conditions, Delaunay surfaces are produced in $\boldsymbol{R}^3$, S^3 and H^3, and this can be seen as follows:

In the case of $\boldsymbol{R}^3$, under the mapping $z \to R_{\theta_0}(z)$, we have that f as in (2.18) changes as

$$f \to M_{\theta_0} f M_{\theta_0}^{-1} - i(\partial_\lambda M_{\theta_0})|_{\lambda=1} M_{\theta_0}^{-1} \,. \tag{4.8}$$

One can check that Equation (4.8) represents a rotation of angle θ_0 about the line

$$\{x \cdot (-s-t, 0, l) + 2(s-t) \cdot (2l, 0, 2s+2t) \,|\, x \in \boldsymbol{R}\} \,,$$

hence f is a surface of revolution, and thus a Delaunay surface in $\boldsymbol{R}^3$.

In the case of S^3, under the mapping $z \to R_{\theta_0}(z)$, f as in (2.21) changes by

$$f \to (M_{\theta_0}|_{\lambda=e^{-i\gamma}}) f (M_{\theta_0}^{-1}|_{\lambda=e^{i\gamma}}) \,. \tag{4.9}$$

One can check that Equation (4.9) represents a rotation of angle θ_0 about the geodesic line

$$\{(x_1, x_2, 0, x_4) \in \mathbb{S}^3 \mid \sin(\gamma)(s-t)x_1 + r x_2 - \cos(\gamma)(s+t)x_4 = 0\} \,. \tag{4.10}$$

So we have a surface of revolution in this case also (since the geodesic line (4.10) does not depend on θ_0), and hence a Delaunay surface in S^3.

In the case of H^3, under the mapping $z \to R_{\theta_0}(z)$, f as in (2.23) changes by

$$f \to (M_{\theta_0}|_{\lambda=e^{q/2}}) f (\overline{M_{\theta_0}}^t|_{\lambda=e^{q/2}}) \,. \tag{4.11}$$

One can check that Equation (4.11) represents a rotation of angle θ_0 about the geodesic line

$$\{(x_1, 0, x_3, x_0) \in H^3 \mid \sinh(q)(s-t)x_0 - r x_1 + \cosh(q)(s+t)x_3 = 0\} \,. \tag{4.12}$$

Therefore f is a surface of revolution (since the geodesic line (4.12) does not depend on θ_0), and hence a Delaunay surface in H^3.

We summarize this in the following theorem.

Theorem 4.1. *The holomorphic potential ξ defined in Equation* (4.1) *with the condition* (4.5) (*resp.* (4.6), (4.7)) *produces a Delaunay surface in R^3* (*resp. S^3, H^3*).

Which Delaunay surface one gets depends on the choice of r, s and t. An unduloid is produced when $st > 0$. A nodoid is produced when $st < 0$, and for the limiting singular case of a sphere, $st = 0$. A cylinder is produced when $s = t$ and $l = 0$. In the next subsection, we will show that we can explicitly compute f in the case of cylinders.

4.2 Cylinders via DPW

We choose $l = 0$ and $s = t$ for D in Equation (4.1). Thus ξ is

$$\xi = (\lambda^{-1} + \lambda) \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \frac{dz}{z} .$$

By (4.5), (4.6) and (4.7), $s = 1/4$ or $s = 1/(4\cos(\gamma))$ or $s = 1/(4\cosh(q/2))$ in the respective space form. Furthermore, the ϕ in Equation (4.2) is

$$\begin{aligned} \phi &= \exp\left(\log z \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} (\lambda^{-1} + \lambda)\right) \\ &= \begin{pmatrix} \cosh(s(\lambda^{-1} + \lambda)\log z) & \sinh(s(\lambda^{-1} + \lambda)\log z) \\ \sinh(s(\lambda^{-1} + \lambda)\log z) & \cosh(s(\lambda^{-1} + \lambda)\log z) \end{pmatrix} , \end{aligned}$$

which has the explicit r-Iwasawa splitting

$$\phi = FB , \quad \text{where } B = \exp\left(\lambda(\log z + \log \bar{z}) \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}\right) \text{ and}$$

$$F = \exp\left((\lambda^{-1} \log z - \lambda \log \bar{z}) \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}\right) ,$$

for any $r \in (0, 1]$.

Inserting this F into equations (2.18), (2.21), and (2.23), one can explicitly compute the parametrizations for the surfaces and see that cylinders are produced. In the case of S^3, the cylinder wraps around onto itself to become of torus, since the geodesic lines in S^3 are closed loops.

In the case of $\boldsymbol{R}^3$, from Section 4.4 in [4] Delaunay surfaces are obtained from the dressing of cylinders. Similar arguments show the following results for S^3 and H^3.

Theorem 4.2. *Delaunay surfaces in S^3* (*resp. H^3*) *are obtained from the dressing of cylinders in S^3* (*resp. H^3*).

5 Bubbletons

5.1 Bubbletons via DPW

Let $\tilde{\Sigma}$ be the universal cover of the Riemann surface $\boldsymbol{C}^*$. Let $\phi(z, \lambda)$ be a solution of $d\phi = \phi\xi$ on $\tilde{\Sigma}$ with some initial condition $\phi(z_*, \lambda)$ at $z = z_*$ and let $\phi = F \cdot B$ be the

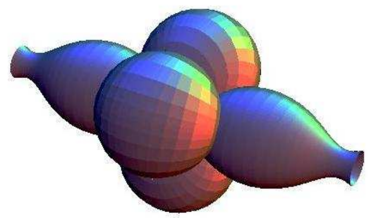

Figure 2: A Delaunay bubbleton in $\boldsymbol{R}^3$. (A cylinder bubbleton in R^3 is shown in Figure 1.) This figure was made by Y. Morikawa.

r-Iwasawa splitting of ϕ, where ξ is as in (4.1). Choose $D/z \in \Lambda_r sl(2,\boldsymbol{C})$ for some $r \in (0,1]$ satisfying either (4.5) or (4.6) or (4.7), depending on the ambient space form. Let f be as in the Sym-Bobenko formula (2.18) or (2.21) or (2.23), respectively, made from the extended frame F. By Theorem 4.1, f is well defined on $\boldsymbol{C}^*$.

Consider the dressing $\phi \to \tilde{\phi} = h \cdot \phi$, where h is the matrix

$$h = \begin{pmatrix} \sqrt{\frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2}} & 0 \\ 0 & \sqrt{\frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2}} \end{pmatrix}, \ \alpha \in \boldsymbol{C}^*. \tag{5.1}$$

Let $\tilde{\phi} = \tilde{F} \cdot \tilde{B}$ be the r-Iwasawa splitting of $\tilde{\phi}$ and let $\tilde{f}$ be the Sym-Bobenko formula (2.18) or (2.21) or (2.23), respectively, made from the extended frame $\tilde{F}$. We note that if $|\alpha| < r$ or $r^{-1} < |\alpha|$, then $h \in \Lambda_r SU(2)$. So the surface $\tilde{f}$ differs from f by only a rigid motion. Therefore we assume $r < |\alpha| < 1$. We note that in general $\tilde{f}$ is not well defined on $\boldsymbol{C}^*$.

Definition 3. Let $f : \boldsymbol{C}^* \longrightarrow \mathcal{M}^3$ and $\tilde{f} : \tilde{\Sigma} \longrightarrow \mathcal{M}^3$, where $\mathcal{M}^3$ is $\boldsymbol{R}^3$ (resp. S^3 or H^3), be CMC immersions derived from the above solutions ϕ and $\tilde{\phi}$. Let M_τ be the monodromy matrix of ϕ defined in Equation (4.4). Then $\tilde{f}$ is a *bubbleton* of the Delaunay surface f in $\boldsymbol{R}^3$ (resp. S^3, H^3) if $hM_\tau h^{-1} \in \Lambda_r SU(2)$.

Lemma 5.1. *The bubbleton $\tilde{f}$ satisfies the closing condition* (3.4) *or* (3.5) *or* (3.6), *so is defined on $\boldsymbol{C}^*$.*

Proof. In the $\boldsymbol{R}^3$ case, we show that since $M_\tau|_{\lambda=1} = \pm id$ and $\partial_\lambda M_\tau|_{\lambda=1} = 0$ are satisfied, thus $(hM_\tau h^{-1})|_{\lambda=1} = \pm id$ and $\partial_\lambda (hM_\tau h^{-1})|_{\lambda=1} = 0$ are also satisfied. This follows from the following computations:

$$\left(hM_\tau h^{-1}\right)|_{\lambda=1} = h|_{\lambda=1}\,(\pm id)\,h^{-1}|_{\lambda=1} = \pm id \ ,$$

$$\partial_\lambda \left(hM_\tau h^{-1}\right)|_{\lambda=1} =$$

$$\left(\left(\partial_\lambda h\right) M_\tau h^{-1}\right)|_{\lambda=1} + \left(h\left(\partial_\lambda M_\tau\right) h^{-1}\right)|_{\lambda=1} + \left(hM_\tau\left(-h^{-1}\left(\partial_\lambda h\right) h^{-1}\right)\right)|_{\lambda=1} = 0 \quad .$$

The H^3 and S^3 cases are similar, in fact they are even simpler, because no derivatives with respect to λ are involved. □

Remark. We saw in Lemma 5.1 that Definition 3 implies the bubbleton is topologically a cylinder.

Lemma 5.2. *$hM_\tau h^{-1}$ is in $\Lambda_r SU(2)$ if and only if M_τ is a lower triangular matrix at $\lambda = \pm\alpha$ and an upper triangular matrix at $\lambda = \pm\bar{\alpha}^{-1}$.*

Proof. Let m_{ij} be the entries of M_τ. We have

$$hM_\tau h^{-1} = \begin{pmatrix} m_{11} & \frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2} m_{12} \\ \frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2} m_{21} & m_{22} \end{pmatrix} .$$

Thus $hM_\tau h^{-1}$ is in $\Lambda_r SU(2)$ if and only if $\frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2} m_{12}(\lambda)$ and $\frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2} m_{21}(\lambda)$ are holomorphic on $r < |\lambda| < r^{-1}$. This happens if and only if M_τ is a lower triangular matrix at $\lambda = \pm\alpha$ and an upper triangular matrix at $\lambda = \pm\bar{\alpha}^{-1}$. □

Theorem 5.3. *There exist round cylinder bubbleton and Delaunay bubbleton surfaces for all three space forms.*

Proof. The monodromy matrix M_τ is

$$M_\tau = \begin{pmatrix} \cos(2\pi X) + ilX^{-1}\sin(2\pi X) & iX^{-1}\sin(2\pi X)(s\lambda^{-1} + t\lambda) \\ iX^{-1}\sin(2\pi X)(s\lambda + t\lambda^{-1}) & \cos(2\pi X) - ilX^{-1}\sin(2\pi X) \end{pmatrix},$$

where

$$X = \sqrt{\frac{1}{4} - a + st(\lambda - \lambda^{-1})^2} \quad ,$$

and

$$\begin{cases} \boldsymbol{R}^3 \text{ case: } a = 0 \\ S^3 \text{ case: } a = -4st\sin^2(\gamma) \\ H^3 \text{ case: } a = 4st\sinh^2(q/2) \end{cases} .$$

M_τ is in $\Lambda_r SU(2)$ for all $r \in (0, 1]$ and satisfies the closing conditions. We take

$$\alpha = \frac{\sqrt{\delta+4} - \sqrt{\delta}}{2} \in \boldsymbol{R} \cup i\boldsymbol{R} \setminus \{0, \pm 1, \pm i\} \text{ with } \delta = \frac{1}{st}\left(\frac{k^2-1}{4} + a\right) \quad , \tag{5.2}$$

$$k^2 \geq \max\{-16st - 4a + 1, -4a + 1, 4\} \text{ and } k \in N \quad .$$

We can immediately compute $M_\tau|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} = -id$. We can choose r so that α satisfies $r < |\alpha| < 1$. Thus Lemma 5.2 and Definition 3 imply existence of bubbletons of cylinders and Delaunay surfaces. □

5.2 Computing the change of frame for the simple type dressing

Now we consider the explicit Iwasawa factorization of $h\phi$ with the simple type dressing h defined in Equation (5.1). This will lead to an explicit parametrization of the bubbletons of round cylinders in all three space forms. In this section, we allow ξ to be a general potential. Let Σ be a Riemann surface with coordinate z. Let ϕ be a solution of $d\phi = \phi\xi$ on Σ with some initial condition $\phi(z_*, \lambda)$ at z_* and let $\phi = F \cdot B$ be the r-Iwasawa splitting of ϕ, where $\xi \in \Lambda_r sl(2, \boldsymbol{C})$ and $r \in (0, 1]$. We assume $r < |\alpha| < 1$, for the same reason as in Section 5.1.

We consider $\boldsymbol{C}^2$ with the standard inner product $\langle \cdot, \cdot \rangle$, and e_1, e_2 forming the orthonormal basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of $\boldsymbol{C}^2$. We define two subspaces V_1, V_2 spanned by specific vectors v_1, v_2 in $\boldsymbol{C}^2$:

$$V_1 = \left\{ a \cdot v_1 \;\middle|\; v_1 = \begin{pmatrix} \bar{A} \\ \lambda^{-1}\bar{\alpha}^{-1}\bar{B} \end{pmatrix}, a \in \boldsymbol{C} \right\},$$

$$V_2 = \left\{ a \cdot v_2 \;\middle|\; v_2 = \begin{pmatrix} -\lambda\alpha^{-1}B \\ A \end{pmatrix}, a \in \boldsymbol{C} \right\},$$

where

$$F|_{\lambda=\alpha} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad F \in \Lambda_r SU(2)$$

We now define projections $\pi_1, \pi_2, \tilde{\pi}_1, \tilde{\pi}_2$ and linear combinations $h, \tilde{h}$ of these projections.

$$(5.3) \qquad \begin{cases} \pi_1 & = \text{orthogonal projection to the span of } e_1 \\ \pi_2 & = \text{orthogonal projection to the span of } e_2 \\ h & = f^{-1/2}\pi_1 + f^{1/2}\pi_2 \end{cases} ,$$

$$(5.4) \qquad \begin{cases} \tilde{\pi}_1 & = \text{projection to } V_1 \text{ parallel to } V_2 \\ \tilde{\pi}_2 & = \text{projection to } V_2 \text{ parallel to } V_1 \\ \tilde{h} & = f^{-1/2}\tilde{\pi}_1 + f^{1/2}\tilde{\pi}_2 \end{cases} ,$$

where

$$f = \frac{\lambda^2 - \alpha^2}{1 - \bar{\alpha}^2\lambda^2}, \quad \alpha \in \boldsymbol{C}^* \quad .$$

Note that in general $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are non-orthogonal projections.

We have the following two lemmas, the first of which is obvious.

Lemma 5.4.

$$\pi_1 \circ \pi_1 = \pi_1 \, , \pi_1 \circ \pi_2 = 0 \, , \pi_2 \circ \pi_1 = 0 \, , \pi_2 \circ \pi_2 = \pi_2 \; .$$

$$\tilde{\pi}_1 \circ \tilde{\pi}_1 = \tilde{\pi}_1 \, , \tilde{\pi}_1 \circ \tilde{\pi}_2 = 0 \, , \tilde{\pi}_2 \circ \tilde{\pi}_1 = 0 \, , \tilde{\pi}_2 \circ \tilde{\pi}_2 = \tilde{\pi}_2 \; .$$

Lemma 5.5.

$$h^{-1} = f^{1/2}\pi_1 + f^{-1/2}\pi_2\ , \qquad \tilde{h}^{-1} = f^{1/2}\tilde{\pi}_1 + f^{-1/2}\tilde{\pi}_2 \quad .$$

Proof.

$$\begin{aligned}&(f^{-1/2}\pi_1 + f^{1/2}\pi_2)\circ(f^{1/2}\pi_1 + f^{-1/2}\pi_2)\\ &= \pi_1\circ\pi_1 + f^{-1}\pi_1\circ\pi_2 + f\pi_2\circ\pi_1 + \pi_2\circ\pi_2\\ &= \pi_1 + \pi_2 = id\ ,\end{aligned}$$

by Lemma 5.4. Similarly $\left(f^{1/2}\pi_1 + f^{-1/2}\pi_2\right)\circ\left(f^{-1/2}\pi_1 + f^{1/2}\pi_2\right) = id$. Replacing π_1 by $\tilde{\pi}_1$ and π_2 by $\tilde{\pi}_2$, we get the analogous result for $\tilde{h}^{-1}$. □

Lemma 5.6. *In terms of the basis e_1, e_2, we can write $\pi_j, \tilde{\pi}_j$ $(j = 1, 2)$ in the following matrix forms:*

$$\pi_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ \ \pi_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\ ,$$

$$\tilde{\pi}_1 = \frac{1}{|A|^2 + |\alpha|^{-2}|B|^2}\begin{pmatrix}|A|^2 & \lambda\alpha^{-1}\bar{A}B\\ \lambda^{-1}\bar{\alpha}^{-1}A\bar{B} & |\alpha|^{-2}|B|^2\end{pmatrix}\ ,$$

$$\tilde{\pi}_2 = \frac{1}{|A|^2 + |\alpha|^{-2}|B|^2}\begin{pmatrix}|\alpha|^{-2}|B|^2 & -\lambda\alpha^{-1}\bar{A}B\\ -\lambda^{-1}\bar{\alpha}^{-1}A\bar{B} & |A|^2\end{pmatrix}\quad .$$

Proof. The matrix forms for π_1 and π_2 are evident. Regarding $\tilde{\pi}_1$ and $\tilde{\pi}_2$, we have $\tilde{\pi}_j \cdot w_i = w_i\delta_{ij}$, $\forall w_i \in V_i$ $(i, j = 1, 2)$, where δ_{ij} is the Kronecker δ function, and so for

$$x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \boldsymbol{C}^2,$$

we have

$$\tilde{\pi}_1 \cdot x = \frac{Ax_1 + \lambda\alpha^{-1}Bx_2}{|A|^2 + |\alpha|^{-2}|B|^2}\begin{pmatrix}\bar{A}\\ \lambda^{-1}\bar{\alpha}^{-1}\bar{B}\end{pmatrix}\ ,$$

$$\tilde{\pi}_2 \cdot x = \frac{-\lambda^{-1}\bar{\alpha}^{-1}\bar{B}x_1 + \bar{A}x_2}{|A|^2 + |\alpha|^{-2}|B|^2}\begin{pmatrix}-\lambda\alpha^{-1}B\\ A\end{pmatrix}\quad .$$

Thus $\tilde{\pi}_1$ and $\tilde{\pi}_2$ have matrix forms as in the lemma. □

We now define a matrix $\mathcal{C} \in \Lambda_r\,\mathrm{SU}(2)$:

$$\mathcal{C} = \frac{1}{\sqrt{|T|^2 + 1}}\begin{pmatrix}e^{i\theta} & \lambda e^{i\theta}T\\ -\lambda^{-1}e^{-i\theta}\bar{T} & e^{-i\theta}\end{pmatrix}\ , \tag{5.5}$$

$$\text{where}\ \ T = \frac{\bar{A}B\left(1 + \bar{\alpha}^2\right)}{\alpha|A|^2 - \bar{\alpha}|B|^2}\ \ \text{and}\ \ \theta = \arg\left(|A|^2 - \frac{\bar{\alpha}}{\alpha}|B|^2\right) + \arg\left(\sqrt{-\alpha^2}\right)\quad .$$

Theorem 5.7. *Let ϕ be a solution of $d\phi = \phi\xi$ on Σ with some initial condition $\phi(z_*, \lambda) \in \Lambda_r\,\mathrm{SL}(2, \boldsymbol{C})$ at z_*, and let $\phi = FB$ be the r-*Iwasawa splitting *of ϕ. We consider the dressing $\phi \to h\cdot\phi$. Then $h\phi = (hF\tilde{h}^{-1}\mathcal{C}^{-1})(\mathcal{C}\tilde{h}B)$ is r-*Iwasawa splitting *of $h\phi$, i.e. $hF\tilde{h}^{-1}\mathcal{C}^{-1} \in \Lambda_r\,\mathrm{SU}(2)$ and $\mathcal{C}\tilde{h}B \in \Lambda_{r+}\,\mathrm{SL}(2, \boldsymbol{C})$, where $h, \tilde{h}, \mathcal{C}$ are as in* (5.3)*,* (5.4) *and* (5.5)*.*

Proof. $\mathcal{C}$ is already in $\Lambda_r SU(2)$, so we first show $hF\tilde{h}^{-1} \in \Lambda_r SU(2)$. F satisfies the reality condition $F(\bar{\lambda}^{-1}) = (F^{-1}(\lambda))^*$. We show that h and $\tilde{h}$ also satisfy the same reality condition:

$$\begin{aligned} h(\bar{\lambda}^{-1}) &= f(\bar{\lambda}^{-1})^{-1/2}\pi_1 + f(\bar{\lambda}^{-1})^{1/2}\pi_2 \ , \\ \left(h^{-1}(\lambda)\right)^* &= \left(f(\lambda)^{1/2}\pi_1 + f(\lambda)^{-1/2}\pi_2\right)^* \\ &= f(\bar{\lambda}^{-1})^{-1/2}\pi_1 + f(\bar{\lambda}^{-1})^{1/2}\pi_2 \ . \\ \tilde{h}(\bar{\lambda}^{-1}) &= f(\bar{\lambda}^{-1})^{-1/2}\tilde{\pi}_1(\bar{\lambda}^{-1}) + f(\bar{\lambda}^{-1})^{1/2}\tilde{\pi}_2(\bar{\lambda}^{-1}) \ , \\ \left(\tilde{h}^{-1}(\lambda)\right)^* &= \left(f(\lambda)^{1/2}\tilde{\pi}_1(\lambda) + f(\lambda)^{-1/2}\tilde{\pi}_2(\lambda)\right)^* \\ &= f(\bar{\lambda}^{-1})^{-1/2}\tilde{\pi}_1(\bar{\lambda}^{-1}) + f(\bar{\lambda}^{-1})^{1/2}\tilde{\pi}_2(\bar{\lambda}^{-1}) \ . \end{aligned}$$

Thus we have shown the reality condition for h and $\tilde{h}$. F is holomorphic on $r < |\lambda| < r^{-1}$. h, $\tilde{h}$ are holomorphic on $r < |\lambda| < r^{-1}$ with singularities only at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$. Thus we need only check that $hF\tilde{h}^{-1}$ has no singularities at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$.

$$\begin{aligned} hF\tilde{h}^{-1}\Big|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} &= \left(\left(f^{-1/2}\pi_1 + f^{1/2}\pi_2\right) F \left(f^{1/2}\tilde{\pi}_1 + f^{-1/2}\tilde{\pi}_2\right)\right)\Big|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} \\ &= \left(\pi_1 F\tilde{\pi}_1 + f\pi_2 F\tilde{\pi}_1 + f^{-1}\pi_1 F\tilde{\pi}_2 + \pi_2 F\tilde{\pi}_2\right)\Big|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} \ . \end{aligned}$$

The only possible singularities in this sum of four terms can occur in $f\pi_2 F\tilde{\pi}_1$ when $\lambda = \pm\bar{\alpha}^{-1}$ and in $f^{-1}\pi_1 F\tilde{\pi}_2$ when $\lambda = \pm\alpha$. But the reality condition of F and a calculation shows that in fact such singularities do not occur.

Finally we show $\mathcal{C}\tilde{h}B \in \Lambda_{+r}SL(2,\boldsymbol{C})$. B is in $\Lambda_{+r}SL(2,\boldsymbol{C})$, so we need only check that $\mathcal{C}\tilde{h}$ is in $\Lambda_{+,r}SL(2,\boldsymbol{C})$. We can easily see $\mathcal{C}\tilde{h} \in \Lambda_r SL(2,\boldsymbol{C})$ and is holomorphic on $0 < |\lambda| < r$ and continuous on $0 < |\lambda| \le r$, and a direct computation shows that

$$\mathcal{C}\tilde{h}|_{\lambda=0} = \begin{pmatrix} \rho_1 & 0 \\ 0 & {\rho_1}^{-1} \end{pmatrix} \ ,$$

$$\text{where} \ \ \rho_1 = \sqrt{\frac{|\alpha|^{-1}|A|^2 + |\alpha||B|^2}{|\alpha||A|^2 + |\alpha|^{-1}|B|^2}} \in \boldsymbol{R}^+ \ .$$

Thus the theorem is proven. □

Theorem 5.7 has the following corollary:

Corollary 5.8. *We have explicit parametrizations for round cylinder bubbletons in all three space forms using the* r-Iwasawa *splitting in Theorem 5.7, the extended frame in Section 4.2 and the Sym-Bobenko formulas* (2.18)*,* (2.21) *and* (2.23)*.*

Remark. Let $\xi = \Sigma_{j\ge -1}\lambda^j A_j dz$ be a *holomorphic potential* on Σ and let ϕ be a solution of $d\phi = \phi\xi$ with some initial condition $\phi(z_*,\lambda) \in \Lambda_r\,\mathrm{SL}(2,\boldsymbol{C})$ at z_*. Let $\phi = F\cdot B$ be r-Iwasawa splitting and let f be as in the Sym-Bobenko formula (2.18) or (2.21) or (2.23), respectively, made from the extended frame F. Then the conformal factor of the metric $4e^{2\mu}dzd\bar{z}$ of f is (see [12])

$$4e^{2\mu} = 16\epsilon^2 e^{2u}|a_{-1}|^2 = 16\epsilon^2\rho^4|a_{-1}|^2 \quad , \tag{5.6}$$

where a_{-1} is the upper right entry of A_{-1}, $B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$ and $\epsilon = 1$ (resp. $\epsilon = \sin((\gamma_2 - \gamma_1)/2)$, $\epsilon = \sinh(-q/2)$) in the case of $\boldsymbol{R}^3$ (resp. S^3, H^3).

5.3 Equivalence of the simple type dressing and Bianchi's Bäcklund transformation on the round cylinder

In this section we prove the equivalence of the simple type dressing (5.1) and Bianchi's Bäcklund transformation in $\boldsymbol{R}^3$, when applied to a cylinder. (The latter is a bubbleton surface in the sense of [20].) We show that the metric, Hopf differential and mean curvature of Bianchi's Bäcklund transformation of a round cylinder are the same as those resulting from the simple type dressing of (5.1) with real α. For general CMC surfaces in $\boldsymbol{R}^3$, Burstall [3] has proven the equivalence of the simple type dressing (5.1) for α either real or pure imaginary and the Darboux transformation. Hertrich-Jeromin and Pedit [11] have proven that any of Bianchi's Bäcklund transformations of a CMC surface is a Darboux transformation of the surface, but not the converse.

In the S^3 and H^3 cases, we have not seen a notion of Bianchi's Bäcklund transformation. So we do not prove the equivalence for the S^3 and H^3 cases.

First we introduce the metric, Hopf differential and mean curvature of Bianchi's Bäcklund transformation using [20]. Using the notation in [20], we can write the first and second fundamental forms and the principal curvatures of a CMC surface as follows:

$$\begin{cases} ds^2 & = e^{2u} dw d\bar{w} \\ II & = e^u \left(\sinh(u)\, dx^2 + \cosh(u)\, dy^2\right) \\ k_1 & = e^{-u}\sinh(u)\,, \quad k_2 = e^{-u}\cosh(u) \end{cases} \quad ,$$

where $w = x + iy$. The Gauss equation becomes

$$2u_{w\bar{w}} + \sinh(2u) = 0 \quad . \tag{5.7}$$

In particular, in the round cylinder case we have $u = 0$. We do the Bäcklund transformation on the cylinder, and using Bianchi's Permutability formula ([20]), we have the new solution u_1 satisfying the Gauss equation (5.7):

$$\tanh\left(\frac{u_1}{2}\right) = \tanh(\beta_1)\,\frac{\cos(y\cosh(\beta_1))}{\cosh(x\sinh(\beta_1))} \quad ,$$

where $\beta_1 \in \boldsymbol{R}$. Under Bianchi's Bäcklund transformation, the mean curvature and the Hopf differential do not change. So the mean curvature and the Hopf differential of the bubbleton are $H = 1/2$ and $Q = (-1/4)dw^2$.

We consider the change of coordinate $\log z = w$. Thus we have the following:

$$\tanh\left(\frac{u_1}{2}\right) = \tanh(\beta_1)\,\frac{\cos(\mathrm{Im}(\log z)\cosh(\beta_1))}{\cosh(\mathrm{Re}(\log z)\sinh(\beta_1))} \quad ,$$

$$H = 1/2 \quad ,$$

$$Q = -\frac{1}{4z^2}dz^2 \quad .$$

Theorem 5.9. *Bianchi's Bäcklund transformation of the round cylinder and the simple type dressing with real α of the round cylinder are the same surface.*

Proof. Using Corollary 5.8, and Equations (5.6), (2.2) and (2.3), the simple type dressing by h has the following conformal factor for the metric $4e^{2\mu}dzd\bar{z}$, and the following mean curvature and Hopf differential:

$$\begin{aligned}4e^{2\mu} &= 16e^{2u_1}|a_{-1}|^2 = 16\rho^4|a_{-1}|^2 \\ &= 16\left(\frac{|\alpha|^{-1}|A|^2+|\alpha||B|^2}{|\alpha||A|^2+|\alpha|^{-1}|B|^2}\right)^2|a_{-1}|^2 \\ &= 16\left(\frac{\alpha^{-1}|\cosh(X)|^2+\alpha|\sinh(X)|^2}{\alpha|\cosh(X)|^2+\alpha^{-1}|\sinh(X)|^2}\right)^2|a_{-1}|^2 \ ,\end{aligned}$$

$$H = 1/2 \ ,$$

$$Q = -\frac{1}{4z^2}dz^2 \ ,$$

where $X = \frac{\alpha^{-1}\log z - \alpha\log\bar{z}}{4}$ and $a_{-1} = 1/(4z)$. Then $\tanh(u_1/2) = \frac{e^{u_1}-1}{e^{u_1}+1}$ implies that

$$\tanh\left(\frac{u_1}{2}\right) = \frac{\left(\alpha^{-1}-\alpha\right)\left(|\cosh(X)|^2-|\sinh(X)|^2\right)}{\left(\alpha^{-1}+\alpha\right)\left(|\cosh(X)|^2+|\sinh(X)|^2\right)} \ .$$

Using addition properties for the hyperbolic sine and cosine functions, we can rewrite the equation as follows:

$$\tanh\left(\frac{u_1}{2}\right) = \frac{\left(\alpha^{-1}-\alpha\right)\cosh\left(X-\bar{X}\right)}{\left(\alpha^{-1}+\alpha\right)\cosh\left(X+\bar{X}\right)} \ .$$

We have $X+\bar{X} = \mathrm{Re}(\log z)(\frac{\alpha^{-1}-\alpha}{2})$ and $X-\bar{X} = i\,\mathrm{Im}(\log z)(\frac{\alpha^{-1}+\alpha}{2})$. Thus the equation finally becomes

$$\begin{aligned}\tanh\left(\frac{u_1}{2}\right) &= \frac{\left(\frac{\alpha^{-1}-\alpha}{2}\right)\cosh\left(i\,\mathrm{Im}\left(\log z\right)\frac{\alpha^{-1}+\alpha}{2}\right)}{\left(\frac{\alpha^{-1}+\alpha}{2}\right)\cosh\left(\mathrm{Re}\left(\log z\right)\frac{\alpha^{-1}-\alpha}{2}\right)} \\ &= \frac{\sinh\left(\beta_1\right)\cos\left(\mathrm{Im}\left(\log z\right)\cosh\left(\beta_1\right)\right)}{\cosh\left(\beta_1\right)\cosh\left(\mathrm{Re}\left(\log z\right)\sinh\left(\beta_1\right)\right)} \ ,\end{aligned}$$

where we set $\frac{\alpha^{-1}+\alpha}{2} = \cosh(\beta_1)$ and $\frac{\alpha^{-1}-\alpha}{2} = \sinh(\beta_1)$. Therefore both transformations give the same metric, mean curvature and Hopf differential. So the fundamental theorem of surface theory implies that the two transformations of the round cylinder are the same. □

5.4 Parallel surfaces of the bubbletons

In this section, we prove that the parallel surfaces of the round cylinder bubbletons are the same surface as the original bubbletons.

Theorem 5.10. *The parallel surface of a round cylinder bubbleton is the same surface as the original cylinder bubbleton, up to a rigid motion, in any of the three space forms $\boldsymbol{R}^3$, S^3 and H^3.*

Proof. Using Corollary 5.8 and Equations (5.6), (2.2) and (2.3), we can describe the conformal factor for the metric $4e^{2\mu}dzd\bar{z}$, mean curvature H and Hopf differential $\mathcal{Q}$ of the round cylinder bubbletons as follows:

$$4e^{2\mu} = 16\epsilon^2 e^{2u}|a_{-1}|^2 = 16\epsilon^2 \left(\frac{\alpha^{-1}|A|^2+\alpha|B|^2}{\alpha|A|^2+\alpha^{-1}|B|^2}\right)^2 |a_{-1}|^2 \quad ,$$

$$H = b \quad ,$$

$$\mathcal{Q} = -\frac{1}{4z^2}\epsilon dz^2 \quad ,$$

where $A = \cosh(\frac{\alpha^{-1}\log z-\alpha\log\bar{z}}{4})$, $B = \sinh(\frac{\alpha^{-1}\log z-\alpha\log\bar{z}}{4})$, and $\epsilon = 1$, $a_{-1} = 1/(4z)$ and $b = 1/2$ (resp. $\epsilon = \sin(-2\gamma), a_{-1} = 1/(4z\cos(\gamma))$ and $b = \cot(-\gamma)$, or $\epsilon = \sinh(-q), a_{-1} = 1/(4z\cosh(q/2))$ and $b = \coth(-q/2)$) in the case of $\boldsymbol{R}^3$ (resp. S^3, or H^3), and where $\alpha \in \boldsymbol{R}$ as in (5.2).

Using Theorem 2.1, Theorem 2.2 and Theorem 2.3, we can also describe the conformal factor for the metric $4e^{2\mu^*}dzd\bar{z}$, mean curvature H^* and Hopf differential $\mathcal{Q}^*$ of the bubbleton parallel surface as follows:

$$4e^{2\mu^*} = 16\epsilon^2 e^{-2u}|a_{-1}| = 16\epsilon^2 \left(\frac{\alpha|A|^2+\alpha^{-1}|B|^2}{\alpha^{-1}|A|^2+\alpha|B|^2}\right)^2 |a_{-1}|^2 \quad ,$$

$$H^* = b \quad ,$$

$$\mathcal{Q}^* = -\frac{1}{4z^2}\epsilon dz^2 \quad .$$

We consider the conformal change of the coordinate $z \to z\exp(\frac{2\pi i}{\alpha+\alpha^{-1}})$ on the parallel surface. Under this change, the mean curvature and Hopf differential do not change. For the metric, $|A|^2$ and $|B|^2$ change to $|B|^2$ and $|A|^2$, respectively, thus the conformal factor $4e^{2\mu^*}$ of the metric changes to

$$16\epsilon^2 \left(\frac{\alpha^{-1}|A|^2+\alpha|B|^2}{\alpha|A|^2+\alpha^{-1}|B|^2}\right)^2 |a_{-1}|^2 = 4e^{2\mu} \quad .$$

Thus both surfaces have the same metric, mean curvature and Hopf differential up to this change of coordinate. Hence the fundamental theorem of surface theory implies the two surfaces are the same. □

Remark. The parallel surface of a Delaunay bubbleton in general is not the same surface as the original Delaunay bubbleton. For example, if the bubbles of Delaunay bubbleton attach at a neck of the Delaunay surface, then the bubbles of the parallel Delaunay bubbleton attach at a bulge of the Delaunay surface.

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