Wavelet clustering in time series analysis

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Abstract

In this paper we shortly summarize the many advantages of the discrete wavelet transform in the analysis of time series. The data are transformed into clusters of wavelet coefficients and rate of change of the wavelet coefficients, both belonging to a suitable finite dimensional domain. It is shown that the wavelet coefficients are strictly related to the scheme of finite differences, thus giving information on the first and second order properties of the data. In particular, this method is tested on financial data, such as stock pricings, by characterizing the trends and the abrupt changes. The wavelet coefficients projected into the phase space give rise to characteristic cluster which are connected with the volativility of the time series. By their localization they represent a sufficiently good and new estimate of the risk.

Mathematics Subject Classification: 35A35, 42C40, 41A15, 47A58; 65T60, 65D07.
Key words: Wavelet, Haar function, time series, stock pricing.

1 Introduction

In the analysis of time series such as financial data, one of the main task is to construct a model able to capture the main features of the data, such as fluctuations, trends, jumps, etc. If the model is both well fitting with the already available data and is expressed by some mathematical relations then it can be temptatively used to forecast the evolution of the time series according to the given model. However, in general any reasonable model depends on an extremely large amount of parameters both deterministic and/or probabilistic [10, 18, 23, 2, 1, 29, 24, 25, 3] implying a complex set of complicated mathematical relations. Therefore the complexity seems to be a logical consequence of the modelling. However, the every day (practical) experience and the simple known models of physics which describes even the more complicated natural phœnomena tell us that not necessarely the mathematical modelling of a phœnomena should follows from a particularly complicated framework. In physics, even complicated systems with high degrees of freedom, are described by very simple equations and a few set of variables. Therefore we propose in the following a simple model of (financial) data mining [18, 23], i.e. data analysis aiming to define a mathematical model corresponding to data. Our model is based on the simple idea

Balkan Journal of Geometry and Its Applications, Vol.10, No.2, 2005, pp. 33-44.

to describe the characteristic features of the time series by analysing the corresponding wavelet coefficients, i.e. the coefficients of the discrete wavelet transform of data [4, 9, 16, 20, 10, 12, 26, 19, 22, 1, 30].

It is well known that financial time series have a structure very suitable for the wavelet analysis [16, 10, 12, 28, 23, 1, 24, 25], in fact, usually the financial data might be rouglhly represented by the composition of a sequence of high and low frequency "small" waves with a trend. The small waves have a bounded frequency, and are localized according to the fact that what happen at a given time t, in general, has a very negligible influence (correlation) with the other data at time $t' \gg t$. Therefore wavelets are the most significant functions to represent"small" waves or localized wave. In fact, the most important properties of wavelets [26, 19, 22, 1, 29, 30, 17] are:

- the localization in time and frequency,

- the data compression: any function can be analytically represented by a series expansion (in terms of wavelets) having a small number of coefficients (comparing e.g. with the Fourier series).

In the following we use the Haar wavelets, which are the simplest wavelets, analytically defined, compactly supported in time and symmetric, combined with splines [4, 5, 7, 15, 8, 16, 14, 17]. They fit very well with time series represented by histograms (piecewise constant functions), but they are not convenient for the analysis of data to be represented by smoother functions.

The trend of the time series is simply analysed by a smoothing of the data, done by a moving average. In particular, it is shown that the smoothing of the data coincides with the first wavelet coefficient. The remaining coefficients describe the detail, i.e. the local jumps of the data.

In order to investigate the first order differential properties [5, 7, 8, 16, 10, 14] of the histograms we consider first a smoothing process of the histograms, using splines. The spline is derived (once) and discretized at fixed time decomposition, so to get a time-series (first derivative) with the same cardinality of the original time-series. The wavelet coefficients of the first derivative of the time-series together with the wavelet coefficients of the time-series represent a cluster [4, 9, 16, 20, 12] of points in a suitable finite dimensional space. The projections of this cluster into bi-dimensional spaces give interesting information about the volatility of the data. It is shown that the wavelet coefficients are strictly related to the finite differences of various order, and this implies that the volatility of the data is connected with the finite differences.

As application of this method are considered and compared two time series representing the stock pricings STM (STMICROELECRONICS) and BPCV (Banca Popolare di Bergamo Scrl) of Milano Stock Exchange Market, during the periods 8/06/1998 - 1/01/2001 and 24/08/1998 - 1/01/2001 respectively.

2 Preliminary remarks

Let $\boldsymbol{Y} \stackrel{def}{=} \{Y_i\}$, $i = 0, \ldots, N-1$ be the observed data (eventually corrupted by the noise) of a time-series, at the discrete time spots $t_i = i/(N-1)$ ranging on the regular grid of the (dyadic) points of the interval[1] $[0,1]$.

A (discrete) wavelet transform is the linear operator $W : \Re^N \to \Re^N$ which associates to the vector $\boldsymbol{Y}$ the vector of the wavelet coefficients $W\boldsymbol{Y}$:

$$W\boldsymbol{Y} = \{\alpha,\ \beta_0^0,\ \beta_0^1, \ldots,\ \beta_{2^{N-1}-1}^{N-1}\}\ .$$

The Haar wavelet interpolation Q is the wavelet (series) expansion

$$y(t) = Q\boldsymbol{Y} = \alpha\varphi(t) + \sum_{n=0}^{N-1}\sum_{k=0}^{2^n-1} \beta_k^n \psi_k^n(t) \tag{2.1}$$

such that $Y_i = y(t_i)$, and where $\varphi(t)$, $\psi_k^n(t)$ are the Haar scaling function (characteristic function on $[0,1]$)

$$\begin{cases} \varphi_k^n(t) \stackrel{def}{=} 2^{n/2}\varphi(2^n t - k) \quad , \quad (n, k \in Z) \quad , \\ \varphi(2^n t - k) = \begin{cases} 1 & , t \in \Omega_k^n \quad , \quad \Omega_k^n \stackrel{def}{=} \left[\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\right) \\ 0 & , t \notin \Omega_k^n \quad , \end{cases} \end{cases}$$

and the Haar wavelet basis [17, 19] respectively[2]

$$\begin{cases} \psi_k^n(t) \stackrel{def}{=} 2^{n/2}\psi(2^n t - k)\big|_{k,n\in Z} \quad , ||\psi_k^n(t)||_{L^2} = 1 \ , \\ \psi_k^n(t) = \begin{cases} -2^{-n/2} & , \quad t \in \left[\frac{k}{2^n}, \frac{k+1/2}{2^n}\right) \\ 2^{-n/2} & , \quad t \in \left[\frac{k+1/2}{2^n}, \frac{k+1}{2^n}\right) \\ 0 & , \quad \text{elsewhere} \ , \end{cases} \end{cases} \tag{2.2}$$

there results that the Haar wavelet series (2.1) s a piecewise constant function (histogram).

2.1 Wavelet coefficients and finite differences

Let us consider the unit interval divided into four parts:

$$[0,1) = [0,1/4) \cup [1/4,1/2) \cup [1/2,3/4) \cup [3/4,1)$$

and, at the dyadic points $t_0 = 0$, $t_1 = 1/3$, $t_2 = 2/3$, t_1, be defined the vector $\boldsymbol{Y} = \{Y_0, Y_1, Y_2, Y_3\}$, then the wavelet transform is: $W\boldsymbol{Y} = \{\alpha, \beta_0^0, \beta_0^1, \beta_1^1\}$, where

[1]The time series considered in the following range over more general interval that, for convenience, are normalized to the unit interval.

[2]Usually, the function ψ, it is taken positive in the first half of the interval and negative in the second half, but with our choice the wavelet coefficients have a more direct interpretation.

- α is the mean value of the set $\{Y_0, Y_1, Y_2, Y_3\}$:

$$\alpha = 2^{-2} (Y_0 + Y_1 + Y_2 + Y_3) \; ;$$

- β_0^0 is the jump from the mean value in the first half of the interval to the second half:

$$\beta_0^0 = 2^{-1} [(Y_2 + Y_3) - (Y_0 + Y_1)]$$

or equivalently, is the mean value of the first derivatives from the extreme dyadic points and the interior dyadic points:

$$\beta_0^0 = 2^{-1} [(Y_3 - Y_0) + (Y_2 - Y_1)]$$

and it vanishes in case of a symmetric distribution with respect to the middle point ($Y_0 = Y_3, \; Y_1 = Y_2$). It is also easy to recognize that this coefficient represents the average of the first derivatives in the middle point, being the derivative in the second point $\dfrac{Y_2 - Y_0}{2}$ and in the third point $\dfrac{Y_3 - Y_1}{2}$;

- β_0^1 is the jump from the mean value in the first quart of the interval to the second one:

$$\beta_0^1 = 2^{-1/2} (Y_1 - Y_0) \; ,$$

which gives us the slope (first derivative) in first part of the interval;

- β_1^1 is the jump from the mean value in the third quart of the interval to the fourth one:

$$\beta_1^1 = 2^{-1/2} (Y_3 - Y_2) \; ,$$

which gives us the slope (first derivative) in the final part of the interval.

If we remind the (forward) finite difference expressions approximating the first and second derivatives

$$\begin{cases} \dot{Y}_i \overset{def}{=} \left.\dfrac{\mathrm{d}y(t)}{\mathrm{d}t}\right|_{t=t_i} & \cong \quad \Delta_h Y_i \quad = \quad (Y_{i+1} - Y_i)/h \\ \ddot{Y}_i \overset{def}{=} \left.\dfrac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\right|_{t=t_i} & \cong \quad \Delta_h^2 Y_i \quad = \quad \Delta_h(\Delta_h Y_i) = (Y_{i+2} + 2Y_{i+1} - Y_i)/h^2 \; , \end{cases}$$

with $y(t) = Q\boldsymbol{Y}$, it is

$$
\left\{
\begin{array}{lcl}
\alpha & = & \dfrac{1}{4}(Y_0+Y_1+Y_2+Y_3) \\
\beta_0^0 & = & 2^{-7}\Delta_h^3 Y_0 \ , \qquad (\Delta_h^3 \stackrel{def}{=} \Delta_h \Delta_h^2, \ h=2^{-2}) \\
\beta_0^1 & = & 2^{-5/2}\Delta_h Y_0 \\
\beta_1^1 & = & 2^{-5/2}\Delta_h Y_2 \qquad .
\end{array}
\right.
\tag{2.3}
$$

For this reason the wavelet coefficients β's are also called details coefficients, indeed they express the finite difference, or better the first order approximate derivative of the function $Q\boldsymbol{Y}$.

3 First (spline) derivative of a time-series

The details coefficients are strictly related to the first derivative of the time series, however it is possible to define a differential operator on the data as follows. Using the so-called spline-derivative algorithm [4, 5, 7, 9, 13, 14], it is easy to associate to the time series a time-series having the meaning of the first derivative. The first-derivative gives immediately a description of the rate of changes of the data. The spline-derivative algorithm is based on the following steps:

- the time-series is interpolated by a suitable order spline (usually a cubic spline)
- since the spline is enough smooth, it can be derived,
- the first derivative of the spline is then discretized at dyadic nodes, in doing so we obtain a time-series having the menaing of the first-derivative of the original time-series.

Let $\boldsymbol{Y} \cup Y_4 = \{Y_0, \ Y_1, \ Y_2, \ Y_3\} \cup Y_4$, it is possible to define the first approximate derivative $\dot{\boldsymbol{Y}} = \{\dot{Y}_0, \ \dot{Y}_1, \dot{Y}_2, \ \dot{Y}_3\}$, using the finite difference scheme as $\dot{\boldsymbol{Y}} \cong \Delta_h Y = \{\Delta_h Y_0, \ \Delta_h Y_1, \ \Delta_h Y_2, \ \Delta_h Y_3\} = 4\{Y_1 - Y_0, \ Y_2 - Y_1, \ Y_3 - Y_2, \ Y_4 - Y_3\}$. With a simple computation are easily obtained the wavelet coefficients $\{\dot{\alpha}, \ \dot{\beta}_0^0, \ \dot{\beta}_0^1, \ \dot{\beta}_1^1\}$ of $\dot{\boldsymbol{Y}}$

$$
\left\{
\begin{array}{lcl}
\dot{\alpha} & = & Y_4 - Y_0 = \Delta_h \alpha \ , \qquad (h = 2^{-2}) \\
\dot{\beta}_0^0 & = & 2(Y_0 - 2Y_2 + Y_4) = \dfrac{1}{2}\Delta_{2h}^2 Y_0 \\
\dot{\beta}_0^1 & = & 2\sqrt{2}(Y_0 - 2Y_1 + Y_2) = 2^{-5/2}\Delta_h^2 Y_0 \\
\dot{\beta}_1^1 & = & 2\sqrt{2}(Y_2 - 2Y_3 + Y_4) = 2^{-5/2}\Delta_h^2 Y_2 \qquad .
\end{array}
\right.
\tag{3.4}
$$

So that the wavelet coefficients of the derivative $\dot{\boldsymbol{Y}}$, describe the rate of change of the trend α and the coefficients $\dot{\beta}$'s express the second order finite differences, respectively. However it should be noticed that for the computation of the approximate first (forward) derivative $\Delta_h \boldsymbol{Y}$ it is necessary to add the auxiliary value Y_4.

With the same computation we obtain the following set of relations between the Haar wavelet coefficients of the vectors

$$\boldsymbol{Y} \cup Y_8 = \{Y_0, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7\} \cup Y_8$$

with its derivative $\dot{\boldsymbol{Y}}$ and the finite forward differences

$$\begin{cases} \alpha &= \dfrac{1}{8}(Y_0 + Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7) \\ \beta_0^0 &= 2^{-29/2}\Delta_h^3\Delta_{2h}^2 Y_0 \ , \qquad (h = 2^{-3}) \\ \beta_0^1 &= 2^{-4}\Delta_h^3 Y_0 \\ \beta_1^1 &= 2^{-4}\Delta_h^3 Y_4 \\ \beta_0^2 &= 2^{-7/2}\Delta_h Y_0 \\ \beta_1^2 &= 2^{-7/2}\Delta_h Y_2 \\ \beta_2^2 &= 2^{-7/2}\Delta_h Y_4 \\ \beta_3^2 &= 2^{-7/2}\Delta_h Y_6 \qquad . \end{cases}$$

and for the derivatives

$$\begin{cases} \dot{\alpha} &= \Delta_h \alpha \ , \qquad (h = 2^{-3}) \\ \dot{\beta}_0^0 &= 32^{-9/2}\Delta_{3h}^2 Y_0 \\ \dot{\beta}_0^1 &= 2^{-2}\Delta_{2h}^2 Y_0 \\ \dot{\beta}_1^1 &= 2^{-2}\Delta_{2h}^2 Y_4 \\ \dot{\beta}_0^2 &= 2^{-9/2}\Delta_h^2 Y_0 \\ \dot{\beta}_1^2 &= 2^{-9/2}\Delta_h^2 Y_2 \\ \dot{\beta}_2^2 &= 2^{-9/2}\Delta_h^2 Y_4 \\ \dot{\beta}_3^2 &= 2^{-9/2}\Delta_h^2 Y_6 \qquad . \end{cases}$$

Therefore the significant parameters for the wavelet analysis of the time series are the following:

$$N = 4 : \alpha, \beta_0^1, \beta_1^1, \dot{\alpha}, \dot{\beta}_0^1, \dot{\beta}_1^1$$

$$N = 8 : \alpha, \beta_0^2, \beta_1^2, \beta_2^2, \beta_3^2, \dot{\alpha}, \dot{\beta}_0^2, \dot{\beta}_1^2, \dot{\beta}_2^2, \dot{\beta}_3^2$$

where, for $N = 4$, α represents the average value and $\dot{\alpha}$ its finite difference, β_0^1 the first finite difference at the first point t_0 and β_1^1 the first finite difference at the third point t_2; the derivative $\dot{\beta}_0^1$ is the second finite difference in t_0 and $\dot{\beta}_1^1$ is the second finite difference in t_3, analogously for $N = 8$.

In fig.1 are shown the Haar-wavelet interpolations for the stocks STM and BPCV, together with the corresponding derivatives in fig.2.

Thus, from the spline-derivative of the piecewise interpolation of the shares (see fig.2) one has a direct view of the positive — above the horizontal axis — or negative — under the horizontal axis — changes in the rates: much higher is the column above

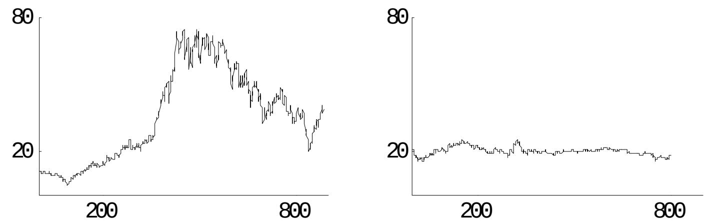

Figure 1: *The stocks STM (left) from 8/06/1998 to 23/11/2001 and BPCV from 24/8/1998 to 23/11/2001 .*

or under the horizontal axis, much faster is the change and, therefore, the *volatility* of the stocks.

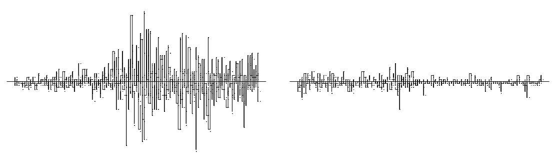

Figure 2: *First spline derivative of the Haar-wavelet interpolation of the stocks STM (left) from 8/06/1998 to 23/11/2001 and BPCV from 24/8/1998 to 23/11/2001.*

3.1 Reduced wavelet interpolation

For a large amount of data, i.e. $N > 8$, we can analyse the longer historical series $\boldsymbol{Y} = \{Y_0,\ Y_1,\ Y_2, \dots\ Y_{N-1}\}$, using only four parameters as follows: we split the total amount of N observations into M segments of four data, so that

$$\boldsymbol{Y} = \bigcup_{i=1}^{M} \boldsymbol{Y}^{(i)}$$

with $Y^{(i)} = \{Y_0^{(i)}, Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)}\}$. Due to the linearity of the wavelet transform, we have $W\boldsymbol{Y} = \bigcup_{i=1}^{M} W\boldsymbol{Y}^{(i)}$ and analogously $W\dot{\boldsymbol{Y}} = \bigcup_{i=1}^{M} W\dot{\boldsymbol{Y}}^{(i)}$ there follows that the wavelet analysis can be done on the cluster of points $\{W\boldsymbol{Y}^{(i)}\}_{i=1,\dots,M}$, belonging to a 4-dimensional space and on the cluster $\{W\boldsymbol{Y}^{(i)}, W\dot{\boldsymbol{Y}}^{(i)}\}_{i=1,\dots,M}$ belonging to a 8-dimensional space.

In fig.3 are represented the STM and BPCV stocks, their spline-derivatives and their corresponding curves of the coefficients α (averaged values), β_0^0, β_0^1,β_1^1.

In fig.4 are represented the projection of the clusters of 3-points and are drawn the ellipsoid quartiles centered on the mean values. It can be seen that nearly the 100 %

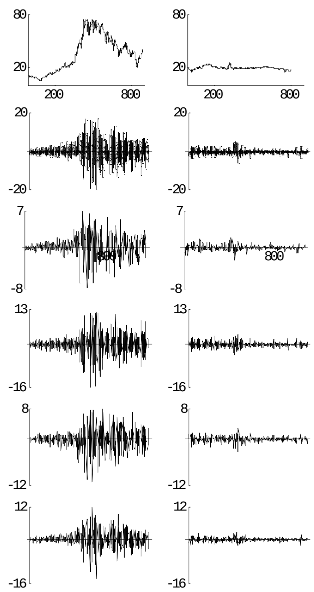

Figure 3: *The stocks STM (top left) and BPCV (top right), the corresponding spline derivatives (second row) and the graphs of the wavelet coefficients (from second row to bottom)* α, β_0^0, β_0^1, β_1^1 *of the spline (first) derivative.*

of the stocks BPCV lies inside the third quantile of the stocks STM. Analogously in fig.5 are projected the clusters of the spline derivatives, while in fig.6 are represented the projections of the cluster of points $\alpha, \beta_0^0, \beta_0^1, \beta_1^1, \dot{\alpha}, \dot{\beta}_0^0, \dot{\beta}_0^1, \dot{\beta}_1^1$

Furthermore, the positive values of the first derivatives data $(A')^I$ correspond to the growing of the data $(A)^I$, while the negative values describe decreasing slopes of the data. Since volatility is strictly connected with the time derivative of the time series, these clusters well represent this feature of the discrete sequences.

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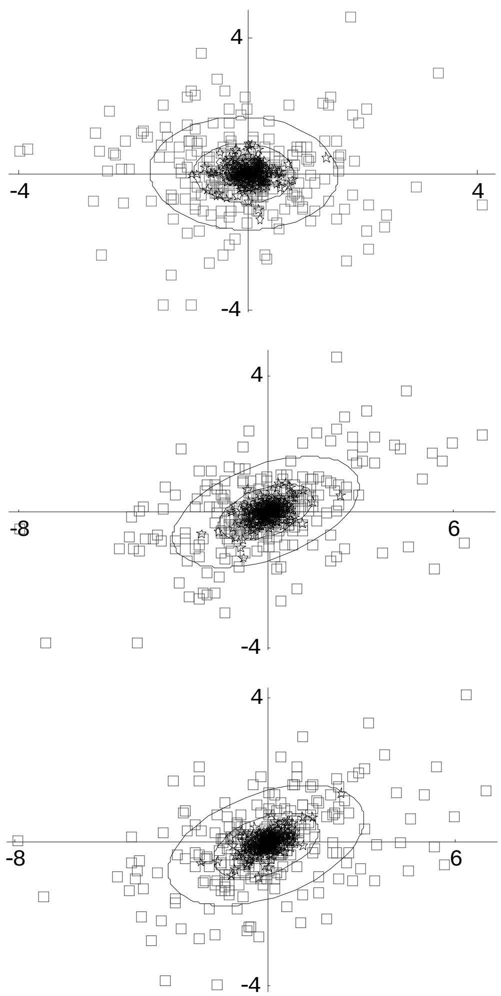

Figure 4: *4-parameter analysis on the 3D clusters (top) of the points having as coordinates the wavelet coefficients* $\beta_0^0, \beta_0^1, \beta_1^1$ *of the data STM (empty squares) and BPCV (stars) in the comparison of their projections on the coordinate planes* $\beta_0^0 = 0$ *(top) ,* $\beta_0^1 = 0$ *(middle)* $\beta_1^1 = 0$ *(bottom), with the ellipsoid quartiles.*

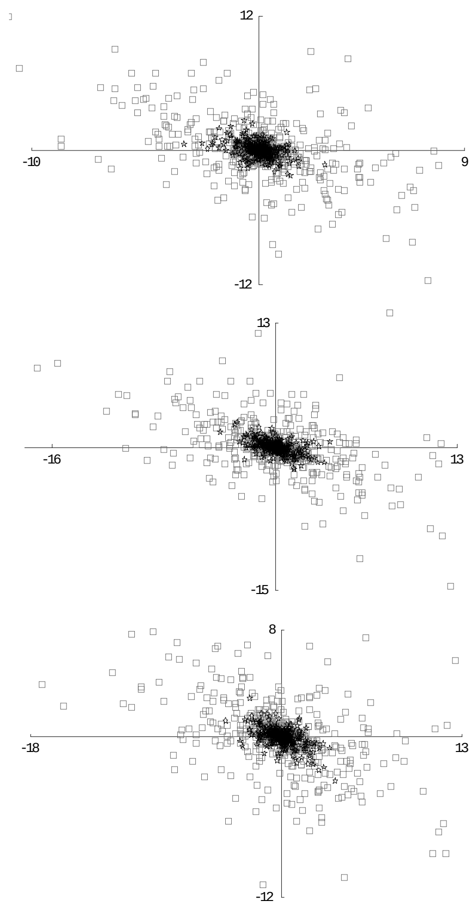

Figure 5: *Projections of the 3D cluster of the points having as coordinates the wavelet coefficients* $\dot{\beta}_0^0, \dot{\beta}_0^1, \dot{\beta}_1^1$ *of the wavelet coefficients of the spline derivatives of the stocks STM (empty squares) and BPCV (stars) on the planes* $\dot{\beta}_0^0 = 0$ *(top)* $\dot{\beta}_0^1 = 0$ *(second row) and* $\dot{\beta}_1^1 = 0$.

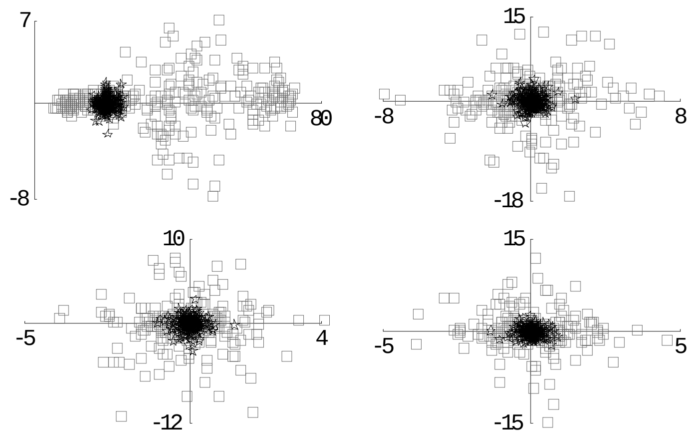

Figure 6: *Projection of STM (empty square) and BPCV (star) with their derivatives into the four planes* $\{\alpha - \alpha'\}$, $\left\{\beta_0^0 - {\beta'}_0^0\right\}$, $\left\{\beta_0^1 - {\beta'}_0^1\right\}$ *and* $\left\{\beta_1^1 - {\beta'}_1^1\right\}$ *(from left to right, from top).*

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