# Cooperative guards in the fortress problem

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**Abstract**

The *fortress problem* asks for the number of guards sufficient to see every point of the exterior of a polygon. A set of guards $S$ is called *cooperative* if the visibility graph $VG(S)$ is connected. In this paper, we investigate the cooperative guard problem in a fortress: tight bounds for vertex and point guards are obtained.

In particular, let $P$ be a fortress of $k$ pockets $p_1, \ldots, p_k$ and of $c$ vertices on the convex hull. Then we show that: (1) if $c = k$ and all $k$ pockets are of even number of vertices, then $\sum_{i=1}^{k} \lfloor \frac{n_{p_i}-2}{2} \rfloor$ cooperative vertex guards are sometimes necessary and always sufficient to cover the exterior of $P$; (2) otherwise, $c - 1 + \sum_{i=1}^{k} (\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative vertex guards are sometimes necessary and always sufficient to cover the exterior of $P$. (3) If guards are not restricted to vertices, then $1 + \sum_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative vertex guards are sometimes necessary and always sufficient to cover the exterior of $P$. Also tight bounds for cooperative vertex guards in orthogonal polygons are provided.

**Mathematics Subject Classification:** 68U05.

**Key words:** computational geometry, art gallery theorem, fortress problem, cooperative guard.

## 1 Introduction

The original *art gallery problem* raised by Klee asks how many guards are sufficient to watch every point of the interior of an $n$-vertex simple polygon. The guard is a stationary point that can *see* any point which can be connected to it with a line segment within the polygon. In 1975, Chvatl [1] proved that $\lfloor \frac{n}{3} \rfloor$ guards are occasionally necessary and always sufficient to cover a polygon with $n$ vertices. Since then many different variations of this problem have arisen; see [8], [9] for more details.

One of a family of guard problem, independently posed by Joseph Mal-kelvitch and Derick Wood, is the *fortress problem*, i.e., one wants to determine the minimal number of guards sufficient to see every point of the exterior of an $n$-vertex simple

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polygon (the guard is a stationary point that can see any point which can be connected to it with a line segment without the polygon.)

In 1983, O'Rourke and Wood [8] solved the fortress problem for vertex guards – they showed that $\lceil \frac{n}{2} \rceil$ vertex guards are sometimes necessary and always sufficient. A tight bound of $\lceil \frac{n}{3} \rceil$ point guards was given by O'Rourke and Aggarwal [8].

Herein we analyze the concept of *cooperative guards* that was proposed by Liaw *et al.* [5]. For a guard set $S$ we define the *visibility graph* $VG(S)$ as follows: the vertex set is $S$ and two vertices $v_1, v_2$ are adjacent if they see each other. The guard set $S$ is said to be *cooperative* if the graph $VG(S)$ is connected. The idea behind this concept is that if something goes wrong with one guard, all the others can be informed.

In 1994, Hernández-Peñalver [4] proved that $\lfloor \frac{n}{2} \rfloor - 1$ cooperative guards are sometimes necessary and always sufficient to cover any point of the interior of an $n$-vertex polygon. One may ask whether this is still true for the fortress problem, however, a convex $n$-gon requires $n-1$ cooperative vertex guards. Thus we have:

**Fact 1.1.** *$n-1$ cooperative vertex guards are sometimes necessary to cover the exterior of a simple polygon with $n$ vertices.*

Yiu [10] considered the number of $k$-consecutive vertex guards that are required to solve the fortress problem. A *$k$-consecutive vertex guard* is a set of vertex guards located at $k$ consecutive vertices of the polygon. He showed that $\lceil \frac{n}{k+1} \rceil$ $k$-consecutive vertex guards always suffice to cover the exterior of any $n$-vertex simple polygon. Thus we have:

**Corollary 1.2.** *$n-1$ cooperative vertex guards always suffice to cover the exterior of a simple polygon with $n$ vertices.*

However, convex polygons are a severely restricted class of polygons, so it is natural to investigate the fortress problem for cooperative guards as a function of a variable other than $n$, the number of vertices of the polygon.

The organization of this paper is as follows. Section 2 is intended to motivate our investigation of a more accurate measure of the number of connected guards sufficient to cover the exterior of the polygon. Section 3 is devoted to notation, terminology, and some basic lemmas. The sufficiency proof for connected vertex guards will be presented in Section 4. Section 5 deals with the case of an orthogonal fortress. In Section 6, we explore point guards. Finally, some related problems are discussed.

## 2 Necessity

Let $c$ denote the number of vertices of an $n$-vertex polygon which are on the convex hull of the set of vertices of the polygon, and for any *pocket* $p$ of the polygon – that is, an exterior polygon interior to the hull and bounded by a hull edge – let $n_p$ denote the number of vertices of the pocket $p$. Let us consider a polygon $P$ with one pocket $p$ that is shown in Fig. 1(a). One can easily check that $P$ requires $c-1+(\lfloor \frac{n_p-1}{2} \rfloor - 1)$ cooperative vertex guards. A simple extension of this polygon, see Fig. 1(b), leads to a class of polygons of $k$ pockets that require $c-1+\sum_{i=1}^{k}(\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative vertex guards, where $n_{p_i}$ is the number of vertices of the pocket $p_i$, $i=1,\ldots,k$.

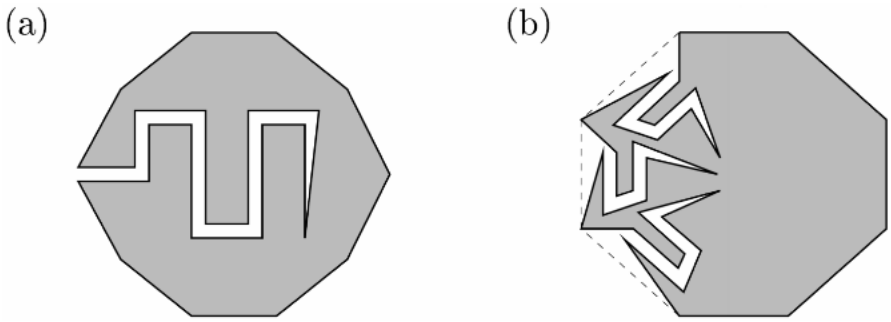

**Fig. 1.** ($a$) A fortress that requires $c-1+(\lfloor\frac{n_p-1}{2}\rfloor-1)$ cooperative vertex guards; here we have $c=11$, $n_p=17$, and the polygon requires 17 cooperative vertex guards.
($b$) A fortress that requires $c-1+\sum_{i=1}^{k}(\lfloor\frac{n_{p_i}-1}{2}\rfloor-1)$ cooperative vertex guards; here the polygon has three pockets, each of 11 vertices, $c=8$, and it requires 19 cooperative vertex guards.

**Lemma 2.1.** *Let $c \geq 3$ and $0 \leq k \leq c$ be integers. Then there exists a fortress of $k$ pockets and $c$ vertices on the convex hull that requires*

$$c-1+\sum_{i=1}^{k}(\lfloor\frac{n_p-1}{2}\rfloor-1)$$

*cooperative vertex guards, where $n_{p_i}$ is the number of vertices of the pocket $p_i$, $i=1,\ldots,k$.* □

But, if $c=k$ and all $n_{p_i}$, $i=1,\ldots,k$, are even, more than $c-1+\sum_{i=1}^{k}(\lfloor\frac{n_p-1}{2}\rfloor-1)$ cooperative vertex guards can be required. Consider a fortress that is shown in Fig. 2: here we have $c=k=4$, all $n_{p_i}=6, i=1\ldots4$, and the fortress requires

$$8=2+2+2+2>c-1+\sum_{i=1}^{k}(\lfloor\frac{n_p-1}{2}\rfloor-1)$$

cooperative vertex guards. Thus we have:

**Lemma 2.2.** *Let $k$ be an integer, $k \geq 3$. Then there exists a fortress of $k$ pockets and no edges on the convex hull that requires*

$$\sum_{i=1}^{k}\lfloor\frac{n_p-2}{2}\rfloor$$

*cooperative vertex guards, where $n_{p_i}$ is the number of vertices of the pocket $p_i$, $i=1,\ldots,k$.* □

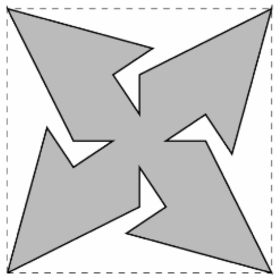

**Fig. 2.** A fortress with no edges on the convex hull that requires $\sum_{i=1}^{k} \lfloor \frac{n_{p_i}-2}{2} \rfloor$ cooperative vertex guards; here the polygon has four pockets, each of 6 vertices, and it requires 8 cooperative vertex guards.

## 3 Definitions

A *fortress* is a (simple) polygon $P$. Let $F(P)$ denote the set of all points on the plane exterior to $P$ or on the boundary of $P$. A *guard* $g$ is any vertex of $P$. A point $x \in F(P)$ is said to be *seen* by a guard $g$ if the line segment with endpoints $x$ and $g$ is a subset of $F(P)$. A collection of guards $S = \{g_1, ..., g_k\}$ is said to *cover* the fortress $P$ if every point $x \in F(P)$ can be seen by some guard $g \in S$.

We define the *visiblity graph* $VG(S)$ as follows: the vertex set is $S$ and two vertices $v_1$ and $v_2$ are incident if they see each other. The guard set $S$ is said to be *cooperative* if the graph $VG(S)$ is connected.

Each connected region inside a convex hull of the polygon $P$ but exterior to $P$ is called a *pocket*; note that the pocket is a simple polygon. A *triangulation graph* of a pocket is a graph whose embedding is a triangulation of the pocket: the vertices correspond to vertices of the pocket and the edges correspond to the edges of the pocket, internal diagonals and the hull edge, called a *pocket lid*.

A *vertex guard* in $G_T$ is a single vertex of $G_T$. A set of guards $S = \{g_1, ..., g_k\}$ is said to *dominate* $G_T$ if every triangular face of $G_T$ has at least one of its vertices assigned as a guard ($\in S$). Finally, the collection of guards $S = \{g_1, ..., g_k\}$ is said to be *cooperative* if the subgraph of $G_T$ induced by set $S$ is connected. Guards in a graph are called *combinatorial cooperative guards.* The reason for introducing triangulation graphs is the following lemma:

**Lemma 3.1.** *Let be a pocket $p$ of $n_p$ vertices, and let $d = \{x_1, x_2\}$ be a pocket lid. Then:*

*a) if $n_p$ is odd, then $\lfloor \frac{n_p-1}{2} \rfloor$ cooperative vertex guards with one guard placed at any endpoint of $d$ suffice to cover the pocket $p$;*

*b) otherwise, $\lfloor \frac{n_p-1}{2} \rfloor$ cooperative vertex guards with one guard placed either at $x_1$ or at $x_2$ suffice to cover the pocket $p$.*

*Proof.* The validity of the lemma for odd $n_p$ follows immediately from Hernández-Peñalver's theorem establishing that $\lfloor \frac{n_p}{2} \rfloor - 1$ cooperative vertex guards suffice to cover all of the pocket [4]. With one additional guard at any endpoints of the pocket lid we will get a coverage of the pocket by $\lfloor\frac{n_p-1}{2}\rfloor$ cooperative vertex guards.

Now, let assume $n_p$ to be even. Consider any triangulation graph $G_T$ of the pocket $p$. Let $G_T^\star$ be a graph that results from adjoining a graph $K_3$ at the pocket lid. It is clear that $G_T^\star$ is a triangulation graph of $(n_p+1)$-vertex polygon, and by [4] it can be dominated by $\lfloor\frac{n_p-1}{2}\rfloor$ cooperative vertex guards. Any triangular face of $G_T^\star$ has at least one of its vertices assigned as a guard, thus without loss of generality there is a guard either at $x_1$ or at $x_2$. The same guards placement in the pocket will cover every point inside the pocket. □

Note that if $n_p$ is equal to 3 or 4, we get the degenerated case – the set of cooperative guards consists of one guard only.

# 4 Vertex guards in a fortress

We will prove in this section that bounds established by Lemmas 2.1-2.2 are tight.

**Theorem 4.1.** *Let $P$ be a fortress of $k$ pockets $p_1,\dots,p_k$ and of $c$ vertices on the convex hull. Then:*

*a) if $c=k$ and all $k$ pockets are of even number of vertices, then $\sum_{i=1}^{k}\lfloor\frac{n_{p_i}-2}{2}\rfloor$ cooperative vertex guards always suffice to cover $F(P)$;*

*b) otherwise, $c-1+\sum_{i=1}^{k}(\lfloor\frac{n_{p_i}-1}{2}\rfloor-1)$ cooperative vertex guards always suffice to cover $F(P)$.*

*Proof.* The proof is by induction on $k$, the number of pockets. Lemma 1.1 establishes the validity of the theorem for $k=0$, so assume that $k\geq 1$ and that the theorem holds for all $\hat{k}<k$. We need to consider three cases. Note that the induction proof is used only in the third case.

*Case* 1: $c=k$ and all $k$ pockets are have even number of vertices. By Lemma 3.1 placing $k$ guards at all vertices of the convex hull permits the reminders of all $k$ pockets to be covered by $\sum_{i=1}^{k}(\lfloor\frac{n_{p_i}-1}{2}\rfloor-1)=\sum_{i=1}^{k}(\lfloor\frac{n_{p_i}-2}{2}\rfloor-1)$ cooperative vertex guards, as all $n_{p_i}$ are even, $i=1,\dots,k$. Therefore, all of $F(P)$ can be covered by $\sum_{i=1}^{k}\lfloor\frac{n_{p_i}-2}{2}\rfloor$ cooperative vertex guards in total.

*Case* 2: $c\neq k$ and all $k$ pockets are of even number of vertices.

*Subcase* 2.*a*: there are two consecutive edges of the polygon on the convex hull. Let these edges be labeled $e_1=\{x_1,x_2\}$ and $e_2=\{x_2,x_3\}$. Then placing $c-1$ guards at all vertices of the convex hull except $x_2$, and applying an argument similar to that in Case 1, we get a coverage of $F(P)$ by $c-1+\sum_{i=1}^{k}(\lfloor\frac{n_{p_i}-1}{2}\rfloor-1)$ cooperative vertex guards.

*Subcase* 2.*b*. Let the vertices on the convex hull be labeled $x_1, x_2, \ldots, x_c$, in a counterclockwise manner. Without loss of generality we can assume $e_1 = \{x_c, x_1\}$ to be an edge of the polygon on the convex hull and $\{x_1, x_2\}$ to be a pocket lid; let this pocket be labeled $p_1$. We shall construct the required vertex cover.

By Lemma 3.1 pocket $p_1$ can be covered by $\lfloor \frac{n_{p_1}-1}{2} \rfloor$ cooperative vertex guards, with one guard either at $x_1$ or at $x_2$. If there is a guard at $x_2$, then placing $c-2$ additional guards at vertices $x_3, \ldots, x_c$ of the convex hull, and applying an argument similar to that in Case 1, we get a coverage of $F(P)$ by $c - 1 + \sum_{i=1}^{k} (\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative vertex guards. Otherwise, if there are no guards $x_2$ and there is a guard at $x_1$, then let us consider vertex $x_2$:

(1) $x_2$ is one of the endpoints of the pocket lid $\{x_2, x_3\}$ of the next pocket $p_2$, in a counterclockwise manner. Again, by Lemma 3.1 pocket $p_2$ can be covered by $\lfloor \frac{n_{p_2}-1}{2} \rfloor$ cooperative vertex guards, with one guard either at $x_2$ or at $x_3$. If there is a guard at $x_3$, then placing $c-3$ additional guards at vertices $x_4, \ldots, x_c$ of the convex hull, together with $\lfloor \frac{n_{p_1}-1}{2} \rfloor$ guards allocated to pocket $p_1$, $\lfloor \frac{n_{p_2}-1}{2} \rfloor$ guards allocated to pocket $p_2$, and by Lemma 3.1 $\sum_{i=3}^{k} (\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ allocated to the remainders of pockets $p_i$, $i = 3, \ldots, k$, we get a coverage of $F(P)$ by $c - 1 + \sum_{i=1}^{k} (\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative vertex guards. Otherwise, apply (1) at the pocket lid $\{x_3, x_4\}$ or (2) at the edge $\{x_3, x_4\}$.

(2) $\{x_2, x_3\}$ is the edge of the polygon on the convex hull, then place the next guard at vertex $x_3$, and apply the reasoning used in (1) in the case of a guard at vertex $x_3$.

It is clear that the above construction will stop either at (1) or when we have considered the last pocket $p_k$ with the pocket lid $\{x_{c-1}, x_c\}$ and a guard at $x_{c-1}$ in pocket $p_k$ was needed. But in this case, we have $c-1$ guards at vertices $x_1, x_2, \ldots, x_{c-1}$ on the convex hull and $\sum_{i=1}^{k} (\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ guards in the remainders of pockets. Again, all of $F(P)$ is covered by $c - 1 + \sum_{i=1}^{k} (\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative vertex guards.

*Case* 3: there is a pocket of an odd number of vertices. Let the considered pocket be labeled $p_k$, and let $d = \{x_1, x_2\}$ be its pocket lid. Then replacing pocket $p_k$ with the new edge $d$ we get a fortress $\widehat{P}$ of $\hat{k} = k - 1$ pockets and the same number $h$ of vertices on the convex hull. By the induction hypothesis $F(\widehat{P})$ can be covered by $c - 1 + \sum_{i=1}^{k-1} (\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative vertex guards. As there is a guard either at vertex $x_1$ or at $x_2$, then by Lemma 3.1

$$c - 1 + \sum_{i=1}^{k-1} (\lfloor \frac{n_{p_i} - 1}{2} \rfloor - 1) + \lfloor \frac{n_{p_k} - 1}{2} \rfloor - 1 = c - 1 + \sum_{i=1}^{k} \lfloor \frac{n_{p_i} - 1}{2} \rfloor$$

cooperative vertex guards suffice to cover all of $F(P)$. □

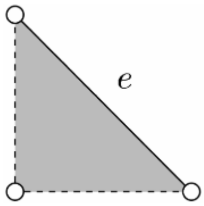


**Fig. 3.** The nose of a slanted edge.

# 5 Vertex guards in an orthogonal fortress

An *orthogonal* polygon is one with only horizontal or vertical edges. Before considering the cooperative guards problem in orthogonal fortresses let us pay attention to the problem of covering the interior of 1-orthogonal polygons.

## 5.1 1-orthogonal polygons

A 1-*orthogonal polygon* is a polygon of no holes with a distinguished edge $e$ called the *slanted edge*, such that the polygon satisfies four conditions:

(1) There are an even number of edges.
(2) Except for possibly $e$, the edges are alternately horizontal and vertical in a traversal of the boundary.
(3) All interior angles are less than or equal to 270°.
(4) The nose of the slanted edge contains no vertices.

The *nose* of a slanted edge is the triangle towards the inside of the polygon whose hypothenuse is $e$; the nose includes the interior of $e$ but exludes the remainder of the boundary, see Fig. 3. The concept of 1-orthogonal polygons was introduced by Lubiv [6].

**Theorem 5.1.** [6] *Any* 1*-orthogonal polygon is convexly quadrilateralizable.*

The existence of a convex quadrilateralization for any 1-orthogonal polygon leads us to the following theorem:

**Theorem 5.2.** $\lfloor \frac{n}{2} \rfloor - 2$ *cooperative vertex guards always suffice to cover the interior of any* 1*-orthogonal polygon with* $n$ *vertices.*

*Proof.* Consider a convex quadrilateralization of an $n$-vertex 1-orthogonal polygon as guaranteed by Theorem 5.1, and let $q$ be the number of quadrilaterals. It easy to check that placing a guard at any endpoint of each internal diagonal that shares two convex quadrilateral we get a guard set $S$ of $P$, with $|S| = q - 1 = \lfloor \frac{n}{2} \rfloor - 2$. It is easy to see that the visibility graph $VG(S)$ is connected. □

## 5.2 $T$-pockets, $F$-pockets and $S$-pockets

Let $P$ be an orthogonal fortress and let us consider the convex hull of $P$ and any pocket $p$. Clearly, the convex hull of $P$ is bounded by four *extermal* edges (northernmost,

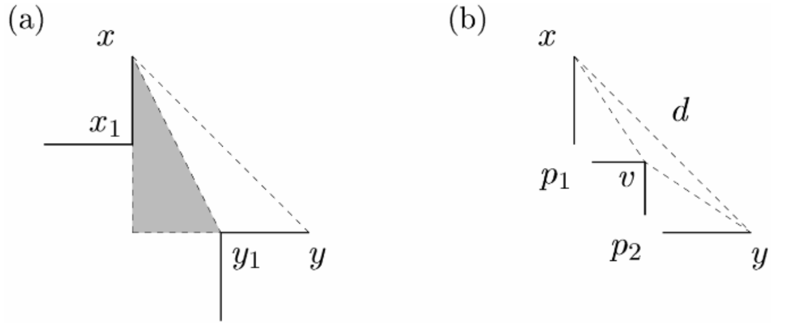


**Fig. 4.** (*a*) Case 1: the nose of a slanted edge $\{x, y_1\}$ is empty; (*b*) Case 2: there is a vertex in the nose of the edge $\{x, y\}$.

westernmost, southernmost, eastermost). As the pocket lid of any pocket of even number of vertices is one of the extremal edges, there are at most four pockets of even number of vertices, all the other pockets are of odd number of vertices. Let $p$ be a pocket with odd number of vertices. If $p$ is of 3 vertices, then we say that it is $T$*-pocket*, if $p$ is of 5 vertices – it is $F$*-pocket*, otherwise – $S$*-pocket*.

**Fact 5.3.** *Any $F$-pocket can be covered by 2 cooperative guards located at the endpoints of its pocket lid.*

**Lemma 5.4.** *Any $S$-pocket $p$ with $n_p$ vertices can be covered by $\lfloor \frac{n_p-1}{2} \rfloor - 1$ cooperative guards with one guard placed at one of the endpoints of its pocket lid.*

*Proof.* Let $p$ be a pocket with $n_p \geq 7$ vertices. We leave to the reader to verify that the lemma holds for $n_p = 7$ and let assume that the lemma holds for all pockets with $\hat{n}_p$ vertices, with $7 \leq \hat{n}_p < n_p$. Let $d = \{x, y\}$ be the pocket lid of $p$ and let $\{x, x_1\}$ and $\{y, y_1\}$ be the edges of the pocket incident to $d$. We need to consider two cases:

*Case* 1: $x$ sees $y_1$ and the nose of the edge $\{y_1, x\}$ is empty, see Fig. 4(a). Replacing the segment $(y_1, y, x)$ with the new edge $\{y_1, x\}$ we get a 1-orthogonal polygon $P$ with $n_p - 1$ vertices, with the slanted edge $\{y_1, x\}$. By Theorem 5.2 polygon $P$ can be covered by $\lfloor \frac{n_p-1}{2} \rfloor - 2$ cooperative vertex guards. The same guard placement in pocket $p$ with one additional guard at $x$ will cover all of $p$ (the triangle $(y_1, y, x)$ is covered be the guard at $x$), and clearly the guard set is cooperative.

*Case* 2: there is a vertex in the nose of the pocket lid $d$, see Fig. 4(b). Let $v$ be the closest vertex to $d$. As $v$ sees both $x$ and $y$, diagonals $\{x, v\}$ and $\{y, v\}$ partition pocket $p$ into the triangle $(x, v, y)$ and two regions $p_1$ and $p_2$, both to be pockets with $n_{p_1}$ and $n_{p_2}$ vertices, respectively.

*Subcase* 2.*a*: $p_1$ and $p_2$ are both $F$-pockets: $n_{p_1} = 5$ and $n_{p_2} = 5$. By Fact 5.3 placing 3 guards at vertices $x$, $v$ and $y$ we get a coverege of the pocekt $p$ by $\lfloor \frac{9-1}{2} \rfloor - 1$ cooperative vertex guards, as $n_{p_1} + n_{p_2} = n_p + 1$.

*Subcase* 2.*b*: $p_1$ is $S$-pocket: $n_{p_1} \geq 7$ and $n_{p_2} \geq 3$. By the induction hypothesis pocket $p_1$ can be covered by $\lfloor \frac{n_{p_1}-1}{2} \rfloor - 1$ cooperative vertex guards with one guard placed

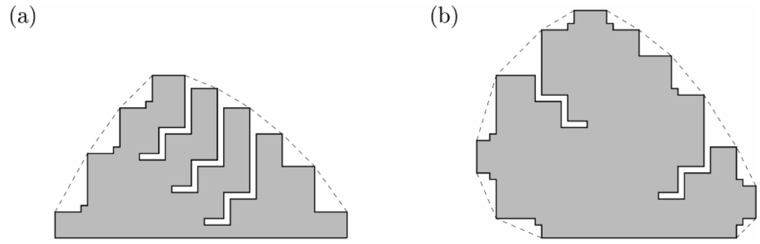


**Fig. 5.** ($a$) An orthogonal fortress that require $3 + t + f + \sum_{i=1}^{s}(\lfloor\frac{n_{p_i}-1}{2}\rfloor - 1)$ cooperative vertex guards; here $t = 2$, $f = 3$, and $s = 3$, each $S$-pocket is of 11 vertices, and the polygon requires 20 cooperative vertex guards.
($b$) An orthogonal fortress that require $4 + t + f + \sum_{i=1}^{s}(\lfloor\frac{n_{p_i}-1}{2}\rfloor - 1)$ cooperative vertex guards, with $f \geq 4$ and $t + s \leq f - 4$; here $t = 1$, $f = 8$, and $s = 2$, each $S$-pocket is of 11 vertices, and the polygon requires 21 cooperative vertex guards.

either at $x$ or at $v$. By Lemma 3.1 pocket $p_2$ can be covered by $\lfloor\frac{n_{p_2}-1}{2}\rfloor$ cooperative vertex guards with one guard placed at $y$. With the same guard placement in $p$ we get a coverage of all of $p$ by

$$\lfloor\frac{n_{p_1}-1}{2}\rfloor - 1 + \lfloor\frac{n_{p_2}-1}{2}\rfloor \leq \lfloor\frac{n_p - 1}{2}\rfloor - 1$$

cooperative vertex guards, as $n_{p_1} + n_{p_2} + 1 = n_p$, and with one guard placed at $y$. □

## 5.3 Theorems

First, let us assume that there are no pockets with even number of vertices. Fig. 5(a) shows a class of orthogonal fortresses that require $3+t+f+\sum_{i=1}^{s}(\lfloor\frac{n_{p_i}-1}{2}\rfloor-1)$ cooperative vertex guards, where $t$ is the number of $T$-pockets, $f$ is the number of $F$-pockets, $s$ is the number of $S$-pockets, and the $S$-pockets $p_i$ are of $n_{p_i}$ vertices, $i = 1, \ldots, s$. Nevertheless, if $f \geq 4$ and $t + s \leq f - 4$, then more guards can be required: Fig. 5(b) shows a class of orthogonal fortresses that require $4+t+f+\sum_{i=1}^{s}(\lfloor\frac{n_{p_i}-1}{2}\rfloor-1)$ cooperative vertex guards (note that any $T$-pocket or $S$-pocket is between two $F$-pockets). We will show these bounds to be tight.

**Theorem 5.5.** *Let $P$ be an orthogonal fortress. Let $t$, $f$ and $s$ be the number of $T$-pockets, $F$-pockets and $S$-pockets in $P$, respectively, and let each of $S$-pockets be of $n_{p_i}$ vertices, $i = 1, \ldots, s$. Then:*

($a$) *if either $f \leq 3$ or $s + t > f - 4$, then $3 + t + f + \sum_{i=1}^{s}(\lfloor\frac{n_{p_i}-1}{2}\rfloor - 1)$ cooperative vertex guards always suffice to cover $F(P)$,*

($b$) *otherwise, $4 + t + f + \sum_{i=1}^{s}(\lfloor\frac{n_{p_i}-1}{2}\rfloor - 1)$ cooperative vertex guards always suffice to cover $F(P)$.*

*Proof.* Let $P$ be an orthogonal polygon and let $e_n, e_e, e_s$ and $e_w$ be its four extremal edges.

$(a)$ As either $f \leq 3$ or $s+t > f-4$, then there are two "consecutive" extremal edges, without loss of generality we can assume that they are $e_w$ and $e_n$, such that between them there are $t_1$, $f_1$ and $s_1$ pockets of type $T$, $F$ and $S$, respectively, and either $f_1 = 0$ or $s_1 + t_1 > f_1 - 4$. Now, applying similar arguments to that in Case 2 of the proof of Theorem 4.1 we can show that there is a vertex $v$ on the convex hull between edges $e_w$ and $e_n$ at which we do not need to place a guard, when we want to guard all of $F(P)$ between edges $e_w$ and $e_n$. Therefore, with $c-1$ guards placed at all vertices on the convex hull, except vertex $v$, together with $\sum_{i=1}^{s} \lfloor \frac{n_{p_i}-1}{2} \rfloor - 2$ cooperative guards for the remainders of all $S$-pockets (by Lemma 5.4), we get a coverage of all of $F(P)$ by $3+t+f+\sum_{i=1}^{s}(\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative guards, since $c-1 = 3+t+f+s$.

$(b)$ If $f \geq 4$ and $s+t \leq f-4$, then with $c$ guards at all vertices of the convex hull, together with $\sum_{i=1}^{s} \lfloor \frac{n_{p_i}-1}{2} \rfloor - 2$ cooperative guards for the remainders of all $S$-pockets (by Lemma 5.4), we get a coverage of all of $F(P)$ by $4+t+f+\sum_{i=1}^{s}(\lfloor \frac{n_{p_i}-1}{2} \rfloor - 1)$ cooperative guards, since $c = 4+t+f+s$. □

If there are (at most four) $m$ pockets of even number of vertices, then by similar arguments to that in the case of no "even"-pockets, by Lemma 3.1, and by the induction on $m$ we get the following:

**Theorem 5.6.** *Let $P$ be an orthogonal fortress. Let $t$, $f$, $s$ and $m$ be the number of $T$-pockets, $F$-pockets, $S$-pockets, and pockets of even number of vertices, respectively, and let each of $S$-pockets be of $n_i$ vertices, $i = 1, \ldots, s$, and let each of "even"-pockets be of $\hat{n}_i$ vertices, $i = 1, \ldots, m$. Then:*

$(a)$ *if either $f \leq 3$ or $s+t > f-4$, then*

$$3+t+f+\sum_{i=1}^{s}(\lfloor \frac{n_i-1}{2} \rfloor - 1) + \sum_{i=1}^{m}(\lfloor \frac{\hat{n}_i}{2} \rfloor - 2)$$

*cooperative vertex guards are sometimes necessary but always sufficient to cover $F(P)$,*

$(b)$ *otherwise,*

$$4+t+f+\sum_{i=1}^{s}(\lfloor \frac{n_i-1}{2} \rfloor - 1) + \sum_{i=1}^{m}(\lfloor \frac{\hat{n}_i}{2} \rfloor - 2)$$

*cooperative vertex guards are sometimes necessary but always sufficient to cover $F(P)$.* □

# 6 Point guards

We have restricted guards to be placed at the vertices of a fortress. However, we can allow guards to be placed at any point of $F(P)$.

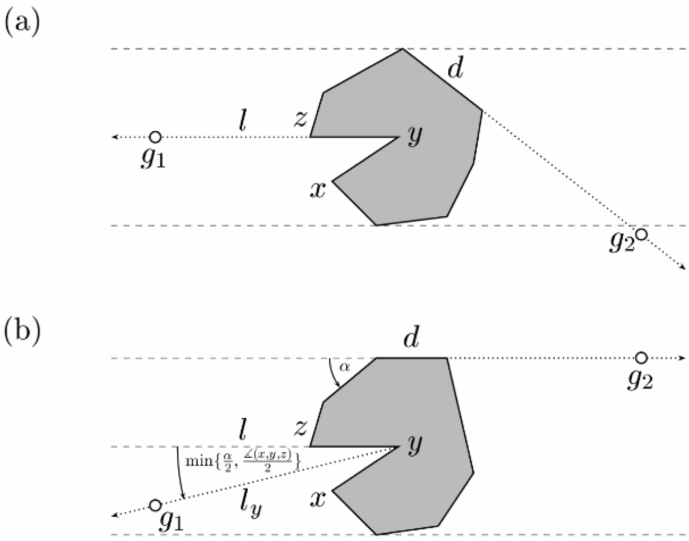


**Fig. 6.** A fortress with one triangle-pocket can be covered be 2 cooperative guards.

First, let us prove a reduced form of the theorem.

**Lemma 6.1.** *An n-vertex fortress with at most one triangle-pocket can be covered be 2 cooperative guards.*

*Proof.* Let $P$ be a convex fortress. Rotate $P$ so that vertex $a$ is uniquely highest and $b$ uniquely lowest. Adding two guards below the lowest vertex of $P$, and both far enough away to see $a$, we will cover all of $F(P)$ by two cooperative guards.

Now, suppose $P$ to be non-convex and that $P$ has only one pocket $(x, y, z)$ of 3 vertices, with $\{x, z\}$ as the pocket lid. Rotate $P$ so that the edge $\{z, y\}$ is horizontal, and let $d$ be the first edge, in the clockwise manner, that is not seen from any point at the line $l$ collinear to the segment $\overline{zy}$, see Fig. 6. We have to consider two cases.

*Case* 1: the edge $d$ is not parallel to line $l$. Then adding two guards, one at line $l$, and the second at the line collinear to edge $d$, both far enough away to see each other and all the edges of $P$, we will get a cover of $F(P)$ by two cooperative guards, see Fig. 6(a).

*Case* 2: the edge $d$ is parallel to line $l$. Let $\alpha$ be an angle between the last edge visible form a point at $l$ and the line collinear to the edge $d$. Let $l_y$ be a line with the angle at $y$ equal to $\min\{\frac{\alpha}{2}, \frac{\measuredangle(x,y,z)}{2}\}$, see Fig. 6(b). Again, adding two guards, one at line $l_y$, and the second at the line collinear to edge $d$, both far enough away to see each other, and to see all the edges of $P$, we will get a cover of $F(P)$ by two cooperative guards. □

**Theorem 6.2.** *Let $P$ be an non-convex fortress of $k$ pockets $p_1, \ldots, p_k$, each of respectively $n_{p_i}$ vertices, $i = 1, \ldots, k$. Then $1 + \sum_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative point guards always suffice to cover $F(P)$.*

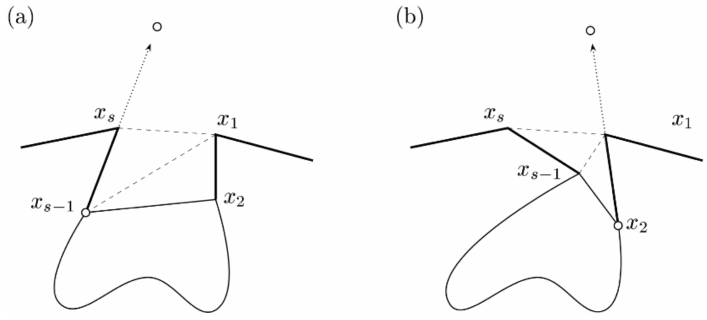


**Fig. 7.** The quadrilateral $Q = (x_1, x_2, x_{s-1}, x_s)$ is empty.

*Proof.* Let $p_1, \ldots, p_k$ be pockets of a fortress $P$, each of respectively $n_{p_i}$ vertices, $i = 1, \ldots, k$. Let us consider the pocket $p_1$. If it is of 3 vertices, then by Lemma 6.1 pocket $p_1$ and the convex hull of $P$ can be covered by 2 cooperative guards. Together with $\sum_{i=2}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards for the others pockets, with one guard per each pocket lid (by Lemma 3.1), we get a coverage of all of $F(P)$ by $2 + \sum_{i=2}^{s} \lfloor \frac{n_{p_i}-1}{2} \rfloor = 1 + \sum_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards.

Next, suppose that $n_{p_1} > 3$ and $n_{p_1}$ is odd. Let us consider a triangulation graph $G_T$ of the pocket $p_1$, and the triangle $t$ of $G_T$ with the pocket lid of $p_1$ as one of its edges. By [4] $\lfloor \frac{n_{p_1}-2}{2} \rfloor$ cooperative guards suffice to dominate $G_T$, and there is a guard at a vertex of $t$. Again, by Lemma 6.1 pocket $p_1$ and the convex hull of $P$ can be covered by $2 + \lfloor \frac{n_{p_1}-2}{2} \rfloor = 1 + \lfloor \frac{n_{p_1}-1}{2} \rfloor$ cooperative guards, as $n_{p_1}$ is odd (these guards are cooperative, as there is a guard in the triangle $t$). Together with $\sum_{i=2}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards for the others pockets, with one guard per each pocket lid (by Lemma 3.1), we get a coverage of all of $F(P)$ by $1 + \sum_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards.

Finally, let us suppose that $n_{p_1}$ is even, and let $x_1, \ldots, x_s$ be the consecutive vertices of the pocket $p_1$, $s = n_{p_1}$, in clockwise manner, with $\{x_s, x_1\}$ as the pocked lid ($s = n_{p_1}$). Let us consider the quadrilateral $Q = (x_1, x_2, x_{s-1}, x_s)$. We have to consider three cases.

*Case* 1: $Q$ is empty and convex, see Fig. 7(a). Then subpocket $(x_2, \ldots, x_{s-1})$ is of $n_{p_1} - 2$ vertices, and by Lemma 3.1 it can be covered by $\lfloor \frac{n_{p_1}-2-1}{2} \rfloor$ cooperative guards, with one guard either at $x_2$ or at $x_{s-1}$ – let us assume, without loss of generality, at $x_{s-1}$. Now, by Lemma 6.1 the triangle $(x_1, x_{s-1}, x_s)$ and the convex hull of $P$ can be covered by 2 cooperative guards. As guard at $x_{s-1}$ covers the triangle $(x_1, x_2, x_{s-1})$, all of the considered pocket $p_1$ is covered. As before, this leads to a coverage of all of $F(P)$ by $2 + \lfloor \frac{n_{p_1}-2-1}{2} \rfloor + \sum_{i=2}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor = 1 + \sum_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards.

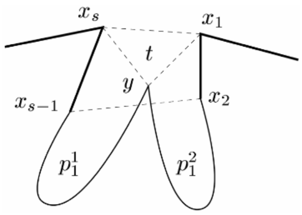


**Fig. 8.** The quadrilateral $Q = (x_1, x_2, x_{s-1}, x_s)$ is not empty.

*Case* 2: $Q$ is empty and non-convex. Let us assume the vertex $x_{s-1}$ to be reflex. If there is a guard at $x_{s-1}$ in a coverage of the subpocket $(x_2, \ldots, x_{s-1})$, we proceed in the same way as in the above case – the guard at $x_{s-1}$ will cover all of the quadrilateral $Q$. Otherwise, all we need is noticing, that a guard at the line collinear to the line segment $\overline{x_1x_2}$ (or close enough to it, the proof of Lemma 6.1, Case 2) will always cover the triangle $(x_1, x_{s-1}, x_s)$, thus all of $Q$ will be covered, see Fig. 7(b).

*Case* 3: there is a vertex in $Q$. Let $y$ be the vertex $\in Q$ closest to the pocket lid $\{x_s, x_1\}$. The pocket $p_1$ can be partitioned into two subpockets $p_1^1$ and $p_1^2$, each of $n_1$ and $n_2$ vertices, respectively, and the triangle $t = (x_1, y, x_s)$, see Fig. 8. As $n_{p_1}$ is even, then either $n_1$ or $n_2$ is odd – let us assume $n_1$ to be odd. By Lemma 3.1 the pocket $p_1^2$ can be covered by $\lfloor \frac{n_2-1}{2} \rfloor$ cooperative guards, with one guard either at $x_1$ or at $y$. If there is a guard at $y$, then by Lemma 3.1 the remainder of the pocket $p_1^1$ can be covered by $\lfloor \frac{n_1-1}{2} \rfloor - 1$ cooperative guards, as $n_1$ is odd. The same construction as in the proof of Lemma 6.1 (we consider the triangle $t$ as a pocket) leads to a coverage of the triangle $t$ (thus, the vertex $y$, as well), and the convex hull of $P$ by 2 cooperative guards. As before, this leads to a coverage of all of $F(P)$ by $2 + \lfloor \frac{n_1-1}{2} \rfloor - 1 + \lfloor \frac{n_2-1}{2} \rfloor + \sum\limits_{i=2}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor \leq 1 + \sum\limits_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards, as $n_1 + n_2 = n_{p_1} + 1$.

If there is a guard at $x_1$, then we have to consider two subcases.

*Subcase* 3.a: the lines $l_1$ and $l_s$ collinear respectively to segments $\overline{x_1x_2}$ and $\overline{x_{s-1}x_s}$ cross in a point $x^* \in F(P)$, see Fig 9. Note that $x^*$ must see $y$. Let us consider a polygon $p_1^*$ that result from replacing the segment $(x_{s-1}, x_s, x_1, x_2)$ in the subpocket $p_1$ by $(x_{s-1}, x^*, x_2)$ – polygon $p_1^*$ is now of $n_{p_1} - 1$ vertices. Next, let us consider a triangulation of $p_1^*$ with $\{y, x^*\}$ as one of its internal diagonals. Then by [4] the polygon $p_1^*$ can be covered by $\lfloor \frac{n_{p_1}-1-2}{2} \rfloor$ cooperative guards, with one guard either at $y$ or at $x^*$. If there is a guard at $y$, again we can proceed in a way similar to that in the proof of Lemma 6.1 (we consider the triangle $(x_1, y, x_s)$ as a pocket). Otherwise, if there is a guard at $x^*$, its clear that two additional cooperative guards will cover all the convex hull of $P$ (and $x^*$). Again, by Lemma 3.1 we will get a coverage of all of $F(P)$ by $1 + \sum\limits_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards.

*Subcase* 3.b: the edges $\{x_1, x_2\}$ and $\{x_{s-1}, x_s\}$ are either parallel or "obtuse". Move $x_1$ along the line collinear to $\overline{x_1x_2}$ far enough to see all possible edges, such transforming the subpocket $p_1^2$, still with $n_2$ vertices, see Fig 9. By Lemma 3.1 the new $p_1^2$ can be

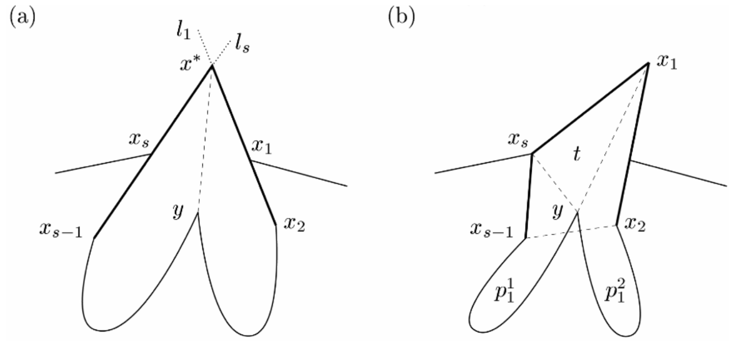


**Fig. 9.** (*a*) Subcase 3.a. (*b*) Subcase 3.b.

covered by $\lfloor\frac{n_2-1}{2}\rfloor$ cooperative guards, with one guard either at $y$ or at new $x_1$. If there is a guard at $y$, then the we can proceed in a way similar to that we have considered before. So assume there is a guard at $x_1$, and let us consider a polygon $p_1^1 \cup (x_1, y, x_s)$ of $n_1 + 1$ vertices. By [4] it can be covered by $\lfloor\frac{n_1+1-2}{2}\rfloor$ cooperative guards, with one guard either at $x_s$ or at $y$, and this guard is seen by the guard at new $x_1$, of course. Thus, all of $p_1^1 \cup (x_1, y, x_s) \cup p_1^2$ can be covered by at most $\lfloor\frac{n_{p_1}-1}{2}\rfloor$ cooperative guards. As the guard at $x_1$ is located far enough, with one additional guard we will cover the pocket $p_1$ and all of the convex hull of $P$ by $1 + \lfloor\frac{n_{p_1}-1}{2}\rfloor$ cooperative guards, only if the first edge not visible from new $x_1$ is not parallel to $\overline{x_1x_2}$ – compare the proof of Lemma 6.1, Case 1. And again, all of $F(P)$ can be covered by $1 + \sum_{i=1}^{k} \lfloor\frac{n_{p_i}-1}{2}\rfloor$ cooperative guards.

Otherwise, if we consider the proof of Lemma 6.1, Case 2, then all we need is the possibility of moving the guard at $x_1$ a small distance $\epsilon > 0$ from $x_1$ along the edge $\{y, x_1\}$ without destroying the co operativeness of guards in the new $p_1^2$ (and so without destroying the cooperativeness of guards in $p_1^1 \cup (x_1, y, x_s) \cup p_1^2$). It can be done by the following argument.

Let $G_T$ be a triangulation graph of an non-degenerated triangulation of an $n$-vertex polygon (there are no triangles with three points on a line), and let $S$ be a guard coverage of an $n$-vertex polygon, with $|S| \leq \lfloor\frac{n-2}{2}\rfloor$, constructed from a cooperative domination of $G_T$ [4]. Let $x$ be a convex vertex with a guard at it – $x$ with all triangles $T_i$ incident to it form a fan $f$. Let $\{x_l, x\}$ and $\{x, x_r\}$ be edges of $P$ incident to $x$, and let $g_1, \ldots, g_k$ be guards incident to $x$ in the fan. As $S$ is constructed form a cooperative domination of $G_T$, it clear that we have to show that the guard at $x$ can be moved without destroying connectivity with these guards only.

For each $g_i$, $i = 1, \ldots, k$, in a sequence:

- rotate $P$ in such a way that the line $s_i$ collinear with the line segment $\overline{xg_i}$ is parallel to y-axis, and $x$ lies below $g_i$;
- consider vertex $r \in f$, closest to the right to line $s_i$, and consider vertex $l \in f$, closest to the left to line $s_i$, respectively (if $g_i = x_l$ or $g_i = x_r$, then assume $r = x_l$, and $l = x_r$, respectively);

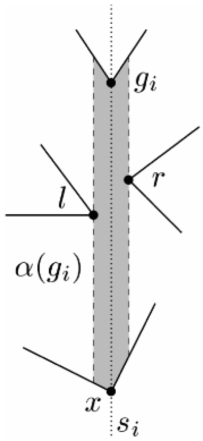


**Fig. 10.** The idea of the construction of the strip $\alpha(g_i)$.

– let $\alpha(g_i)$ be a strip, interior to $P$, delimited by lines going through vertices $l$ and $r$, respectively, and parallel to line $s_i$, see Fig. 10.

It is obvious, that $\bigcap_i \alpha(g_i) \neq \{x\}$, and all $g_i$ are visible from any point $\in \bigcap_i \alpha(g_i)$. Furthermore, $\bigcap_i \alpha(g_i) \cap \{x_l, x\} \neq \{x\}$, and $\bigcap_i \alpha(g_i) \cap \{x, x_r\} \neq \{x\}$. Thus the guard $g$ at $x$ can be moved a small $\epsilon > 0$ either along edge $\{x_l, x\}$ or edge $\{x, x_r\}$, and the guard set $S$ will still remain cooperative.

Thus all of $F(P)$ can be covered by $1 + \sum_{i=1}^{k} \lfloor \frac{n_{p_i}-1}{2} \rfloor$ cooperative guards. □

# 7 Open problems

We have considered the situation when a guard $g_1$ sees another guard $g_2$ if they can be connected with the line segment without the polygon. Nevertheless, we can restrict guards (and only guards) to see each other only when they can be connected with the line segment within the polygon (guards are located at the vertices, of course). This problem seems to be rather different from that one we have considered, and more realistic.

**Conjecture 7.1.** *If guards can see each other only within the polygon,* $\lceil \frac{n}{2} \rceil$ *cooperative guards always suffice to cover* $F(P)$*.*

**Weakly cooperative guards.** Let us recall that a set of guards $S$ is called *weakly cooperative* if the visibility graph $VG(S)$ has no isolated vertices. A convex $n$-gon requires $\lceil \frac{2n}{3} \rceil$ watched vertex guards. From [10] we have:

**Corollary 7.2.** $\lceil \frac{2n}{3} \rceil$ *weakly cooperative vertex guards always suffice to cover the exterior of a simple polygon with* $n$ *vertices.*

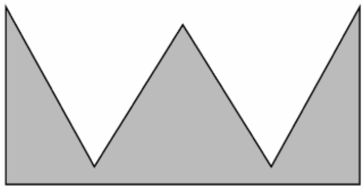

**Fig. 11.** If guards can see each other only within the polygon, then a non-convex fortress can require as many as $\lceil \frac{n}{2} \rceil$ cooperative guards; here $n = 7$ and the fortress requires 4 cooperative guards.

But, as in the case of cooperative guards, it would be desirable to find a more accurate measure of the number of watched guards other than a function of $n$, the number of vertices.

**The Prison Yard Problem.** Finally, it would be interesting to investigate the concept of cooperative guards for *The Prison Yard Problem*, i.e., one wants to determine the number of cooperative guards always sufficient to cover both the interior and the exterior of a polygon. The original problem was solved in 1992 by Füredi and Kleitman [3], who proved that $\lceil \frac{n}{2} \rceil$ vertex guards (respectively $\lfloor \frac{n}{2} \rfloor$) are always sufficient and occasionally necessary to simultaneously guard the interior and the exterior of a convex (respectively non-convex) polygon with $n$ vertices.

# 8 Acknowledgments

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