Slant submanifolds with prescribed scalar curvature into cosymplectic space form

Ram Shankar Gupta, S.M.Khrusheed Haider and A.Sharfuddin

Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. In this paper, we have proved that locally there exist infinitely many three dimensional slant submanifolds with prescribed scalar curvature into cosymplectic space form $\overline{M}^5(c)$ with $c \in \{-4, 4\}$while there does not exist flat minimal proper slant surface in $\overline{M}^5(c)$ with $c \neq 0$. In section 5, we have established an inequality between mean curvature and sectional curvature of the subamnifold and have given an example which satisfies the equality sign.

Mathematics Subject Classification: 53B25, 53C40, 53C42.
Key words: slant submanifolds, cosymplectic space form, prescribed scalar curvature, mean curvature.

1 Introduction

The notion of a slant submanifold of an almost Hermitian manifold was introduced by Chen [9]. Examples of slant submanifolds of C^2 and C^4 were given by Chen and Tazawa [11, 12], while that of slant submanifolds of a Kaehler manifold were given by Maeda, Ohnita and Udagawa [22]. On the other hand, A. Lotta [1] has defined and studied slant submanifolds of an almost contact metric manifold. He has also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-Contact manifolds [2]. Later, L. Cabrerizo and others have investigated slant submanifolds of a Sasakian manifold and obtained many interesting results [15, 16]. It was proved in [17] that every surface in a complex space form $\overline{M}^2(4c)$ is proper slant if it has constant curvature and non-zero parallel mean curvature vector. Existence of minimal proper slant surfaces in C^2 have been proved in [10]. In contrast, It was shown in [6] that there does not exist minimal proper slant surfaces in complex projective and complex hyperbolic planes. There exists a slant surface in C^2 with prescribed Gaussian curvature [7]and existence of slant submanifolds in almost contact metric manifolds have been proved in { [1], [15]}.

Also, Chen has established a sharp inequality between mean curvature and Gauss curvature for proper slant surfaces in a complex space form [19]. Similar to this inequality we have established an inequality in section 5 for proper slant submanifolds of cosymplectic manifolds.

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 54-65.

2 Preliminaries

Let$\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g), where φ is a (1,1) tensor field, ξ a vector field, η a 1-form and g the Riemannian metric on $\overline{M}$. These tensors satisfy [13]

$$(2.2.1)\qquad \begin{cases} \varphi^2 X = -X + \eta(X)\xi, \ \ \varphi\xi = 0, \ \ \eta(\xi) = 1, \ \ \eta(\varphi X) = 0; \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi) \end{cases}$$

for any $X, Y \in T\overline{M}$. A normal almost contact metric manifold is called a cosymplectic manifold [13] if

$$(2.2.2)\qquad (\overline{\nabla}_X \varphi)(Y) = 0, \quad \overline{\nabla}_X \xi = 0$$

where$\overline{\nabla}$ denotes the levi-civita connection of $\overline{M}$.
If a cosymplectic manifold $\overline{M}$ has constant ϕ-sectional curvature c, then $\overline{M}$ is called a cosymplectic-space form. The curvature tensor $\overline{R}$ of cosymplectic manifold $\overline{M}$ is given by [13]

$$(2.2.3)\qquad \overline{R}(X, Y)Z = \frac{1}{4}c(g(\varphi Y, \varphi Z)X - g(\varphi X, \varphi Z)Y + \eta(Y)(X, Z)\xi$$

$$-\eta(X)g(Y, Z)\xi + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y + 2g(X, \varphi Y)\varphi Z)$$

for all $X, Y, Z \in T\overline{M}$.
Now, let M be an m-dimensional immersed submanifold of cosymplectic manifold $\overline{M}$.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$(2.2.4)\qquad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), and$$

$$(2.2.5)\qquad \overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for $X, Y \in TM$, $N \in T^{\perp}M$; where h and A_N are the second fundamental forms related by

$$(2.2.6)\qquad g(A_N X, Y) = g(h(X, Y), N)$$

and $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$ of M.
Denote by R the curvature tensor of M and by $R^{\perp}$ the curvature tensor of the normal connection. The equations of Gauss, Ricci and Codazzi are given,respectively, by

(2.2.7)
$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W))$$

$$(2.2.8)\qquad \overline{R}(X, Y, U, V) = R^{\perp}(X, Y, U, V) - g([A_U, A_V]X, Y)$$

$$(2.2.9)\qquad [\overline{R}(X, Y)Z]^{\perp} = (\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z)$$

for all $X, Y, Z, W \in T\overline{M}$ and $U, V \in T^{\perp}M$ where $[\overline{R}(X,Y)Z]^{\perp}$ denotes the normal component of $\overline{R}(X,Y)Z$ and

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.2.10}$$

For any $X \in TM$ and $N \in T^{\perp}M$, we write

$$\varphi X = PX + FX \text{ and } \varphi N = tN + fN \tag{2.2.11}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX,and tN (resp. fN) denotes the tangential (resp. normal) component of φN.

In what follows,we suppose that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \{\xi\}$.

For each non zero X tangent to M at x such that X is not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and T_xM.

The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$ [1]. The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Now, suppose that M is θ-slant in a cosymplectic manifold $\overline{M}$. Then, for any $X, Y \in TM$, we have [20]

$$P^2 = -\cos^2\theta(X - \eta(X)\xi) \tag{2.2.12}$$

If P is the endomorphism defined by (2.2.11), then

$$g(PX, Y) + g(X, PY) = 0 \tag{2.2.13}$$

On the other hand,the Gauss and Weingarten formulae together with (2.2.6) and (2.2.7) imply

$$(\nabla_X P)Y = A_{FY}X + th(X,Y) \tag{2.2.14}$$

$$\nabla^{\perp}_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X, PY) \tag{2.2.15}$$

for any $X, Y \in TM$

We denote,for each $X \in TM$,

$$X^* = \frac{FX}{\sin\theta} \tag{2.2.16}$$

We define the symmetric bilinear TM-valued form ρ on M by

$$\rho(X,Y) = th(X,Y) \tag{2.2.17}$$

Moreover, from (2.2.2), we can obtain

(2.2.18) $$\rho(X,\xi)=0$$

We have proved in [21] that

(2.2.19) $$h(X,Y)=\csc^2\theta(P\rho(X,Y)-\varphi\rho(X,Y))$$

(2.2.20) $$\begin{aligned}R(X,Y,Z,W)=&\cos^2\theta(g(\rho(X,W),\rho(Y,Z))-g(\rho(X,Z),\rho(Y,W)))\\&+\tfrac{c}{4}\{g(Y,Z)g(X,W)-g(X,W)\eta(Y)\eta(Z)-g(X,Z)g(Y,W)\\&+g(Y,W)\eta(X)\eta(Z)+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)\\&+g(PY,Z)g(PX,W)-g(PX,Z)g(PY,W)+2g(X,PY)g(PZ,W)\}\end{aligned}$$

(2.2.21) $$\begin{aligned}&(\nabla_X\rho)(Y,Z)+\csc^2\theta\{P\rho(X,\rho(Y,Z))+\rho(X,P\rho(Y,Z))\}\\&+\tfrac{c}{4}\sin^2\theta\{g(X,PZ)(Y-\eta(Y)\xi)+g(X,PY)(Z-\eta(Z)\xi)\}\\&=(\nabla_Y\rho)(X,Z)+\csc^2\theta\{P\rho(Y,\rho(X,Z))+\rho(Y,P\rho(X,Z))\}\\&+\tfrac{c}{4}\sin^2\theta\{g(Y,PZ)(X-\eta(X)\xi)+g(Y,PX)(Z-\eta(Z)\xi)\}\end{aligned}$$

We recall the following existence and uniqueness theorem for slant immersion into cosymplectic-space-form.

Theorem A (Existence) Let c and θ be two constants with $0<\theta\leq\frac{\pi}{2}$ and M be a simply connected $(m+1)$-dimensional Riemannian manifold with metric tensor g. Suppose that there exist a unit global vector field ξ on M, an endomorphism P of the tangent bundle TM and a symmetric bilinear TM-valued form ρ on M such that for all X,Y,Z $\in TM$,we have

(*i*) $P(\xi)=0,\quad g(\rho(X,Y),\xi))=0,\quad \nabla_X\xi=0$

(*ii*) $P^2=-\cos^2\theta(X-\eta(X)\xi)$

(*iii*) $g(PX,Y)+g(X,PY)=0$

(*iv*) $\rho(X,\xi)=0$

(*v*) $g((\nabla_XP)Y,Z)=g(\rho(X,Y),Z)-g(\rho(X,Z),Y)$

(*vi*) $$\begin{aligned}R(X,Y,Z,W)=&\cos^2\theta(g(\rho(X,W),\rho(Y,Z))-g(\rho(X,Z),\rho(Y,W)))\\&+\tfrac{c}{4}\{g(Y,Z)g(X,W)-g(X,W)\eta(Y)\eta(Z)-g(X,Z)g(Y,W)\\&+g(Y,W)\eta(X)\eta(Z)+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)\\&+g(PY,Z)g(PX,W)-g(PX,Z)g(PY,W)+2g(X,PY)g(PZ,W)\}\end{aligned}$$

and

(*vii*) $$\begin{aligned}&(\nabla_X\rho)(Y,Z)+\csc^2\theta\{P\rho(X,\rho(Y,Z))+\rho(X,P\rho(Y,Z))\}\\&+\tfrac{c}{4}\sin^2\theta\{g(X,PZ)(Y-\eta(Y)\xi)+g(X,PY)(Z-\eta(Z)\xi)\}\\&=(\nabla_Y\rho)(X,Z)+\csc^2\theta\{P\rho(Y,\rho(X,Z))+\rho(Y,P\rho(X,Z))\}\\&+\tfrac{c}{4}\sin^2\theta\{g(Y,PZ)(X-\eta(X)\xi)+g(Y,PX)(Z-\eta(Z)\xi)\}\end{aligned}$$

where η is a dual 1-form of ξ. Then, there exists a θ-slant immersion from M into $\overline{M}^{2m+1}(c)$ whose second fundamental form h is given by

$$h(X,Y)=\csc^2\theta(P\rho(X,Y)-\varphi\rho(X,Y))$$

Theorem B (Uniqueness) Let x^1,x^2:$M\to\overline{M}(c)$ be two slant immersions with slant angle θ $(0<\theta\leq\frac{\pi}{2})$, of a connected Riemannian manifold M^{m+1} into the cosymplectic space-form $\overline{M}^{2m+1}(c)$. Let h^1, h^2 denote the second fundamental forms of x^1 and x^2

respectively. Let there be a vector field $\overline{\xi}$ on M such that $x^1_{*p}(\overline{\xi_p}) = \xi_{x^i(p)}$, for i = 1, 2 and $p \in M$, and

$$g(h^1(X,Y), \varphi x^1_* Z) = g(h^2(X,Y), \varphi x^2_* Z)$$

for all vector fields X, Y, Z tangent to M. Suppose also that we have one of the following conditions:
(i) $\theta = \frac{\pi}{2}$
(ii) there exists a point p of M such that $P_1 = P_2$
(iii) $c \neq 0$
Then there exists an isometry Ψ of $\overline{M}^{2m+1}(c)$ such that $x^1 = \Psi o x^2$.

3 Some Results

Let $r = r(x)$ be a differentiable function defined on an open interval containing 0. Let c and θ be two constants with $0 < \theta \leq \frac{\pi}{2}$ and M be simply-connected domain R^3 containing origin. Consider the following Ricatti differential equation

(3.3.1) $$\psi^{'}(x) + \psi^2(x) + \frac{r(x)}{2} = 0$$

Suppose

(3.3.2) $$f(x) = exp \int \psi(x)dx$$

(3.3.3) $$\eta = dz$$

(3.3.4) $$g = \eta \otimes \eta + dx \otimes dx + f^2(x) dy \otimes dy$$

and

(3.3.5) $$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{f(x)}\frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z}$$

Now, it is easy to verify that $\{e_1, e_2, \xi\}$ is a local orthonormal frame field of TM and η is the dual 1-form of structure vector field ξ. Also, we can obtain

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0,$$
$$\nabla_{e_2} e_1 = \psi e_2, \quad \nabla_{e_2} e_2 = -\psi e_1, \quad \nabla_{e_2} e_3 = 0,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

We define the tensor φ and endomorphism P by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1 \text{ and } \quad \varphi e_3 = \varphi \xi = 0, \quad P = (\cos\theta)\varphi$$

and also define a symmetric bilinear TM-valued form ρ on M as follows:

(3.3.6) $$\rho(e_1, e_1) = \lambda e_1 + \mu e_2, \quad \rho(e_1, e_2) = \mu e_1 + \phi e_2, \quad \rho(e_2, e_2) = \phi e_1 + \delta e_2$$

(3.3.7) $$\rho(e_1, \xi) = 0, \quad \rho(e_2, \xi) = 0, \quad \rho(\xi, \xi) = 0$$

Then,

$$g(\rho(X,Y),Z) = g(\rho(X,Z),Y)$$

for any X,Y,Z tangent to M.
It is easy to verify that (M, P, ρ) satisfies conditions (i)$\sim$(v) of Theorem A. On the other hand, after a lengthy calculation, we obtain that (M, P, ρ) satisfies the remaining two conditions of the existence theorem if

$$\lambda = \frac{1}{\phi}\{\mu^2 + \phi^2 - \mu\delta + [\frac{r(x)}{2} - \frac{c}{4}(1 + 3\cos^2\theta)]\sin^2\theta\} \tag{3.3.8}$$

$$y_1^{'}(x) = \{2y_3^2 - 2y_1y_2 + [\frac{r(x)}{2} - \frac{c}{4}(1 + 3\cos^2\theta)]\sin^2\theta\}\csc\theta\cot\theta$$
$$-3y_1\psi + \tfrac{3c}{4}\sin^2\theta\cos\theta \tag{3.3.9}$$

$$y_2^{'}(x) = \{2y_1y_2 - 2y_3^2 - [\frac{r(x)}{2} - \frac{c}{4}(1 + 3\cos^2\theta)]\sin^2\theta\}\csc\theta\cot\theta$$
$$-(2y_1 - y_2)\psi + \tfrac{3c}{4}\sin^2\theta\cos\theta \tag{3.3.10}$$

$$y_3^{'}(x) = \frac{\psi}{y_3}\{y_1^2 + y_3^2 - y_1y_2 + [\frac{r(x)}{2} - \frac{c}{4}(1 + 3\cos^2\theta)]\sin^2\theta\} - 2y_3\psi$$
$$+\{\tfrac{y_2}{y_3}[y_1^2 + y_3^2 - y_1y_2 + \{\tfrac{r(x)}{2} - \tfrac{c}{4}(1 + 3\cos^2\theta)\}\sin^2\theta - y_1y_3\}\csc\theta\cot\theta \tag{3.3.11}$$

where $\mu = y_1$, $\phi = y_2$, $\delta = y_3$, with initial conditions $y_1(0)=c_1$, $y_2(0)=c_2$ and $y_3(0)=c_3\neq 0$. Thus by applying the Existence Theorem, we know that there exists a θ slant isometric immersion from M into cosymplectic space form $\overline{M}^5(c)$, whose second fundamental form is given by

$$h(X,Y) = \csc^2\theta(P\rho(X,Y) - \varphi\rho(X,Y)). \tag{3.3.12}$$

From (3.3.1) and (3.3.8)$\sim$ (3.3.11), we know that the scalar curvature of the slant submanifold is given by $r(x)$.

Now, we have the following:

Theorem 3.1. *Locally, for any given* θ ($0<\theta\leq\frac{\pi}{2}$) *and for any given function* $r = r(x)$ *there exist infinitely many* θ*-slant submanifolds in complex projective space and in the complex hyperbolic space* $\overline{M}^5(c)$ *with r as prescribed scalar curvature.*

Since for any prescribed scalar curvature $r = r(x)$, the function ψ can be chosen to be any of the solutions of the Riccati equation (3.3.1) and with c_1, c_2, c_3, as any of the three real numbers with $c_3\neq 0$, we have the above theorem.

Now, we give a theorem which shows that the above theorem is not true in general.

Theorem 3.2. *For any* $\theta \in (0, \frac{\pi}{2})$ *, there does not exist* θ*-slant submanifold in the cosymplectic space form* $\overline{M}^5(c)$ *with zero prescribed mean curvature.*
Or, we can also restate it as:
There does not exist flat minimal proper slant surface in $\overline{M}^5(c)$ *with* $c \neq 0$*.*

Proof. Assume that M is a three dimensional flat minimal proper slant submanifold in a non-flat cosymplectic-space form $\overline{M}^5(c)$. Since M is flat, the metric tensor g of M is given by

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz$$

and

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{f(x)}\frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z}$$

Thus $\nabla_{e_i} e_j$ =0. Let θ be the slant angle of M in $\overline{M}^5(c)$. Then

$$Pe_1 = \cos\theta e_2, \quad Pe_2 = -\cos\theta e_1 \quad and \quad P\xi = 0 \tag{3.3.13}$$

Since M is minimal, the second fundamental form h of M in $\overline{M}^5(c)$ takes the following form

$$h(e_1, e_1) = ae_1^* + be_2^*, \quad h(e_1, e_2) = be_1^* - ae_2^*, \quad h(e_2, e_2) = -ae_1^* - be_2^*, \tag{3.3.14}$$

$$h(e_1, e_3) = 0, \quad h(e_2, e_3) = 0, \quad h(e_3, e_3) = 0.$$

for some functions a and b.
Thus, from (3.3.14) and (2.2.17), we have

$$\rho(e_1, e_1) = -\sin\theta(ae_1 + be_2), \quad \rho(e_1, e_2) = -\sin\theta(be_1 - ae_2), \tag{3.3.15}$$

$$\rho(e_2, e_2) = \sin\theta(ae_1 + be_2), \quad \rho(e_1, e_3) = 0, \quad \rho(e_2, e_3) = 0, \quad \rho(e_3, e_3) = 0.$$

Putting $X = Y = e_1$, and $Z = e_2$ in (2.2.21) and using (3.3.13) and (3.3.14), we obtain

$$e_1 b - ae_2 = -\frac{3c}{4}\sin\theta\cos\theta \tag{3.3.16}$$

Similarly, by putting $X = Z = e_2$, and $Y = e_1$ in (2.2.21), we find

$$e_1 b - ae_2 = -\frac{3c}{4}\sin\theta\cos\theta \tag{3.3.17}$$

Combining (3.3.16) and (3.3.17), we get $c\sin\theta\cos\theta$=0, which is a contradiction, since $c\neq 0$ and $\theta\neq 0$ or $\frac{\pi}{2}$, by hypothesis.
Therefore, theorem 3.1 is not true in general. For example, if we replace the scalar curvature by mean curvature, then from theorem 3.2, there does not exist θ-slant submanifold in the cosymplectic space form $\overline{M}^5(c)$ with zero prescribed mean curvature. □

4 Some Explicit solution of Differential system:

Consider the differential system (3.3.1), (3.3.9)∼(3.3.11) with c =±4. Then Ψ=0 is the trivial solution of Ricatti equation (3.3.1) when $r = 0$ and from (3.3.9)∼(3.3.11) , we have

$$y_1'(x) = \{2y_3^2 - 2y_1 y_2\}\csc\theta\cot\theta - \frac{c}{4}(1 + 3\cos 2\theta)\cos\theta \tag{4.4.1}$$

$$y_2'(x) = \{2y_1y_2 - 2y_3^2\}\csc\theta\cot\theta + c\cos\theta \tag{4.4.2}$$

$$y_3y_3'(x) = [y_2\{y_1^2 - y_1y_2 - \frac{c}{4}(1+3\cos^2\theta)\sin^2\theta\} + (y_2 - y_1)y_3^2]\csc\theta\cot\theta \tag{4.4.3}$$

Combining (4.4.1) and (4.4.2), we get

$$y_1'(x) + y_2'(x) = \frac{3c}{2}\cos\theta\sin^2\theta \tag{4.4.4}$$

On integrating (4.4.4), we have

$$y_1(x) + y_2(x) = \frac{3c}{2}\cos\theta\sin^2\theta x - b_1, \text{for some constant } b_1. \tag{4.4.5}$$

Combining (4.4.1) and (4.4.5), we obtain

$$y_1'(x) = \{2y_3^2 + 2y_1^2 + 2b_1y_1\}\csc\theta\cot\theta - 3xy_1c\cos^2\theta - \frac{c}{4}(1+3\cos 2\theta)\cos\theta \tag{4.4.6}$$

Differentiating (4.4.6), we find

$$y_1''(x) = 2\{(b_1 + 2y_1)y_1' + 2y_3y_3'\}\csc\theta\cot\theta \ - 3y_1c\cos^2\theta - 3xy_1'c\cos^2\theta \tag{4.4.7}$$

Therefore, substituting (4.4.3), (4.4.5) and (4.4.6) into (4.4.8), we get

$$y_1''(x) = c\{2b_1 - 3cx\cos\theta + 3xc\cos 3\theta\}\cot^2\theta \tag{4.4.8}$$

Solving (4.4.8), we obtain

$$y_1(x) = b_2 + b_3x + b_1cx^2\cot^2\theta - 8x^3\cos^3\theta \tag{4.4.9}$$

for some constants.

From (4.4.5) and (4.4.9), we have

$$y_2(x) = \frac{3cx}{2}\cos\theta\sin^2\theta - b_1cx^2\cot^2\theta - 8x^3\cos^3\theta - b_1 - b_2 - b_3x \tag{4.4.10}$$

Hence, substituting (4.4.9) and (4.4.10) in (4.4.6), we find

$$\begin{aligned} y_3^2(x) = &-(b_1cx^2\cot^2\theta - 8x^3\cos^3\theta + b_2 + b_3x) \\ &\times(b_1cx^2\cot^2\theta - 8x^3\cos^3\theta + b_1 + b_2 + b_3x) \\ &+\tfrac{c}{8}[1+3\cos 2\theta+12x\cos\theta(b_1cx^2\cot^2\theta-8x^3\cos^3\theta+b_2+b_3x)]\sin^2\theta \\ &+\tfrac{1}{2}(2cb_1x\cot^2\theta - 24x^2\cos^3\theta + b_3)\sin\theta\tan\theta \end{aligned} \tag{4.4.11}$$

If $c = 4$, and b_1=b_2=b_3=0, then we have

$$y_1 = -8x^3\cos^3\theta \tag{4.4.12}$$

$$y_2 = 6x\cos\theta\sin^2\theta + 8x^3\cos^3\theta \tag{4.4.13}$$

(4.4.14)
$$y_3^2(x) = -64x^6\cos^6\theta + \frac{1}{2}[1+3\cos 2\theta - 96x^4\cos^4\theta]\sin^2\theta - 12x^2\cos^3\theta\sin\theta\tan\theta$$

Conversely, it is easy to verify that (4.4.9)∼ (4.4.11) satisfies the differential system (4.4.1)∼ (4.4.3).

5 An Inequality between Mean Curvature and Scalar Curvature for Slant Submanifold.

In the following theorem we have established an inequality between mean curvature and scalar curvature of slant submanifold of a cosymplectic manifold.

Theorem 5.1. *Let M be a proper slant subamnifold in a cosymplectic space-form $\overline{M}^5(c)$ with slant angle θ. Then the squared mean curvature and the scalar curvature of M satisfy*

$$H^2(p) \geq \frac{4}{9}r(p) - (1 + 3\cos^2\theta)\frac{2c}{9} \tag{5.5.1}$$

at each point $p \in M$.

The equality sign of (5.5.1) holds at a point $p \in M$ if and only if, the shape operators of M at p take the following form with respect to a suitable adapted orthonormal frame $\{e_1, e_2, \xi = e_3, e_4, e_5\}$:

$$A_{e_4} = \begin{pmatrix} 3\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.5.2}$$

Proof. Suppose that M is proper slant with slant angle θ in the cosymplectic space form $\overline{M}^5(c)$. Then, for a unit tangent vector field e_1 of M perpendicular to ξ, we put

$$e_2 = (\sec\theta)Pe_1, \quad e_3 = \xi, \quad e_4 = (\csc\theta)Fe_1, \quad e_5 = (\csc\theta)Fe_2.$$

Also, from Corollary 3.1 of [20], we have

$$g(A_{FY}X, Z) = g(A_{FX}Y, Z) \tag{5.5.3}$$

for any $X, Y, Z \in TM$

Then, with respect to adapted orthonormal frame $\{e_1, e_2, \xi = e_3, e_4, e_5\}$ and using (5.5.3), we get

$$A_{e_4} = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} b & c & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.5.4}$$

From (2.2.20) and (5.5.4), we find

$$9H^2 = (a + c)^2 + (b + d)^2, \quad \frac{r}{2} = ac - b^2 + bd - c^2 + (1 + 3cos^2\theta)\frac{c}{4},$$

Or,

$$9H^2(p) - 4r(p) + 2(1 + 3\cos^2\theta)c = (a - 3c)^2 + (3b - d)^2 \geq 0 \tag{5.5.5}$$

and consequently, we get (5.5.1). From (5.5.5), we know that the equality case of (5.5.1) holds at a point p if and only if $a = 3c$, $d = 3b$. Hence, if we choose e_1 in such a way such that Fe_1 is in the direction of the mean curvature vector H , then the shape operators take the form (5.5.2). The converse can be proved by applying (2.2.20).

□

The following result shows that the inequality (5.5.1) is sharp for $\theta\in(0, \frac{\pi}{2})$.

Proposition 5.2. *There exists a three dimensional non-totally geodesic proper slant subamnifold M in cosymplectic space-form* $\overline{M}^5(c)$ *with slant angle* θ *which satisfies the equality sign of (5.5.1) at some points in M.*

Proof. Let $\phi{=}\phi(x)$ and $\phi_i{=}\phi_i(x)$, $i = 1, 2, 3$, be four functions defined on an open interval containing 0. Let $\phi{=}\phi(x)$ be defined such that $\phi(0){=}0$, $b{\neq}0$. Consider the system of first order ordinary differential equations

$$y_1' = -3y_1y_3 + \cot\theta\csc\theta(y_2^2 + y_2\phi)$$

$$y_2' = \phi y_3 - 2y_3y_2 - \cot\theta\csc\theta(y_1\phi + y_2y_1) \tag{5.5.6}$$

$$y_3' = -y_3^2 - \csc^2\theta(\phi y_2 - 2y_1^2 - y_2^2),$$

with the initial conditions $y_1(0) = d_1$, $y_2(0) = d_2$, $y_3(0) = d_3$. Let ϕ_1, ϕ_2 and ϕ_3 be the components of the unique solution of this differentiable system on some open interval containing 0. Let M be a simply connected open neighbourhood of the origin $(0, 0, 0){\in}\Re^3$ endowed with the metric

$$f(x) = exp\int\phi_3(x)dx \tag{5.5.7}$$

$$\eta = dz \tag{5.5.8}$$

$$g = \eta\otimes\eta + dx\otimes dx + f^2(x)dy\otimes dy \tag{5.5.9}$$

and

$$e_1 = \frac{\partial}{\partial x},\quad e_2 = \frac{1}{f(x)}\frac{\partial}{\partial y},\quad e_3 = \xi = \frac{\partial}{\partial z} \tag{5.5.10}$$

Now, it is easy to verify that $\{e_1, e_2, \xi\}$ is a local orthonormal frame field of TM such that

$$\nabla_{e_1}e_1{=}0,\quad \nabla_{e_1}e_2{=}0,\quad \nabla_{e_1}e_3{=}0,$$

$$\nabla_{e_2}e_1 = \phi_3e_2,\quad \nabla_{e_2}e_2 = -\phi_3e_1,\quad \nabla_{e_2}e_3 = 0, \tag{5.5.11}$$

$$\nabla_{e_3}e_1{=}0,\quad \nabla_{e_3}e_2{=}0,\quad \nabla_{e_3}e_3{=}0.$$

We define a symmetric bilinear TM-valued form ρ on M as follows:

$$\rho(e_1, e_1) = \phi e_1 + \phi_1e_2,\ \ \rho(e_1, e_2) = \phi_1e_1 + \phi_2e_2,\ \ \rho(e_2, e_2) = \phi_2e_1 - \phi_1e_2 \tag{5.5.12}$$

$$\rho(e_1, \xi) = 0,\quad \rho(e_2, \xi) = 0,\quad \rho(\xi, \xi) = 0 \tag{5.5.13}$$

It is easy to check that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold and $(\nabla_X\varphi)Y{=}0$, for any X, Y $\in$ TM. We put $P = \cos\theta\varphi$, and after a lengthy calculation, we can show that it satisfy the conditions of Existence Theorem for $c = 0$.

By applying Theorem A, we obtain that there exists a θ-slant isometric immersion from M in $\overline{M}^5(c)$, whose second fundamental form is given by

$$h(X,Y) = \cos^2\theta(P\rho(X,Y) - \varphi\rho(X,Y))$$

From the initial conditions it follows that the shape operators of M take the form of (3.3.2) at the point p=$(0,0,0)$ and satisfy the equality sign of (5.5.1). Also it follows from (5.5.11) that the second fundamental form does not vanish identically. Hence, the submanifold is non-totally geodesic. □

References

[1] A.Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie 39 (1996), 183-198.

[2] A.Lotta, *Three dimensional slant submanifolds of K-contact manifolds*, Balkan J. Geom. Appl. 3, 1 (1998), 37-51.

[3] B.Y.Chen, and L. Vrancken, *Slant surfaces with prescribed Gaussian curvature*, Balkan Journal of Geometry and Its Applications 7, 1 (2002), 29-36.

[4] B.Y.Chen, *Classification of flat slant surfaces in complex Euclidean plane*, J. Math. Soc. Japan 54 (2002), 719-746.

[5] B.Y.Chen, *Flat slant surfaces in complex projective and complex hyperbolic planes*, Results Math. (to appear).

[6] B.Y.Chen and Y. Tazawa, *Slant submanifolds of complex projective and complex hyperbolic spaces*, Glasgow Math. J. 42 (2000), 439-454.

[7] B.Y.Chen and L.Vrancken, *Existence and uniqueness theorem for slant immersions and its applications*, Results Math. 31(1997), 28-39; Addendum, ibid 39 (2001), 18-22.

[8] B.Y.Chen, *On slant surfaces*, Taiwanese J. of Mathematics. Soc. 3, 2 (1999), 163-179.

[9] B.Y.Chen, *Slant immersions*, Bull. Australian Math. Soc. 41 (1990), 135-147.

[10] B.Y.Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, 1990.

[11] B.Y.Chen and Y. Tazawa, *Slant surfaces with codimension 2*, Ann. Fac. Sci. Toulouse Math. XI 3 (1990), 29-43.

[12] B.Y.Chen and Y. Tazawa, *Slant submanifolds in complex Euclidean spaces*, Tokyo J. Math. 14, 1 (1991), 101-120.

[13] D.E.Blair, *Contact manifolds in Riemannian geometry*, Lect. Notes in Math. Springer Verlag, Berlin- New York, 509, 1976.

[14] J.L.Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Existence and uniqueness theorem for slant immersions in Sasakian-space-froms*, Publ. Math.Debrecen 58 (2001), 559-574.

[15] J.L.Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez,*Slant submanifolds in Sasakian manifolds*, Glasgow Math.J. 42 (2000), 125-138.

[16] J.L.Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Structure on a slant submanifold of a contact manifold*, Indian J.pure and appl. Math. 31, 7 (2000), 857-864.

[17] K.Kenmotsu, and D. Zhou, *Classification of the surfaces with parallel mean curvature vector in two dimensional complex space forms*, Amer. J. Math. 122 (2000), 295-317.

[18] K.Matsumoto, I. Mihai and A.Oiaga, *Shape operator for slant submanifolds in complex space forms*, Yamagata Univ. Natur. Sci. 14 (2000), 169-177.

[19] A.Oiaga, and I. Mihai, *B. Y. Chen inequalities for slant submanifolds in complex space forms*, Demonstratio Math. 32 (1999), 835-846.

[20] R.S. Gupta, S. M. Khursheed Haider and M. H. Shahid, *Slant submanifolds of cosymplectic manifolds*, An. Stint. Univ. Iasi, tom. L, s. I. a , (f.1) (2004), 33-50.

[21] R.S. Gupta, S. M. Khursheed Haider and A. Sharfuddin, *Existence and uniqueness theorem for slant immersion and its application into cosymplectic space form*, Publ. Math.Debrecen 67 (2005), 169-188.

[22] S.Maeda, Y. Ohnita and S. Udagawa, *On Slant immersions into Kaehler manifolds*, Kodai Math. J. 16 (1993), 205-219.

Ram Shankar Gupta
Department of Mathematics, Amity School of Engineering, Sector 125, Noida-201301,India.
e-mail:guptarsgupta@rediffmail.com

Authors' address:

S.M.Khrusheed Haider and A.Sharfuddin
Department of Bioscience, Faculty of Natural Sciences,
Jamia Millia Islamia, New Delhi-110025, India.
email: smkhaider@yahoo.co.in