The L-dual of an (α, β) Finsler space of order two

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Abstract. In ([12]), ([8]) the prolongation to Osc^2M of Riemannian, Finslerian and Lagrangian structures were introduced. These allow us to construct, in this paper, a Randers, Kropina and Matsumoto space of second order and also to give the $\mathcal{L}$-dual of these special Finsler spaces of order two, using Legendre transformation of second order.

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1 Introduction

The $\mathcal{L}$-duality of Finsler and Lagrange spaces was introduced by R. Miron ([10]) and was intensively studied by others, including the last author of this article.

Concrete cases of Hamiltonians obtained by $\mathcal{L}$-duality methods were also constructed. In special the $\mathcal{L}$-duals of some (α, β)-metrics like Randers and Kropina are quite interesting ([4]), ([5]). Moreover, very recently ([13]) have succeeded to compute the $\mathcal{L}$-dual of another famous (α, β)-metric, namely the Matsumoto metric. We have, actually, shown that the $\mathcal{L}$-dual of a Matsumoto metric is a Hamiltonian written by means of four quadratic forms and a 1-form.

A natural question arises: what are the duals of second order Randers, Kropina and Matsumoto metrics? In the present paper this is the question we are going to answer.

By means of the prolongation of a Riemannian metric to second order introduced by R. Miron ([10]) we define the second order Randers, Kropina and Matsumoto spaces. Using then the second order Legendre transformation ([9]) we compute the $\mathcal{L}$-duals of these metrics. The $\mathcal{L}$-duals obtained are the first order Randers, Kropina and Matsumoto spaces, respectively. Initially we hoped to obtain same second order metrics as the duals of second order Randers, Kropina and Matsumoto metrics, respectively. The present paper is a kind of rigidity result showing that using the definitions of the second order (α, β)-metrics and the $\mathcal{L}$-duality defined by (2.12) the

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$\mathcal{L}$-duals of these second order metrics can be only first order (α, β)-metrics. The main reasons behind the rigidity is the definition (2.12) of $\mathcal{L}$-duality.

The present rigidity results easily extend to the higher-order.

2 The Legendre transformation

Let us consider a Lagrange space of order two ([8]) denoted by $L^{(2)n} = (M, L(x, y^{(1)}, y^{(2)}))$, where $L : (x, y^{(1)}, y^{(2)}) \in T^2M \longrightarrow L(x, y^{(1)}, y^{(2)}) \in R$ is the fundamental function and c_{ij} is the fundamental metric tensor given by:

$$c_{ij}(x, y^{(1)}, y^{(2)}) = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(2)i} \partial y^{(2)j}}. \tag{2.1}$$

If M is paracompact manifold, the existence of second order Lagrange spaces, with positively defined fundamental tensor field is always assured ([8]). In this case, there is also a Riemannian metric a on M. Then, the Liouville d-vector

$$z^{(2)i} = y^{(2)i} + \frac{1}{2}\gamma^i_{jk} y^{(1)j} y^{(1)k}, \tag{2.2}$$

is globally defined on $\tilde{E}$, where

$$\tilde{E} = Osc^2M \backslash \{0\} = \{(x, y^{(1)}, y^{(2)}) \in Osc^2M | rank \|y^{(1)i}\| = 1\}$$

and it depends only on the metric a. Here γ^i_{jk} are the Christoffel symbols of Riemannian metric a.
The Liouville d-vector $z^{(2)i}$ allows us to construct not only the regular Lagrangian:

$$L(x, y^{(1)}, y^{(2)}) = \frac{1}{2}\big(a_{ij}(x) z^{(2)i} z^{(2)j}\big)^2,$$

but also some others, for example, putting $\alpha^2 = a_{ij}(x) z^{(2)i} z^{(2)j}$ and $\beta = b_i(x) z^{(2)i}$ a differential linear function in $z^{(2)i}$. This is the Prolongation of a Riemannian metric to Osc^2M, introduced by R. Miron in ([10]).

It is known, ([9]), a Finsler space of order two $F^{(2)n} = (M, F(x, y^{(1)}, y^{(2)}))$ is a Lagrange space of second order $L^{(2)n} = (M, L(x, y^{(1)}, y^{(2)}))$ with

$$L(x, y^{(1)}, y^{(2)}) = F^2(x, y^{(1)}, y^{(2)}), \tag{2.3}$$

having the fundamental function F positively, 2-homogeneous with respect to $y^{(2)i}$, the fundamental tensor c_{ij} positively defined. In this way, we can define an (α, β) **Finsler spaces of order two** as follows:

1. **a Randers space of second order** having the fundamental function:

$$F(x, y^{(1)}, y^{(2)}) = \alpha(x, y^{(1)}, y^{(2)}) + \beta(x, y^{(1)}, y^{(2)}), \tag{2.4}$$

2. **a Kropina space of order two** with fundamental function:

$$F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2(x, y^{(1)}, y^{(2)})}{\beta(x, y^{(1)}, y^{(2)})}, \tag{2.5}$$

3. **a Matsumoto space of order two** with:

$$F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2(x, y^{(1)}, y^{(2)})}{\alpha(x, y^{(1)}, y^{(2)}) - \beta(x, y^{(1)}, y^{(2)})}. \tag{2.6}$$

The fundamental function is called, like in classical case, an (α, β)-metric if F is homogeneous of α and β of degree two.

Let us consider a Hamilton space of order two $H^{(2)n} = (M, H(x, y, p))$ with the regular Hamiltonian $H : T^{*2}M \longrightarrow R$, differentiable on $T^{*\tilde{2}}M$ and continuous on the zero section of the projection $\pi^{*2} : T^{*2}M \longrightarrow M$, having the fundamental tensor field:

$$g^{ij}(x, y, p) = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}, \tag{2.7}$$

with constant signature on the manifold $T^{*2}M$.
Let $C^{(2)n} = (M, K(x, y, p))$ be a Cartan space of order two. From ([12]) is known, that is a Hamilton space of second order $H^{(2)n}$ for which the fundamental function $H(x, y, p)$ is 2-homogeneous with respect to momenta p_i and

$$H(x, y, p) = K^2(x, y, p). \tag{2.8}$$

Proposition 1.1 ([12]) *For any Cartan space of order two we have:*

1. *The components $g^{ij}(x, y, p)$ of the fundamental tensor are 0-homogeneous with respect to p_i.*

2. $$\frac{1}{2} \frac{\partial K^2}{\partial p_i} = g^{ij} p_j, \tag{2.9}$$

3. $$g^{ij}(x, y, p) p_i p_j = K^2(x, y, p), \tag{2.10}$$

4. $$p_i C^{ijk} = 0, \quad C^{ijk} = -\frac{1}{4} \frac{\partial^3 K^2}{\partial p_i \partial p_j \partial p_k}. \tag{2.11}$$

Let $g_{ij}(x, y, p)$ be the covariant tensor of $g^{ij}(x, y, p)$.
A point $(x, y^{(1)}, y^{(2)})$ of the manifold T^2M will be denoted by $(y^{(0)}, y^{(1)}, y^{(2)})$ and its coordinates by $(y^{(0)i}, y^{(1)i}, y^{(2)i})$ ([9]).

For a Lagrange space of order two, one can consider ([12]) a local diffeomorphism $\varphi : T^2M \longrightarrow T^{*2}M$ which preserves the fibers, in the following:
Proposition 1.2 ([12]) *If L is a fundamental function of a Lagrange space of order two, $L^{(2)n}$, then, the following mapping:*

$$\varphi : (y^{(0)}, y^{(1)}, y^{(2)}) \in T^{\tilde{2}}M \longrightarrow (x, y, p) \in T^{*\tilde{2}}M, \tag{2.12}$$

given by:

$$x^i = y^{(0)i}, \quad y^i = y^{(1)i}, \quad p_i = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}},$$

is a local diffeomorphism which preserves the fibers.

This local diffeomorphism is called the **Legendre transformation of second order**.
It is also known, ([12]),

$$p_i = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}} = \varphi_i(y^{(0)}, y^{(1)}, y^{(2)}),$$

and φ_i is a d-covector field on $L^{(2)n}$.
The inverse local diffeomorphism $\xi = \varphi^{-1}$ is given by:

$$y^{(0)i} = x^i, \quad y^{(1)i} = y^i, \quad y^{(2)i} = \xi^i(x, y, p).$$

The mappings φ and ξ satisfy the conditions:

$$\xi \circ \varphi = 1_{\check{U}}, \quad \varphi \circ \xi = 1_{\hat{U}}, \quad \check{U} = (\pi^2)^{-1}(U), \quad \hat{U} = (\pi^{*2})^{-1}(U), \quad U \subset M.$$

The following identities hold good:

$$c_{ij}(y^{(0)}, y^{(1)}, y^{(2)}) = \frac{\partial \varphi_i}{\partial y^{(2)j}}, \quad c^{ij}(x, y, \xi(x, y, p)) = \frac{\partial \xi^i}{\partial p_j},$$

and

$$\begin{aligned} \frac{\partial \varphi_i}{\partial x^j} &= -c_{is}\frac{\partial \xi^s}{\partial x^j}; \quad \frac{\partial \varphi_i}{\partial y^j} = -c_{is}\frac{\partial \xi^s}{\partial y^j}; \quad \frac{\partial \varphi_i}{\partial y^{(2)j}} = c_{ij}; \\ \frac{\partial \xi^i}{\partial x^j} &= -c^{is}\frac{\partial \varphi_s}{\partial x^j}; \quad \frac{\partial \xi^i}{\partial y^j} = -c^{is}\frac{\partial \varphi_s}{\partial y^j}; \quad \frac{\partial \xi^i}{\partial p_j} = c^{ij}. \end{aligned}$$

By means of the mapping φ, ([12]) a regular Lagrangian $L(y^{(0)}, y^{(1)}, y^{(2)})$ is transformed into a regular Hamiltonian. Notice that $y^{(2)i} = \xi^i(x, y, p)$ is not a vector field. Therefore, the product $p_i\xi^i(x, y, p)$ is not a scalar field as in the classical case of the Hamiltonian space $H^{(1)n} = (M, H(x, p))$.
Still, the Liouville d-vector in T^2M:

$$z^{(2)i} = y^{(2)i} + \frac{1}{2}\gamma^i_{jk}y^{(1)j}y^{(1)k}.$$

is transformed by φ in the following d-vector field on $T^{*2}M$:

$$\check{z}^{(2)i} = \xi^i(x, y, p) + \frac{1}{2}\gamma^i_{jk}y^{(1)j}y^{(1)k}, \tag{2.13}$$

Theorem 1.1 ([12]) *The Hamiltonian*

$$H(x,y,p) = 2p_i\check{z}^{(2)i} - L(x,y,\xi(x,y,p)) \tag{2.14}$$

is the fundamental function of the Hamiltonian space $H^{(2)n}$ and its fundamental tensor field $c^{ij}(x,y,\xi(x,y,p))$ is the contravariant of the fundamental tensor field c_{ij} of the space $L^{(2)n} = (M,L)$.

This space, $H^{(2)n} = (M,H)$ is called the dual of the space $L^{(2)n} = (M,L)$.
In addition, ([12]),

$$\frac{1}{2}\frac{\partial H}{\partial p_i} = \check{z}^{(2)i}, \tag{2.15}$$

and

$$g^{ij}(x,y,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j} = \frac{\partial \check{z}^{(2)j}}{\partial p_i} = \frac{\partial \xi^j}{\partial p_i} = c^{ij}(x,y,\xi(x,y,p)). \tag{2.16}$$

Moreover, for a Cartan space of order two $C^{(2)n} = (M,K(x,y,p))$ and a Finsler space of order two $F^{(2)n} = (M,F(x,y^{(1)},\xi(x,y,p)))$, using the property of homogeneity of K^2 and F^2 with respect to momenta p_i, respective $y^{(p_i)}$, from Proposition 1.1, we have:

$$K^2 = 2p_i\check{z}^{(2)i} - F^2 = p_i\frac{\partial K^2}{\partial p_i} = 2g^{ij}p_ip_j - F^2 = 2K^2 - F^2,$$

i.e.

$$K^2(x,y,p) = F^2(x,y^{(1)},\xi(x,y,p)) \tag{2.17}$$

3 The $\mathcal{L}$-dual of an (α,β) space of order two

Let $F^{(2)n} = (M,F)$ be an (α,β)-metric Finsler space of second order defined as above, with $\alpha^2 = a_{ij}(x)z^{(2)i}z^{(2)j}$ and $\beta = b_i(x)z^{(2)i}$ and $z^{(2)i}$ from (2.2).
Inspired by ([14]), let us choose:

$$\begin{aligned} z_i &= \frac{1}{2}\frac{\partial F^2}{\partial z^{(2)i}}, \quad l_i = \frac{1}{\alpha}a_{ij}z^{(2)j} = \frac{1}{\alpha}z_i, \quad l^i = a^{ij}l_j, \\ b^i &= a^{ij}b_j, \quad d_i = b_i + l_i, \quad h_{ij} = a_{ij} - l_il_j, \end{aligned}$$

where h_{ij} is the angular metric tensor of the space (M,a_{ij}) .
We also choose like in ([12]):

$$p^i = a^{ij}p_j, \quad \alpha^* = p_ip^i = a^{ij}p_ip_j, \quad \beta^* = b_ip^i$$

Theorem 2.1 *Let (M,F) be a Randers space of order two and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then:*

1. If $b^2 = 1$, the $\mathcal{L}$-dual of (M, F) is a Kropina space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\frac{a^{ij}p_ip_j}{2b^ip_i}\right)^2. \tag{3.1}$$

2. If $b^2 \neq 1$, the $\mathcal{L}$-dual of (M, F) is a Randers space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\sqrt{\tilde{a}^{ij}p_ip_j} \pm \tilde{b}^ip_i\right)^2, \tag{3.2}$$

where

$$\tilde{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^ib^j; \quad \tilde{b}^i = \frac{1}{1-b^2}b^i,$$

(in (3.2) $'-'$ corresponds to $b^2 < 1$ and $'+'$ corresponds to $b^2 > 1$).

Proof: By using the Theorem 1.1, the fundamental tensor field r^{ij} of space $H^{(2)n}$ is the contravariant of the fundamental tensor field r_{ij} of the space $F^{(2)n}$. Therefore, we have to find the fundamental and its contravariant tensor field of $F^{(2)n}$. Now, inspired by ([14]), using the same model, we find for $F = \alpha + \beta$:

$$r_{ij} = \frac{\alpha+\beta}{\alpha}h_{ij} + d_id_j, \tag{3.3}$$

and

$$r^{ij} = \frac{\alpha}{\alpha+\beta}a^{ij} - \frac{\alpha^2}{(\alpha+\beta)^2}(b^il^j + b^jl^i) + \frac{\alpha^2(\alpha b^2+\beta)}{(\alpha+\beta)^3}l^il^j. \tag{3.4}$$

¿From Proposition 1.1 we know that $r^{ij}(x,y,p)p_ip_j = K^2(x,y,p)$. So, we compute $r^{ij}p_ip_j$ and we find:

$$K^2 = \frac{\alpha}{F} - 2\alpha\beta^* + F\alpha(b^2-1) + F^2.$$

Using now formula (2.2), we find, like in classical case $C^{(1)n} = (M, K)$ ([12]):

$$F^2(b^2-1) - 2\beta^*F + \alpha^{2*} = 0,$$

and for $b^2 = 1$ we get $F = \frac{\alpha^{2*}}{2\beta^*}$ and $b^2 \neq 1$

$$\left(F - \frac{\beta^*}{b^2-1}\right)^2 = \frac{\beta^*}{(b^2-1)^2} - \frac{\alpha^*}{b^2-1}$$

which means:

$$F = \frac{\beta^*}{b^2-1} \pm \sqrt{\frac{\beta^*}{(b^2-1)^2} - \frac{\alpha^*}{b^2-1}}.$$

Setting now, for first case $\tilde{b}^i = 2b^i$ we get statement 1, and for the second case $\tilde{b}^i = \frac{1}{b^2-1}b^i$, $\tilde{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^ib^j$ we get 2.

Remark: One can prove Theorem 2.1 in the same as in the classical case $C^{(1)n} = (M, K)$, namely:

$$p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^{(2)i}} = F\Big(\frac{a_{ij}z^{(2)j}}{\alpha} + b_i\Big), \tag{3.5}$$

$$\xi^i = \frac{\alpha}{F}a^{ij}(p_j - Fb_j) - \frac{1}{2}\gamma^i_{jk}y^{(1)j}y^{(1)k}, \tag{3.6}$$

$$\check{z}^{(2)i} = \frac{\alpha}{F}a^{ij}(p_j - Fb_j). \tag{3.7}$$

Now, contracting in (3.5) by p^i and b_i we get:

$$\alpha^* = F\Big(\frac{F^2}{\alpha} + \beta^*\Big), \tag{3.8}$$

$$\beta^* = F\Big(\frac{\beta}{\alpha} + b^2\Big). \tag{3.9}$$

Therefore,

$$\beta^* = F\Big(\frac{F}{\alpha} + b^2 - 1\Big). \tag{3.10}$$

For (3.8) and (3.9) by substitution, like in case $C^{(1)n} = (M, K)$, ([12]), for $b^2 = 1$ we get $F = \frac{\alpha^{2*}}{2\beta^*}$ and for $b^2 \neq 1$

$$\Big(F + \frac{\beta^*}{1-b^2}\Big)^2 = \frac{\beta^*}{(1-b^2)^2} + \frac{\alpha^*}{1-b^2},$$

proving in this way our theorem.

In other words, using α^* and β^* the Theorem 2.1 can be written:

Remark 2.1 ([13]) *Let (M, F) be a Randers space of order two and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then:*

1. *If $b^2 = 1$, the $\mathcal{L}$-dual of (M, F) is a Kropina space on T^*M with:*

$$H(x,p) = \frac{1}{2}\left(\frac{\alpha^{*2}}{2\beta^*}\right)^2. \tag{3.11}$$

2. *If $b^2 \neq 1$, the $\mathcal{L}$-dual of (M, F) is a Randers space on T^*M with:*

$$H(x,p) = \frac{1}{2}\left(\alpha^* \pm \beta^*\right)^2, \tag{3.12}$$

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_ip_j}$ and $\beta^* = \tilde{b}^ip_i$ where

$$\tilde{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^ib^j; \quad \tilde{b}^i = \frac{1}{1-b^2}b^i,$$

(in (3.12) $'-'$ corresponds to $b^2 < 1$ and $'+'$ corresponds to $b^2 > 1$).

Remarks:

1. The $\mathcal{L}$-dual of a Randers space of order two is a Kropina space (for $b^2 = 1$) and a Randers space (for $b^2 \neq 1$), both in T^*M.

2. It is known that a Randers space $F = \alpha + \beta$ is positively defined if and only if $b^2 < 1$ or $a_{ij} - b_i b_j$ is positively defined, where $b^2 = a^{ij} b_i b_j$ ([1]).
   In ([2]), it has been proved that the above condition is necessary and sufficient for the Randers space to have the fundamental tensor positively defined. We are going to use the same techniques to prove the same result for a Randers space of order two.
   Proposition 2.1 *A Randers metric of second order $F = \alpha + \beta$ is positive-valued for any $z^{(2)}$ if and only if the length b of b_i with respect to α is less than 1 or $a_{ij} - b_i b_j$ is positively defined, provided a_{ij} is positively defined.*

   Proof. The proof is the same as in case of Randers space of first order ([1]).

3. The condition $\|b\| < 1$, which guarantees the positivity of F, also ensures the strong convexity.

4. The $\mathcal{L}$-dual of a strongly convex Randers space of order two is also a strongly convex Randers space ([14]).

5. The $\mathcal{L}$-dual of a dual of a Randers space of second order $(M, F = \sqrt{a_{ij} z^{(2)i} z^{(2)j}} + b_i z^{(2)i})$ is:
   a) If $b^2 = 1$, a Randers space of second order on T^2M having the fundamental function:

$$F(x, y^{(1)}, y^{(2)}) = \tilde{b}_i z^{(2)i} \pm \sqrt{\tilde{a}_{ij}(x) z^{(2)i} z^{(2)j}}, \tag{3.13}$$

   where $\tilde{b}_i = 2b_i$ and $\tilde{a}_{ij} = 4a_{ij}$.
   b) If $b^2 \neq 1$, a Randers space of second order with the fundamental function:

$$F(x, y^{(1)}, y^{(2)}) = \sqrt{\tilde{a}_{ij}(x) z^{(2)i} z^{(2)j}} \pm \tilde{b}_i z^{(2)i}, \tag{3.14}$$

   where $\tilde{a}_{ij} = 2a_{ij}|b^2 - 1| + \left(\frac{2b^2-3}{2|b^2-1|}\right)^2 b_i b_j$ and $\tilde{b}_i = \frac{2b^2-3}{2|b^2-1|} b_i$,
   where $|a|$ means the absolute value of number a.
   (in (3.14) '+' corresponds to $b^2 > 1$ and '-' corresponds to $b^2 < 1$).
   This last remark can be proved in the same as the classical case $C^{(1)n} = (M, K)$ knowing that ([12]) $z^{(2)i} = \frac{1}{2} \frac{\partial H}{\partial p_i}$.

Theorem 2.2 *The $\mathcal{L}$-dual of a Kropina space of order two is a Randers space on T^*M with the Hamiltonian:*

$$H(x, p) = \frac{1}{2} \left(\sqrt{\tilde{a}^{ij} p_i p_j} \pm \tilde{b}^i p_i \right)^2, \tag{3.15}$$

where

$$\tilde{a}^{ij} = \frac{b^2}{4} a^{ij}; \quad \tilde{b}^i = \frac{1}{2} b^i,$$

(in (3.15) $'-'$ corresponds to $\beta < 0$ and $'+'$ corresponds to $\beta > 0$).

Proof: We prove this theorem by using Theorem 1.1. For $F = \frac{\alpha^2}{\beta}$ like in ([14]) the covariant tensor and its contravariant tensor for $F^{(2)n}$:

$$k_{ij} = 2\frac{\alpha^2}{\beta^2}h_{ij} + \frac{3\alpha^2(2\beta^2-\alpha^2)}{\beta^4}l_il_j + 3\frac{\alpha^4}{\beta^4}d_id_j - 3\frac{\alpha^3(\alpha+4\beta)}{\beta^4}(b_il_j+b_jl_i), \tag{3.16}$$

and

$$k^{ij} = \frac{1}{2}\frac{\beta^2}{\alpha^2}a^{ij} - \frac{1}{2b^2}\frac{\beta^2}{\alpha^2}b^ib^j + \frac{1}{b^2}(b^il^j+b^jl^j) - \frac{\beta^2}{\alpha^2}\Big(\frac{\beta^2}{\alpha^2}\frac{2}{b^2}-1\Big)l^il^j. \tag{3.17}$$

Contracting (3.17) by p_ip_j we get:

$$4F^2 - 4F\beta^* + \beta^{*2} = \alpha^{*2}b^2,$$

i.e.

$$(2F-\beta^*)^2 = \alpha^{*2},$$

and

$$F = \frac{1}{2}(\beta^* \pm \alpha^* b).$$

Setting now $a^{ij} = \frac{b^2}{4}a^{ij}$ and $\tilde{b}^i = \frac{1}{2}b^i$, we easily get (3.15).

Remark: Another way of proving this result is by finding:

$$p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^{(2)i}} = \frac{F}{\beta}(a_{ij}z^{(2)j} - Fb_i), \tag{3.18}$$

$$\xi^i = \frac{1}{2}a^{ij}(p_j\frac{1}{F} + Fb_j) - \frac{1}{2}\gamma^i_{jk}y^{(1)j}y^{(1)k}, \tag{3.19}$$

$$\check{z}^{(2)i} = \frac{1}{2}a^{ij}(\frac{1}{F}p_j + Fb_j). \tag{3.20}$$

Contracting now in (3.18) by p^i and b^i we get:

$$\alpha^{*2} = \frac{F^2}{\beta}(2F-\beta^*), \tag{3.21}$$

$$\beta^* = \frac{F}{\beta}(2\beta - Fb^2). \tag{3.22}$$

After a simple computation, we obtain, like in the previous case, $(2F-\beta^*)^2 = \alpha^{*2}$ and (3.15).

Using again α^* and β^* Theorem 2.2 becomes:
Remark 2.2 ([13]) *The $\mathcal{L}$-dual of a Kropina space of order two is a Randers space on T^*M with the Hamiltonian:*

$$H(x,p)=\frac{1}{2}\Big(\alpha^*\pm\beta^*\Big)^2, \tag{3.23}$$

with $\alpha^*=\sqrt{\tilde{a}^{ij}(x)p_ip_j}$ *and* $\beta^*=\tilde{b}^ip_i$ *where*

$$\tilde{a}^{ij}=\frac{b^2}{4}a^{ij};\quad \tilde{b}^i=\frac{1}{2}b^i,$$

(in (3.23) $'-'$ *corresponds to* $\beta<0$ *and* $'+'$ *corresponds to* $\beta>0$*).*

Remarks:

1. The $\mathcal{L}$-dual of a Kropina space of order two is a Randers space in T^*M.

2. The $\mathcal{L}$-dual of a dual of a Kropina space of second order $\left(M,F=\frac{a_{ij}z^{(2)i}z^{(2)j}}{b_iz^{(2)i}}\right)$ is a Kropina space of second order having the fundamental function:

$$F=\pm\frac{8}{b^2}\frac{a_{ij}z^{(2)i}z^{(2)j}}{b_iz^{(2)i}}, \tag{3.24}$$

(in (3.24) '+' corresponds to $\beta>0$ and '-' corresponds to $\beta<0$).
This last remark can be proved in the same as the classical case $C^{(1)n}=(M,K)$ knowing that ([12]) $z^{(2)i}=\frac{1}{2}\frac{\partial H}{\partial p_i}$.

Theorem 2.3 *Let* (M,F) *be a Matsumoto space of order two and* $b=(a_{ij}b^ib^j)^{\frac{1}{2}}$ *the Riemannian length of* b_i*. Then*

1. *If* $b^2=1$*, the* $\mathcal{L}$*-dual of* (M,F) *is the space having the fundamental function:*

$$H(x,p)=\frac{1}{2}\left(-\frac{b^ip_i}{2}\frac{\left(\sqrt[3]{a^{ij}p_ip_j}+\sqrt[3]{\left(b^ip_i+\sqrt{\tilde{a}^{ij}p_ip_j}\right)^2}\right)^3}{a^{ij}p_ip_j+\left(b^ip_i+\sqrt{\tilde{a}^{ij}p_ip_j}\right)^2}\right)^2, \tag{3.25}$$

where

$$\tilde{a}^{ij}\;=\;b^ib^j-a^{ij}.$$

2. *If* $b^2\neq1$*, the* $\mathcal{L}$*-dual of* (M,F) *is the space on* T^*M *having the fundamental function:*

$$H(x,p)=\frac{1}{2}\left(-\frac{b^ip_i}{200}\frac{25\Big(2\sqrt{d_2^{ij}p_ip_j}+\sqrt{d_4^{ij}p_ip_j}\Big)^2+d_8^{ij}p_ip_j}{\sqrt{d_2^{ij}p_ip_j}\sqrt{d_4^{ij}p_ip_j}+d_9^{ij}p_ip_j}\right)^2, \tag{3.26}$$

where

$$
\begin{aligned}
c_1^{ij} &= (b^i b^j + 2\varepsilon_1 a^{ij})^2 + (2a^{ij})^2\varepsilon_3,\\
c_2^{ij} &= a^{ij}(\theta_4^2 b^i b^j + a^{ij}\varepsilon_2),\\
c_3^{ij} &= (2a^{ij})^2\theta_5^3,\\
\sqrt[3]{\tilde{a}^{ij}}^2 &= \sqrt[3]{c_1^{ij}} - 2\sqrt[3]{c_2^{ij}} + \sqrt[3]{c_3^{ij}},\\
d_1^{ij} &= d_3^{ij} + 4m(a^{ij}b^2 - b^i b^j),\\
d_2^{ij} &= \sqrt{d_3^{ij}a^{ij}} + 4\sqrt{d_1^{ij}a^{ij}} - d_3^{ij},\\
d_3^{ij} &= 2\sqrt[3]{2a^{ij}(\tilde{a}^{ij})^2},\\
\sqrt{d_4^{ij}} &= \sqrt{d_3^{ij}} + 3\sqrt{a^{ij}},\\
\sqrt{d_5^{ij}} &= \sqrt{d_3^{ij}a^{ij}},\\
d_6^{ij} &= d_1^{ij}a^{ij},\\
\sqrt{d_7^{ij}} &= 2\sqrt{d_2^{ij}} + \sqrt{d_4^{ij}},\\
d_8^{ij} &= 200\Big(\sqrt{d_6^{ij}} + 2na^{ij}\Big) - 5\Big(4\sqrt{d_3^{ij}} + \sqrt{d_4^{ij}}\Big),\\
d_9^{ij} &= 4\sqrt{d_6^{ij}} + 4a^{ij}p + 9\sqrt{d_5^{ij}},
\end{aligned}
$$

and

$$
\begin{aligned}
m &= 1 - b^2,\\
n &= \frac{20b^2 - 29}{29},\\
p &= \frac{1 - 2b^2}{2},\\
\theta_1 &= -\frac{712b^6 - 452b^4 + 24b^2 + 1}{1728},\\
\theta_2 &= \frac{576b^4 - 2232b^2 + 2628}{1728},\\
\theta_3 &= -\left(\frac{8b^2 + 1}{12}\right)^2,\\
\theta_4 &= \frac{2b^2 + 1}{6},
\end{aligned}
$$

$$
\begin{aligned}
\theta_5 &= \frac{11b^2 + 1}{12},\\
\varepsilon_1 &= 2(\theta_4^2 - \theta_2),\\
\varepsilon_2 &= 3\theta_3\theta_4^2 + \theta_2^2,\\
\varepsilon_3 &= 4\varepsilon_2 - 2\theta_1 - \varepsilon_1.
\end{aligned}
$$

Proof: Let us prove this theorem in the same way as the classical case. So, we have:

$$p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^{(2)i}} = F\frac{a_{ij}z^{(2)j}(\alpha-2\beta)+\alpha^2 b_i}{(\alpha-\beta)^2}, \tag{3.27}$$

$$\xi^i = \frac{1}{\alpha-2\beta}a^{ij}\Big(\frac{(\alpha-2\beta)^2}{F}p_j+\alpha^2 b_j\Big)-\frac{1}{2}\gamma^i_{jk}y^{(1)j}y^{(1)k}, \tag{3.28}$$

$$\check{z}^{(2)i} = \frac{1}{\alpha-2\beta}a^{ij}\Big(\frac{(\alpha-2\beta)^2}{F}p_j+\alpha^2 b_j\Big). \tag{3.29}$$

Contracting now in (3.27) by p^i and b_i and setting $s=\frac{\beta}{\alpha}$ ([15]) we get:

$$\begin{aligned}\alpha^{\star 2} &= F^2\frac{1-2s}{(1-s)^3}+F\frac{1}{(1-s)^2}\beta^\star \\ \beta^\star &= Fs\frac{1-2s}{(1-s)^2}+F\frac{1}{(1-s)^2}b^2.\end{aligned} \tag{3.30}$$

Now we put $1-s=t$, i.e. $s=1-t$ and both equations become:

$$\alpha^{\star 2} = F^2\frac{2t-1}{t^3}+F\frac{1}{t^2}\beta^\star \tag{3.31}$$

$$\beta^\star = F(1-t)\frac{2t-1}{t^2}+F\frac{1}{t^2}b^2. \tag{3.32}$$

We get

$$\beta^* t^2 = M(-2t^2+3t+b^2-1). \tag{3.33}$$

For $b^2=1$ from (3.32) we obtain:

$$F=-\frac{\beta^* t}{2t-3} \tag{3.34}$$

and by substitution of F in (3.32), after some computations we get a cubic equation:

$$t^3-3t+\frac{9}{4}t-\frac{\beta^\star}{2\alpha^{\star 2}}=0. \tag{3.35}$$

Using Cardano's method for solving cubic equation ([16]), we get:

$$F=-\frac{\beta^\star}{2}\frac{(2P-1)^2}{3P^2+(P-1)^2}, \tag{3.36}$$

where for P we have:

$$P=\frac{1}{2}\sqrt[3]{\left(\frac{\beta^\star+\sqrt{\beta^{\star 2}-\alpha^{\star 2}}}{\alpha^\star}\right)^2}. \tag{3.37}$$

After some computations, for F we get:

$$F=-\frac{\beta^\star}{2}\frac{\left(\sqrt[3]{\alpha^{\star 2}}+\sqrt[3]{(\beta^\star+\sqrt{\beta^{\star 2}-\alpha^{\star 2}})^2}\right)^3}{\alpha^{\star 2}+(\beta^\star+\sqrt{\beta^{\star 2}-\alpha^{\star 2}})^2}. \tag{3.38}$$

Substituting now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$ we can easily get (3.25).

If $b^2 \neq 1$ (3.34) is more complicated because:

$$F = \frac{\beta^* t^2}{-2t^2 + 3t + b^2 - 1}, \tag{3.39}$$

and by substituting this in (3.31) we obtain the quadric equation:

$$t^4 - 3t^3 + t^2 \frac{13 - 4b^2}{4} + t \frac{6\alpha^{*2}(b^2 - 1)}{4\alpha^{*2}} + \frac{\alpha^{*2}(b^2 - 1)^2 + \beta^{*2}(1 - b^2)}{4\alpha^{*2}} = 0. \tag{3.40}$$

After a quite long computation, formula (3.40) becomes a cubic equation and solving it we get:

$$\begin{aligned} F &= -\frac{\beta^*}{2}\Bigg(\Bigg(\sqrt{-A^2 + 3A + 2\sqrt{A^2 + m\Big(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\Big)}} + \frac{A}{2} + \frac{3}{4}\Bigg)^2 \\ &+ \sqrt{A^2 + m\Big(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\Big)} - \frac{5}{4}\Big(A + \frac{3}{10}\Big)^2 + n\Bigg)/ \\ &/ \Bigg(\Big(\frac{3}{2} + 2A\Big)(\sqrt{-A^2 + 3A + 2\sqrt{A^2 + m\Big(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\Big)}} \\ &+ 2\sqrt{A^2 + m\Big(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\Big)} + \frac{9}{2}A + p\Bigg), \end{aligned} \tag{3.41}$$

where

$$A^2 = \sqrt[3]{\Big(\frac{1}{2}\frac{\beta^{*2}}{\alpha^{*2}} + \varepsilon_1\Big)^2 + \varepsilon_3} + \sqrt[3]{-4\Big(\theta_4^3 \frac{\beta^{*2}}{\alpha^{*2}} + \varepsilon_2\Big)} + \theta_5. \tag{3.42}$$

By substituting now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$, after some computations, we obtain (3.26).

Using now α^* and β^* the Theorem 2.3 becomes:

Remark 2.3 *Let (M, F) be a Matsumoto space of second order and $b = (a_{ij} b^i b^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then*

1. *If $b^2 = 1$, the $\mathcal{L}$-dual of (M, F) is the space having the fundamental function:*

$$H(x,p) = \frac{1}{2}\left(-\frac{\beta^\star}{2}\frac{\left(\sqrt[3]{\alpha^{\star 2}} + \sqrt[3]{\left(\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}}\right)^2}\right)^3}{\alpha^{\star 2} + (\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}})^2}\right)^2. \tag{3.43}$$

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x) p_i p_j}$ and $\beta^* = b^i p_i$ where

$$\tilde{a}^{ij} = b^i b^j - a^{ij}.$$

2. *If $b^2 \neq 1$, the $\mathcal{L}$-dual of (M, F) is the space on T^*M having the fundamental function:*

$$H(x,p) = \frac{1}{2}\left(-\frac{\beta^*}{200} \frac{25\big(2\alpha_2^* \ + \alpha_4^*\big)^2 + \big(\alpha_8^*\big)^2}{\alpha_2^*\alpha_4^* + \big(\alpha_9^*\big)^2} \right)^2, \tag{3.44}$$

where

$$\alpha_2^* = \sqrt{d_2^{ij} p_i p_j}, \quad \alpha_4^* = \sqrt{d_4^{ij} p_i p_j},$$
$$\alpha_8^* = \sqrt{d_8^{ij} p_i p_j}, \quad \alpha_9^* = \sqrt{d_9^{ij} p_i p_j}.$$

Remarks:

1. In (3.25) $\tilde{a}^{ij}$ is positive-definite and the Randers metric on T^*M $p_i b^i + \sqrt{p_i p_j \tilde{a}^{ij}}$ is positive-valued for any p.

2. Some of the values for which α_2^*, α_4^*, α_8^*, α_9^* exist are:$b^2 < \frac{1}{2}$ and $a^{ij} > 2b^i b^j$. Certainly, there are many other values for b^2, a^{ij}, b^i, b^j which justify the existence of (3.44).

3. The $\mathcal{L}$-dual of a Matsumoto metric of second order is given by means of four distinct quadratic forms on T^*M. The coefficients of the quadratic forms are constructed only from the Riemannian metric, a_{ij} and the 1-forms β's coefficients $b_i(x)$.

4. As dual in T^2M of the above space, the $\mathcal{L}$-dual of a Matsumoto space of second order $(M, F = \frac{a_{ij}z^{(2)i}z^{(2)j}}{\sqrt{a_{ij}z^{(2)i}z^{(2)j}} - b_i z^{(2)i}})$, we find the Matsumoto space of second order with the fundamental function:

$$F = \frac{\tilde{a}_{ij} z^{(2)i} z^{(2)j}}{\sqrt{b^2 a_{ij} z^{(2)i} z^{(2)j}} - \tilde{b}_i z^{(2)i}}, \tag{3.45}$$

where

$$\begin{aligned} \tilde{b}_i &= 4b^2 b_i, \\ \tilde{a}_{ij} &= a_{ij}^2 b_i b_j (7 + 8b^2) - \sqrt{a_{ij}} b_i [a_{ij}(1 + 2b^2) - 12 b_i b_j] \\ &\pm m_{ij}[a_{ij}^2 b_i (7 + 8b^2) - \sqrt{a_{ij}}(a_{ij} - 12 b_i b_j)], \end{aligned}$$

with

$$m_{ij} = \sqrt{b_i b_j - b^2 a_{ij}}.$$

This last remark can be proved in the same as the classical case $C^{(1)n} = (M, K)$ knowing that ([12]) $z^{(2)i} = \frac{1}{2}\frac{\partial H}{\partial p_i}$.

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