About the Dirac operator

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. We apply a result of Kim about the eigenvalue estimation of the Dirac operator on a Riemannian compact spin manifold (M, g), considering $M = N \times \mathbf{S}^1$, where N is a Riemannian compact spin manifold admitting a parallel vector field.

We show that the lower bounds given in a theorem of Hijazi and Zhang for the eigenvalues of the so called (submanifold) twisted Dirac operator D_H in the case when $H \neq 0$ is true for $H = 0$ also.

As an example, we consider every spin Kähler manifold as a totally geodesic submanifold of its twistor space and we study its twisted Killing spinors.

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1 Introduction

In general, the term "Dirac operator" is used to refer to any first-order operator which factorizes the "Laplacian" for a given quadratic space of arbitrary signature.

In 1928, Dirac [10] introduced a first-order linear operator in order to express the square root of the wave operator $\Box = \frac{\partial^2}{\partial x_0^2} - (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2})$.

Dirac constructed this first-order operator using the Dirac algebra which is a particular realization of the Clifford algebra associated to the real quadratic form of signature (1,3). We briefly review the construction of the Dirac algebra, using the same method that Dirac did. Let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

denote the Pauli matrices. The Pauli matrices are used to construct the Dirac $\gamma-$matrices

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, j = 1, 2, 3.$$

The $\gamma-$matrices satisfy $\gamma_0^2 = I, \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -I, \gamma_j\gamma_k = -\gamma_k\gamma_j, j \neq k$.

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As a differential operator on $\mathbf{C}^4-$valued (i.e. spinor-valued) function on an open set $U \subseteq \mathbf{R}^4$, the Dirac operator $D = \gamma_0 \frac{\partial}{\partial x_0} + \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + \gamma_3 \frac{\partial}{\partial x_3}$ has the desired property $D^2 = \Box$.

Not long after Dirac's original construction, Brauer and Weyl generalized it to any arbitrary finite-dimensional quadratic space of arbitrary signature, using the universal Clifford algebra associated with a real non-degenerate quadratic space [9].

The natural operations for vector spaces with quadratic forms carry over to vector bundles with metrics. In particular, suppose that $\pi : TM \to M$ is the tangent vector bundle of the Riemannian manifold (M, g). Then in each fibre $T_x(M) = \pi^{-1}(x)$, the quadratic form g_x can be used to construct the associated Clifford algebra $Cl(g_x)$. The result is the a bundle $Cl(M) \to M$ of algebras over M called the Clifford bundle of M. In the light of representation theory of Clifford algebra, it is natural to ask whether one can also find a vector bundle $S(M) \to M$ with the property that each fibre S_x is an irreducible module over $Cl(g_x)$. The answer is in general no. But, with some obstructions, if M is an oriented Riemannian manifold and the second Stiefel -Whitney class $w_2(M)$ is vanish, the vector bundle $S(M) \to M$ exist. Then, this will lead to the notion of a spin structure and spin manifold M, using also the theory of morphism fibre bundles.

Let (M, g) be a compact Riemannian spin manifold and $S(M)$ a spinor fibre bundle on M. We can define a canonical first-order differential operator $D : \Gamma(S(M)) \to \Gamma(S(M))$ called the Dirac operator on M, by setting $D\sigma = \sum_{i=1}^{n} e_i.\nabla_{e_i}\sigma$, at $x \in M$, where $\{e_1, ..., e_n\}$ is an orthonormal basis of T_xM, where ∇ denotes the covariant derivative on $S(M)$ determined by the connection and where "." denotes the Clifford module multiplication. The operator D^2 is called the Dirac laplacian.

In 1963, André Lichnerowicz was the first to consider estimating eigenvalues of Dirac operator. He was also the first to provide a rough approximation of this eigenvalues. By integration, from the Schrödinger-Lichnerowicz formula [21]

$$D^2 = \nabla^*\nabla + \frac{s}{4} \tag{1.1}$$

it is clear that all eigenvalues λ of D must satisfy the inequality

$$\lambda^2 \geq \frac{s_0}{4} \tag{1.2}$$

where $s_0 = \min\{s(x) \mid x \in M\}$ is the minimum of the scalar curvature s. Of course, this is interesting only if the scalar curvature is positive but then the estimate (1.2) is never sharp.

In 1980, the first sharp estimation was given by Friedrich [12] who proved [11] that

$$\lambda^2 \geq \frac{n}{4(n-1)} s_0. \tag{1.3}$$

Friedrich's proof is based on a modification of the Levi-Civita connection and a formula similar to (1.1). The boundary of this estimate is characterized by the existence of a non-trivial real Killing spinor on M. The manifolds with non-trivial real Killing spinors have constant positive scalar curvature. They are furthermore Einstein and do not admit any parallel $k-$ form for $k \neq 0, n$. This shows that the estimation (1.3) cannot be sharp in some situations.

In 1986, Hijazi [17] proved that in dimension $n \geq 3, \lambda^2 \geq \frac{n}{4(n-1)}\mu_1$ where μ_1 is the first eigenvalue of the Yamabe operator $Y = 4\frac{n-1}{n-2}\Delta + s$.

Ch. Bär showed [3], [4] that for a closed and connected Riemannian spin manifold of dimension $n = 2$, we have $\lambda^2 \geq \frac{2\pi\chi(M)}{area(M)}$, where $\chi(M)$ is the Euler Poincaré number of M.

B. Alexandrov, G. Grantcharov and S. Ivanov [2] showed that if M admits a parallel $1-$ form, then

$$\lambda^2 \geq \frac{n-1}{n-2}\frac{s_0}{4}. \tag{1.4}$$

Moroianu and Ornea [24] proved that the same holds true under the weaker assumption that the $1-$ form is harmonic and has constant length.

In [1] B. Alexandrov proved that if $TM = \oplus_{i=1}^{k} T^i$, where T^i are parallel distributions of dimension n_i, with $n_1 < ... < n_k$, then

$$\lambda^2 \geq \frac{n_k}{n_k - 1}\frac{s_0}{4} \tag{1.5}$$

and this result is a generalization of one of Kim [19] for $k = 2$.

E. Witten [30] has introduced the hypersurface Dirac operator to prove the positive mass theorem. The spinorial background that has developed to extend the classical estimates to hypersurfaces has now become a powerful tool to investigate extrinsec geometry and manifolds with boundary problems.

Using Hijazi-Zhang's theorem, in [15], [14] lower bounds are given for the eigenvalues of the so called (submanifold) twisted Dirac operator D_H and are discussed their limiting cases if the mean curvature $H \neq 0$. We showed that this study is very natural, because when the considered spin submanifold is minimal, and therefore $H = 0$, Hijazi-Zhang's theorem is true as well. Furthermore, the corresponding (real) *twisted* Killing spinors play the same role as the Killing spinors for spin manifolds. As an example, we have considered every spin Kähler manifold as a totally geodesic submanifold of its twistor space and we study its twisted Killing spinors.

2 First application

Let (N, g) be a $n-$ dimensional Riemannian manifold.

In [29] we have showed:

Theorem 2.1. *Let ξ be a global field on N. Let $D_i : N \times \mathbf{S}^1 \to T_p(N \times \mathbf{S}^1), i = 1, 2$ be the maps $D_1 : p = (x, a) \to T_p^1$, $D_2 : p = (x, a) \to T_p^2$ where*

$$\begin{aligned} T_p^1 &= Sp\{(-\xi_p, 1), (\xi_p, 1)\} \\ T_p^2 &= \{(X, 0) \mid X \in T_p(N), g_p(\xi_p, X) = 0\} \end{aligned}$$

$T_p(N)$ being the tangent space of N on the point p. Then, the maps D_1, D_2 gives rise respectively to two smooth distributions T^1 and T^2 on $NX\mathbf{S}^1$ of dimension 2 and $n-1$ respectively.

Proposition 2.1. *The distributions given by D_1, D_2 are parallel if the vector field ξ is parallel.*

Let (N, g) be a Riemannian, compact $n-$dimensional spin manifold, $n \geq 2$, with fixed spin structure and ξ a parallel vector field on N. On the compact Riemannian spin manifold $(N \times \mathbf{S}^1, G)$, where $G = g + g_0$, g_0 being the standard metric of S^1, there is the global parallel vector field $(\xi, 1)$. This fact allows as to use the formula (1.4), and we have the following estimation for an eigenvalue λ of the Dirac operator on $N \times \mathbf{S}^1$

$$\lambda^2 \geq \frac{n}{n-1}\frac{s_0+1}{4}, \tag{2.1}$$

where s_0 is the minimum of the scalar curvature s of N.

In the other hand, the tangent space $T(N \times \mathbf{S}^1) = T^1 \oplus T^2$, where T^1, T^2 are parallel distributions of dimension $2, n-1$, respectively. Then the first eigenvalue λ of the Dirac operator satisfies, according with (1.5), the inequality

$$\lambda^2 \geq \begin{cases} \frac{n-1}{n-2}\frac{s_0+1}{4} \text{ if } n \geq 3 \\ \frac{s_0+1}{2} \text{ if } n = 2. \end{cases} \tag{2.2}$$

This last estimation is better than (2.1).

The only compact, orientable and therefore spin surface having a global parallel vector field is the two dimensional torus T and we may apply the formula (2.2) for it.

3 Second application

Let $(\widetilde{M}, G)$ be a Riemannian $m+n-$ dimensional spin manifold and let M be an immersed oriented $m-$ dimensional submanifold in $\widetilde{M}$ with the induced Riemannian structure $g = G_{|M}$. Assume that (M, g) is spin. Denote by NM the normal vector bundle of M in $\widetilde{M}$. There exists also a spin structure on NM [23].

Let $\Sigma M, \Sigma N$ and $\Sigma\widetilde{M}$ be the spinor bundles over M, NM and $\widetilde{M}$ respectively. The restricted spinor bundle $S := \Sigma\widetilde{M}_{|M}$ may be considered and is possible to identify it with $\Sigma =: \Sigma M \otimes \Sigma N$ [14] if mn is even. If m and n are both odd, one has to take Σ as the direct sum of two copies of that bundle.

Denote by $(e_1, ..., e_m, \upsilon_1, ..., \nu_n)$ a positively oriented local orthonormal basis of $T\widetilde{M}_{|M}$ such that $(e_1, ..., e_m)$ (resp. $(\upsilon_1, ..., \nu_n)$) is a positively oriented local orthonormal basis of TM (respectively of NM). If $\widetilde{\nabla}$ denotes the Levi-Civita connection of $(\widetilde{M}, G)$, then for all $X \in \Gamma(TM)$, for all $A \in \Gamma(TN)$ and $i = 1, ..., m$, the Gauss formula may be written, as $\widetilde{\nabla}_i(X + A) = \nabla_i(X + A) + h(e_i, X) - h^*(e_i, A)$ where $\nabla_i(X + A) = \nabla^M_{e_i} X + \nabla^N_{e_i} A$, $h^*(e_i, .)$ is the transpose of second fundamental form h viewed as a linear map from TM to NM, and $\widetilde{\nabla}_i$ stands for $\widetilde{\nabla}_{e_i}$.

Denote also by $\widetilde{\nabla}$ and ∇ the induced spinorial covariant derivative on $\Gamma(S)$. Therefore, on $\Gamma(S)$ we have:

$$\widetilde{\nabla} = \begin{cases} (\nabla^{\Sigma M} \otimes Id + Id \otimes \nabla^{\Sigma N}) \oplus (\nabla^{\Sigma M} \otimes Id + Id \otimes \nabla^{\Sigma N}) \text{ if } m, n \text{ odd} \\ \nabla^{\Sigma M} \otimes Id + Id \otimes \nabla^{\Sigma N} \text{ otherwise} \end{cases} \tag{3.1}$$

The spinorial Gauss formula [7] is

$$\widetilde{\nabla}_i\varphi = \nabla_i\varphi + \frac{1}{2}\sum_{j=1}^{2m} e_j.h_{ij}.\varphi, (\forall)\varphi \in \Gamma(S).$$

The following submanifold Dirac's operators [14], may be introduced:

$$\widetilde{D} = \sum_{i=1}^{m} e_i \cdot \widetilde{\nabla}_i, D = \sum_{i=1}^{m} e_i.\nabla_i, D_H = (-1)^n\omega_\perp \cdot D + \frac{1}{2}H.\omega_\perp, \tag{3.2}$$

where we have denoted by $H =: \sum_{i=1}^{m} h(e_i, e_i)$ the mean curvature vector field and where:

$$\omega_\perp = \begin{cases} \omega_n & \text{for } n \text{ even,} \\ i\omega_n & \text{for } n \text{ odd,} \end{cases}$$

ω_n denoting the complex volume form:

$$\omega_n = i^{[\frac{n+1}{2}]}\nu_1 ... \nu_n.$$

In both cases $(\ \omega_\perp)^2 = (-1)^n$.

We consider that (M, g) is a *minimal submanifold* of $(\widetilde{M}, G)$, therefore we have $H = 0$ and for all $\varphi \in \Gamma(S)$

$$D_H\varphi = (-1)^n\omega_\perp D\varphi, \tag{3.3}$$

Recall that there exists a hermitian inner product on $\Gamma(S)$, denoted by $< .,. >$, such that Clifford multiplication by a vector of $T\widetilde{M}_{|M}$ is skew-symmetric. In the following, we write $(.,.) = Re(< .,. >)$.

For any spinor field $\varphi \in \Gamma(S)$ is defined the application $R^N_\varphi : M_\varphi \to \mathbf{R}$ with

$$R^N_\varphi := 2\sum_{i,j=1}^{m}(e_i.e_j.Id \otimes R^N_{e_ie_j}\varphi, \frac{\varphi}{\mid \varphi \mid^2}) \tag{3.4}$$

and $M_\varphi := \{x \in M \mid \varphi(x) \neq 0\}$, where $R^N_{e_ie_j}$ stand for *spinorial normal curvature tensor* [14].

Theorem 3.1. *Let (M, g) be a compact $m-$ dimensional immersed, minimal spin submanifold in the spin manifold $(\widetilde{M}, G)$. Then, denoting the scalar curvature of (M, g) by R, we have*

$$\lambda^2 \geq \inf_{M_\varphi} \frac{1}{4}(R_0 + R^N_\varphi), \tag{3.5}$$

where $R_0 = \inf_M R$ and λ is an eigenvalue of the twisted Dirac operator D_H.

Proof. Using (3.1), the Schrödinger-Lichnerowicz formula for the twisted Dirac operator D_H becomes

$$D_H^2\varphi = \frac{R}{4}(Id \otimes Id)\varphi + \frac{1}{2}\sum_{i,j=1}^{m} e_i e_j Id \otimes R^N_{e_i e_j}\varphi - \sum_{i=1}^{m}\nabla_i\nabla_i\varphi \tag{3.6}$$

and, therefore, we obtain

$$(D_H^2\varphi, \varphi) = \frac{R + R^N_\varphi}{4} \mid \varphi \mid^2 + \mid \nabla\varphi \mid^2 . \tag{3.7}$$

Suppose that $\varphi \in \Gamma(S)$ is a non-zero eigenvalue spinor of the twisted Dirac operator, so that $D_H\varphi = \lambda\varphi$. Therefore, (3.7) gives

$$(\lambda^2 - \frac{R + R^N_\varphi}{4}) \mid \varphi \mid^2 = \mid \nabla\varphi \mid^2 .$$

Assuming by absurd that

$$\lambda^2 - \frac{R + R^N_\varphi}{4} < 0$$

then it results $\varphi = 0$ in contradiction with the hypothesis. Hence the inequality is verified. ■

The estimation (3.5) is not optimal. We have the following:

Theorem 3.2. *Let (M, g) be a compact $m-$ dimensional immersed minimal spin submanifold in the spin manifold $(\widetilde{M}, G)$ and λ an eigenvalue of the twisted Dirac operator D_H. Then, if*

$$R + R^N_\varphi > 0$$

i) the following inequality holds

$$\lambda^2 \geq \frac{m}{m-1} \inf_{M_\varphi} \frac{R + R^N_\varphi}{4}. \tag{3.8}$$

ii) If $\lambda = \pm\sqrt{\frac{1}{2}\frac{m}{m-1}\inf_{M_\varphi}(R + R^N_\varphi)}$ is an eigenvalue corresponding to the eigenspinor φ for the twisted Dirac operator D_H, then the following equations are satisfied:

$$\nabla_X\varphi + \frac{1}{2m}\sqrt{\frac{m}{m-1}(R + R^N_\varphi)}X.\omega_\perp.\varphi = 0 \tag{3.9}$$

$$\nabla_X\varphi - \frac{1}{2m}\sqrt{\frac{m}{m-1}(R + R^N_\varphi)}X.\omega_\perp.\varphi = 0 \tag{3.10}$$

for all $X \in \Gamma(TM)$.

Proof. i) Let φ be an eigenspinor for the twisted Dirac operator D_H, so that

$$D_H\varphi = \lambda\varphi.$$

We consider the modified connection

$$\nabla_i^{\frac{\lambda}{m}} = \nabla_i + (-1)^{[\frac{n}{2}]}\frac{\lambda}{m}e_i.\omega_\perp, (\forall)i = 1, ..., m. \tag{3.11}$$

We can easily compute

$$| \nabla^{\frac{\lambda}{m}}\varphi |^2 = \sum_{i=1}^{m}(\nabla_i^{\frac{\lambda}{m}}\varphi, \nabla_i^{\frac{\lambda}{m}}\varphi) = | \nabla\varphi |^2 - \frac{\lambda^2}{m} | \varphi |^2 . \tag{3.12}$$

The equality (3.12) is a consequence of the fact that the Clifford multiplication with $e_i, i = 1, ..., m$, and $\gamma_j, j = 1, ..., m$ is orthogonal and so

$$(\sum_{i=1}^{m}\nabla_i\varphi, e_i\omega_\perp\varphi) = (-1)^{[\frac{n}{2}]+1}(D_H\varphi, \varphi) = (-1)^{[\frac{n}{2}]+1}\lambda | \varphi |^2 .$$

On the other hand, using Schrödinger- Lichnerowicz formula (3.6), (3.12) and (3.7) we obtain

$$((D_H - \frac{\lambda}{m})^2\varphi, \varphi) = [\frac{R + R_\varphi^N}{4} - \frac{m-1}{m^2}\lambda^2] | \varphi |^2 + | \nabla^{\frac{\lambda}{m}}\varphi |^2$$

and by a direct calculus

$$((D_H - \frac{\lambda}{m})^2\varphi, \varphi) = \frac{(m-1)^2}{m^2}\lambda^2 | \varphi |^2 .$$

Comparing the last two relations, we obtain

$$[\frac{R + R_\varphi^N}{4} - \frac{m-1}{m}\lambda^2] | \varphi |^2 + | \nabla^{\frac{\lambda}{m}}\varphi |^2 = 0. \tag{3.13}$$

Because $\varphi \neq 0$, this equality (3.13) implies (3.8).

ii) If $\lambda = \pm\frac{1}{2}\sqrt{\frac{m}{m-1}\inf_{M_\varphi}(R + R_\varphi^N)}$, the equality (3.13) implies that $R + R_\varphi^N$ is constant, $M_\varphi = M$ and $| \nabla^{\frac{\lambda}{m}}\varphi |^2 = 0$. So, $\nabla^{\frac{\lambda}{m}}\varphi = 0$, and the definition (3.11) implies respectively the equations (3.9), (3.10). ■

Remark 3.1. *Compare theorem 4 with Hijazi-Zhang's theorem [15], [14], which is proved when $H \neq 0$, note that this is also true for $H = 0$, as shown in formula (3.8).*

Let (M, g) be an oriented Riemannian manifold of dimension $m = 2p$ and $Z(M)$ its corresponding twistor space. It is well know that the twistor space $Z(M)$ may be endowed with a natural metric G such that $\pi : Z(M) \to M$ results to be a Riemannian submersion and the vertical and horizontal parts are orthogonal.

Assume that (M, g) is a Kähler manifold. This hypotheses implies that $U(n)$ is the structure group of its principal fibre bundle of orthogonal frames and, moreover, the connection form of the Levi-Civita connection of the metric g is $u(n)-$valued. We have showed [28]

Theorem 3.3. *Let (M, g) be a $2p-$ dimensional Kähler manifold and $Z(M)$ its corresponding space of twistors. Then, there exists $(\widehat{M}, G)$ an immersed oriented submanifold in the twistor manifold $(Z(M), G)$, such that $\pi : \widehat{M} \to M$ is an isometric transformation.*■

Remark 3.2. *Via the projection π we may identify the manifolds $(\widehat{M}, G)$ and (M, g).*

We may consider the normal vector fibre bundle NM of M in $Z(M)$, with respect to the metric G . Assume that (M, g) is spin. It is proved [25], [26] that if M admits a spin structure, then its twistor space $Z(M)$ also admits a spin structure. There exists also a spin structure on NM [23].

Let $\Sigma M, \Sigma N$ and $\Sigma Z(M)$ be the spinor bundles over M, NM and $Z(M)$ respectively. The restricted spinor bundle $S := \Sigma Z(M)_{|M}$ may be considered and is possible to be identified with $\Sigma =: \Sigma M \otimes \Sigma N$ [14].

We have showed [28] that if M is a Kähler manifold, then $\widehat{M}$ is a totally geodesic submanifold of $Z(M)$.

Therefore, the theorems 3 and 4 holds with $m = 2p$ and $n = p(p-1)$, because $H = 0$.

Recall [14] that a non-zero section $\varphi \in \Gamma(S)$ satisfying

$$\nabla_X \varphi = -\frac{\mu}{m} X.\omega_\perp.\varphi, (\forall) X \in \Gamma(TM), \tag{3.14}$$

for a given real constant μ is called a *twisted (real) Killing spinor*.

If $\varphi \in \Gamma(S)$ is a twisted Killing spinor, so that we have (3.14), then it is easy to check that $D_H\varphi = \mu\varphi$ and so, φ is an eigenspinor of the twisted Dirac operator D_H, corresponding to the eigenvalue μ. We have:

Theorem 3.4. *Let (M, g) be a Kähler spin manifold of dimension $m = 2p$, with a twisted Killing spinor φ so that we have (3.14). Then the manifold (M, g) is an Einstein space. Moreover the scalar curvature function R is positive and*

$$\mu^2 = \frac{1}{4}\frac{m}{m-1}R. \tag{3.15}$$

Remark 3.3. *An analysis similar to the one above can be carried out for a Riemannian spin manifold considered as a submanifold of its manifold of euclidean inner products [27], but this submanifold is not minimal.*

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