Jet geometrical extension of the KCC-invariants

Vladimir Balan and Mircea Neagu

Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. In this paper we construct the jet geometrical extensions of the KCC-invariants, which characterize a given second-order system of differential equations on the 1-jet space $J^1(\mathbb{R}, M)$. A generalized theorem of characterization of our jet geometrical KCC-invariants is also presented.

M.S.C. 2000: 58B20, 37C10, 53C43.
Key words: 1-jet spaces; temporal and spatial semisprays; nonlinear connections; SODE; jet $h-$KCC-invariants.

1 Geometrical objects on 1-jet spaces

We remind first several differential geometrical properties of the 1-jet spaces. The 1-jet bundle

$$\xi = (J^1(\mathbb{R}, M), \pi_1, \mathbb{R} \times M)$$

is a vector bundle over the product manifold $\mathbb{R} \times M$, having the fibre of type $\mathbb{R}^n$, where n is the dimension of the *spatial* manifold M. If the spatial manifold M has the local coordinates $(x^i)_{i=\overline{1,n}}$, then we shall denote the local coordinates of the 1-jet total space $J^1(\mathbb{R}, M)$ by (t, x^i, x_1^i); these transform by the rules [13]

$$\begin{cases} \widetilde{t} = \widetilde{t}(t) \\ \widetilde{x}^i = \widetilde{x}^i(x^j) \\ \widetilde{x}_1^i = \dfrac{\partial \widetilde{x}^i}{\partial x^j} \dfrac{dt}{d\widetilde{t}} \cdot x_1^j. \end{cases} \tag{1.1}$$

In the geometrical study of the 1-jet bundle, a central role is played by the *distinguished tensors* ($d-$tensors).

Definition 1.1. A geometrical object $D = \left(D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\right)$ on the 1-jet vector bundle, whose local components transform by the rules

$$D^{1i(j)(1)\ldots}_{1k(1)(u)\ldots} = \widetilde{D}^{1p(m)(1)\ldots}_{1r(1)(s)\ldots} \frac{dt}{d\widetilde{t}} \frac{\partial x^i}{\partial \widetilde{x}^p} \left(\frac{\partial x^j}{\partial \widetilde{x}^m} \frac{d\widetilde{t}}{dt} \right) \frac{d\widetilde{t}}{dt} \frac{\partial \widetilde{x}^r}{\partial x^k} \left(\frac{\partial \widetilde{x}^s}{\partial x^u} \frac{dt}{d\widetilde{t}} \right) \ldots, \tag{1.2}$$

is called a *$d-$tensor field.*

Balkan Journal of Geometry and Its Applications, Vol.15, No.1, 2010, pp. 8-16.

Remark 1.2. The use of parentheses for certain indices of the local components

$$D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}$$

of the distinguished tensor field D on the 1-jet space is motivated by the fact that the pair of indices " $\begin{smallmatrix}(j)\\(1)\end{smallmatrix}$ " or " $\begin{smallmatrix}(1)\\(l)\end{smallmatrix}$ " behaves like a single index.

Example 1.3. The geometrical object

$$\mathbf{C} = \mathbf{C}^{(i)}_{(1)} \frac{\partial}{\partial x^i_1},$$

where $\mathbf{C}^{(i)}_{(1)} = x^i_1$, represents a $d-$tensor field on the 1-jet space; this is called the *canonical Liouville $d-$tensor field* of the 1-jet bundle and is a global geometrical object.

Example 1.4. Let $h = (h_{11}(t))$ be a Riemannian metric on the relativistic time axis $\mathbb{R}$. The geometrical object

$$\mathbf{J}_h = J^{(i)}_{(1)1j} \frac{\partial}{\partial x^i_1} \otimes dt \otimes dx^j,$$

where $J^{(i)}_{(1)1j} = h_{11}\delta^i_j$ is a $d-$tensor field on $J^1(\mathbb{R}, M)$, which is called the *h-normalization $d-$tensor field* of the 1-jet space and is a global geometrical object.

In the Riemann-Lagrange differential geometry of the 1-jet spaces developed in [12], [13] important rôles are also played by geometrical objects as the *temporal* or *spatial semisprays*, together with the *jet nonlinear connections.*

Definition 1.5. A set of local functions $H = \left(H^{(j)}_{(1)1}\right)$ on $J^1(\mathbb{R}, M)$, which transform by the rules

$$2\widetilde{H}^{(k)}_{(1)1} = 2H^{(j)}_{(1)1} \left(\frac{dt}{d\widetilde{t}}\right)^2 \frac{\partial \widetilde{x}^k}{\partial x^j} - \frac{dt}{d\widetilde{t}} \frac{\partial \widetilde{x}^k_1}{\partial t}, \tag{1.3}$$

is called a *temporal semispray* on $J^1(\mathbb{R}, M)$.

Example 1.6. Let us consider a Riemannian metric $h = (h_{11}(t))$ on the temporal manifold $\mathbb{R}$ and let

$$H^1_{11} = \frac{h^{11}}{2} \frac{dh_{11}}{dt},$$

where $h^{11} = 1/h_{11}$, be its Christoffel symbol. Taking into account that we have the transformation rule

$$\widetilde{H}^1_{11} = H^1_{11} \frac{dt}{d\widetilde{t}} + \frac{d\widetilde{t}}{dt} \frac{d^2 t}{d\widetilde{t}^2}, \tag{1.4}$$

we deduce that the local components

$$\mathring{H}^{(j)}_{(1)1} = -\frac{1}{2} H^1_{11} x^j_1$$

define a temporal semispray $\mathring{H} = \left(\mathring{H}^{(j)}_{(1)1}\right)$ on $J^1(\mathbb{R}, M)$. This is called the *canonical temporal semispray associated to the temporal metric $h(t)$.*

Definition 1.7. A set of local functions $G = \left(G^{(j)}_{(1)1}\right)$, which transform by the rules

$$2\widetilde{G}^{(k)}_{(1)1} = 2G^{(j)}_{(1)1}\left(\frac{dt}{d\widetilde{t}}\right)^2\frac{\partial\widetilde{x}^k}{\partial x^j} - \frac{\partial x^m}{\partial\widetilde{x}^j}\frac{\partial\widetilde{x}^k_1}{\partial x^m}\widetilde{x}^j_1, \tag{1.5}$$

is called a *spatial semispray* on $J^1(\mathbb{R}, M)$.

Example 1.8. Let $\varphi = (\varphi_{ij}(x))$ be a Riemannian metric on the spatial manifold M and let us consider

$$\gamma^i_{jk} = \frac{\varphi^{im}}{2}\left(\frac{\partial\varphi_{jm}}{\partial x^k} + \frac{\partial\varphi_{km}}{\partial x^j} - \frac{\partial\varphi_{jk}}{\partial x^m}\right)$$

its Christoffel symbols. Taking into account that we have the transformation rules

$$\widetilde{\gamma}^p_{qr} = \gamma^i_{jk}\frac{\partial\widetilde{x}^p}{\partial x^i}\frac{\partial x^j}{\partial\widetilde{x}^q}\frac{\partial x^k}{\partial\widetilde{x}^r} + \frac{\partial\widetilde{x}^p}{\partial x^l}\frac{\partial^2 x^l}{\partial\widetilde{x}^q\partial\widetilde{x}^r}, \tag{1.6}$$

we deduce that the local components

$$\overset{\circ}{G}{}^{(j)}_{(1)1} = \frac{1}{2}\gamma^j_{kl}x^k_1x^l_1$$

define a spatial semispray $\overset{\circ}{G} = \left(\overset{\circ}{G}{}^{(j)}_{(1)1}\right)$ on $J^1(\mathbb{R}, M)$. This is called the *canonical spatial semispray associated to the spatial metric* $\varphi(x)$.

Definition 1.9. A set of local functions $\Gamma = \left(M^{(j)}_{(1)1}, N^{(j)}_{(1)i}\right)$ on $J^1(\mathbb{R}, M)$, which transform by the rules

$$\widetilde{M}^{(k)}_{(1)1} = M^{(j)}_{(1)1}\left(\frac{dt}{d\widetilde{t}}\right)^2\frac{\partial\widetilde{x}^k}{\partial x^j} - \frac{dt}{d\widetilde{t}}\frac{\partial\widetilde{x}^k_1}{\partial t} \tag{1.7}$$

and

$$\widetilde{N}^{(k)}_{(1)l} = N^{(j)}_{(1)i}\frac{dt}{d\widetilde{t}}\frac{\partial x^i}{\partial\widetilde{x}^l}\frac{\partial\widetilde{x}^k}{\partial x^j} - \frac{\partial x^m}{\partial\widetilde{x}^l}\frac{\partial\widetilde{x}^k_1}{\partial x^m}, \tag{1.8}$$

is called a *nonlinear connection* on the 1-jet space $J^1(\mathbb{R}, M)$.

Example 1.10. Let us consider that $(\mathbb{R}, h_{11}(t))$ and $(M, \varphi_{ij}(x))$ are Riemannian manifolds having the Christoffel symbols $H^1_{11}(t)$ and $\gamma^i_{jk}(x)$. Then, using the transformation rules (1.1), (1.4) and (1.6), we deduce that the set of local functions

$$\overset{\circ}{\Gamma} = \left(\overset{\circ}{M}{}^{(j)}_{(1)1}, \overset{\circ}{N}{}^{(j)}_{(1)i}\right),$$

where

$$\overset{\circ}{M}{}^{(j)}_{(1)1} = -H^1_{11}x^j_1 \quad \text{and} \quad \overset{\circ}{N}{}^{(j)}_{(1)i} = \gamma^j_{im}x^m_1,$$

represents a nonlinear connection on the 1-jet space $J^1(\mathbb{R}, M)$. This jet nonlinear connection is called the *canonical nonlinear connection attached to the pair of Riemannian metrics* $(h(t), \varphi(x))$.

In the sequel, let us study the geometrical relations between *temporal* or *spatial semisprays* and *nonlinear connections* on the 1-jet space $J^1(\mathbb{R}, M)$. In this direction, using the local transformation laws (1.3), (1.7) and (1.1), respectively the transformation laws (1.5), (1.8) and (1.1), by direct local computation, we find the following geometrical results:

Theorem 1.11. *a) The* temporal semisprays $H = (H^{(j)}_{(1)1})$ *and the sets of* temporal components of nonlinear connections $\Gamma_{temporal} = (M^{(j)}_{(1)1})$ *are in one-to-one correspondence on the 1-jet space* $J^1(\mathbb{R}, M)$*, via:*

$$M^{(j)}_{(1)1} = 2H^{(j)}_{(1)1}, \qquad H^{(j)}_{(1)1} = \frac{1}{2}M^{(j)}_{(1)1}.$$

b) The spatial semisprays $G = (G^{(j)}_{(1)1})$ *and the sets of* spatial components of nonlinear connections $\Gamma_{spatial} = (N^{(j)}_{(1)k})$ *are connected on the 1-jet space* $J^1(\mathbb{R}, M)$*, via the relations:*

$$N^{(j)}_{(1)k} = \frac{\partial G^{(j)}_{(1)1}}{\partial x^k_1}, \qquad G^{(j)}_{(1)1} = \frac{1}{2}N^{(j)}_{(1)m}x^m_1.$$

2 Jet geometrical KCC-theory

In this Section we generalize on the 1-jet space $J^1(\mathbb{R}, M)$ the basics of the KCC-theory ([1], [4], [7], [14]). In this respect, let us consider on $J^1(\mathbb{R}, M)$ a second-order system of differential equations of local form

$$\frac{d^2x^i}{dt^2} + F^{(i)}_{(1)1}(t, x^k, x^k_1) = 0, \quad i = \overline{1, n}, \tag{2.1}$$

where $x^k_1 = dx^k/dt$ and the local components $F^{(i)}_{(1)1}(t, x^k, x^k_1)$ transform under a change of coordinates (1.1) by the rules

$$\widetilde{F}^{(r)}_{(1)1} = F^{(j)}_{(1)1}\left(\frac{dt}{d\widetilde{t}}\right)^2 \frac{\partial \widetilde{x}^r}{\partial x^j} - \frac{dt}{d\widetilde{t}}\frac{\partial \widetilde{x}^r_1}{\partial t} - \frac{\partial x^m}{\partial \widetilde{x}^j}\frac{\partial \widetilde{x}^r_1}{\partial x^m}\widetilde{x}^j_1. \tag{2.2}$$

Remark 2.1. The second-order system of differential equations (2.1) is invariant under a change of coordinates (1.1).

Using a temporal Riemannian metric $h_{11}(t)$ on $\mathbb{R}$ and taking into account the transformation rules (1.3) and (1.5), we can rewrite the SODEs (2.1) in the following form:

$$\frac{d^2x^i}{dt^2} - H^1_{11}x^i_1 + 2G^{(i)}_{(1)1}(t, x^k, x^k_1) = 0, \quad i = \overline{1, n},$$

where

$$G^{(i)}_{(1)1} = \frac{1}{2}F^{(i)}_{(1)1} + \frac{1}{2}H^1_{11}x^i_1$$

are the components of a spatial semispray on $J^1(\mathbb{R}, M)$. Moreover, the coefficients of the spatial semispray $G^{(i)}_{(1)1}$ produce the spatial components $N^{(i)}_{(1)j}$ of a nonlinear connection Γ on the 1-jet space $J^1(\mathbb{R},M)$, by putting

$$N^{(i)}_{(1)j}=\frac{\partial G^{(i)}_{(1)1}}{\partial x^j_1}=\frac{1}{2}\frac{\partial F^{(i)}_{(1)1}}{\partial x^j_1}+\frac{1}{2}H^1_{11}\delta^i_j.$$

In order to find the basic jet differential geometrical invariants of the system (2.1) (see Kosambi [11], Cartan [9] and Chern [10]) under the jet coordinate transformations (1.1), we define the *$h-$KCC-covariant derivative of a $d-$tensor of kind* $T^{(i)}_{(1)}(t,x^k,x^k_1)$ on the 1-jet space $J^1(\mathbb{R},M)$ via

$$\begin{aligned}\frac{\overset{h}{D}T^{(i)}_{(1)}}{dt} &= \frac{dT^{(i)}_{(1)}}{dt}+N^{(i)}_{(1)r}T^{(r)}_{(1)}-H^1_{11}T^{(i)}_{(1)}=\\ &= \frac{dT^{(i)}_{(1)}}{dt}+\frac{1}{2}\frac{\partial F^{(i)}_{(1)1}}{\partial x^r_1}T^{(r)}_{(1)}-\frac{1}{2}H^1_{11}T^{(i)}_{(1)},\end{aligned}$$

where the Einstein summation convention is used throughout.

Remark 2.2. The *$h-$KCC-covariant derivative* components $\dfrac{\overset{h}{D}T^{(i)}_{(1)}}{dt}$ transform under a change of coordinates (1.1) as a $d-$tensor of type $\mathcal{T}^{(i)}_{(1)1}$.

In such a geometrical context, if we use the notation $x^i_1=dx^i/dt$, then the system (2.1) can be rewritten in the following distinguished tensorial form:

$$\begin{aligned}\frac{\overset{h}{D}x^i_1}{dt} &= -F^{(i)}_{(1)1}(t,x^k,x^k_1)+N^{(i)}_{(1)r}x^r_1-H^1_{11}x^i_1=\\ &= -F^{(i)}_{(1)1}+\frac{1}{2}\frac{\partial F^{(i)}_{(1)1}}{\partial x^r_1}x^r_1-\frac{1}{2}H^1_{11}x^i_1,\end{aligned}$$

Definition 2.3. The distinguished tensor

$$\overset{h}{\varepsilon}{}^{(i)}_{(1)1}=-F^{(i)}_{(1)1}+\frac{1}{2}\frac{\partial F^{(i)}_{(1)1}}{\partial x^r_1}x^r_1-\frac{1}{2}H^1_{11}x^i_1$$

is called the *first $h-$KCC-invariant* on the 1-jet space $J^1(\mathbb{R},M)$ of the SODEs (2.1), which is interpreted as an *external force* [1], [7].

Example 2.4. It can be easily seen that for the particular first order jet rheonomic dynamical system

$$\frac{dx^i}{dt}=X^{(i)}_{(1)}(t,x^k)\Rightarrow\frac{d^2x^i}{dt^2}=\frac{\partial X^{(i)}_{(1)}}{\partial t}+\frac{\partial X^{(i)}_{(1)}}{\partial x^m}x^m_1, \tag{2.3}$$

where $X^{(i)}_{(1)}(t,x)$ is a given $d-$tensor on $J^1(\mathbb{R},M)$, the first $h-$KCC-invariant has the form

$$\overset{h}{\varepsilon}{}^{(i)}_{(1)1}=\frac{\partial X^{(i)}_{(1)}}{\partial t}+\frac{1}{2}\frac{\partial X^{(i)}_{(1)}}{\partial x^r}x^r_1-\frac{1}{2}H^1_{11}x^i_1.$$

In the sequel, let us vary the trajectories $x^i(t)$ of the system (2.1) by the nearby trajectories $(\overline{x}^i(t,s))_{s\in(-\varepsilon,\varepsilon)}$, where $\overline{x}^i(t,0)=x^i(t)$. Then, considering the *variation $d-$tensor field*

$$\xi^i(t)=\left.\frac{\partial\overline{x}^i}{\partial s}\right|_{s=0},$$

we get the *variational equations*

$$\frac{d^2\xi^i}{dt^2}+\frac{\partial F^{(i)}_{(1)1}}{\partial x^j}\xi^j+\frac{\partial F^{(i)}_{(1)1}}{\partial x^r_1}\frac{d\xi^r}{dt}=0. \tag{2.4}$$

In order to find other jet geometrical invariants for the system (2.1), we also introduce the *$h-$KCC-covariant derivative of a $d-$tensor of kind $\xi^i(t)$* on the 1-jet space $J^1(\mathbb{R},M)$ via

$$\frac{\overset{h}{D}\xi^i}{dt}=\frac{d\xi^i}{dt}+N^{(i)}_{(1)m}\xi^m=\frac{d\xi^i}{dt}+\frac{1}{2}\frac{\partial F^{(i)}_{(1)1}}{\partial x^m_1}\xi^m+\frac{1}{2}H^1_{11}\xi^i.$$

Remark 2.5. The *$h-$KCC-covariant derivative* components $\dfrac{\overset{h}{D}\xi^i}{dt}$ transform under a change of coordinates (1.1) as a $d-$tensor $T^{(i)}_{(1)}$.

In this geometrical context, the variational equations (2.4) can be rewritten in the following distinguished tensorial form:

$$\frac{\overset{h}{D}}{dt}\left[\frac{\overset{h}{D}\xi^i}{dt}\right]=\overset{h}{P}{}^i_{m11}\xi^m,$$

where

$$\begin{aligned}\overset{h}{P}{}^i_{j11} = & -\frac{\partial F^{(i)}_{(1)1}}{\partial x^j}+\frac{1}{2}\frac{\partial^2F^{(i)}_{(1)1}}{\partial t\partial x^j_1}+\frac{1}{2}\frac{\partial^2F^{(i)}_{(1)1}}{\partial x^r\partial x^j_1}x^r_1-\frac{1}{2}\frac{\partial^2F^{(i)}_{(1)1}}{\partial x^r_1\partial x^j_1}F^{(r)}_{(1)1}+\\ & +\frac{1}{4}\frac{\partial F^{(i)}_{(1)1}}{\partial x^r_1}\frac{\partial F^{(r)}_{(1)1}}{\partial x^j_1}+\frac{1}{2}\frac{dH^1_{11}}{dt}\delta^i_j-\frac{1}{4}H^1_{11}H^1_{11}\delta^i_j.\end{aligned}$$

Definition 2.6. The $d-$tensor $\overset{h}{P}{}^i_{j11}$ is called the *second $h-$KCC-invariant* on the 1-jet space $J^1(\mathbb{R},M)$ of the system (2.1), or the *jet $h-$deviation curvature $d-$tensor.*

Example 2.7. If we consider the second-order system of differential equations of the *harmonic curves associated to the pair of Riemannian metrics* $(h_{11}(t),\varphi_{ij}(x))$, system which is given by (see the Examples 1.6 and 1.8)

$$\frac{d^2x^i}{dt^2}-H^1_{11}(t)\frac{dx^i}{dt}+\gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt}=0,$$

where $H^1_{11}(t)$ and $\gamma^i_{jk}(x)$ are the Christoffel symbols of the Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$, then the second $h-$KCC-invariant has the form

$$\overset{h}{P}{}^i_{j11}=-R^i_{pqj}x^p_1x^q_1,$$

where

$$R^i_{pqj} = \frac{\partial \gamma^i_{pq}}{\partial x^j} - \frac{\partial \gamma^i_{pj}}{\partial x^q} + \gamma^r_{pq}\gamma^i_{rj} - \gamma^r_{pj}\gamma^i_{rq}$$

are the components of the curvature of the spatial Riemannian metric $\varphi_{ij}(x)$. Consequently, the variational equations (2.4) become the following *jet Jacobi field equations*:

$$\frac{\overset{h}{D}}{dt}\left[\frac{\overset{h}{D}\xi^i}{dt}\right] + R^i_{pqm}x^p_1 x^q_1 \xi^m = 0,$$

where

$$\frac{\overset{h}{D}\xi^i}{dt} = \frac{d\xi^i}{dt} + \gamma^i_{jm}x^j_1\xi^m.$$

Example 2.8. For the particular first order jet rheonomic dynamical system (2.3) the jet $h-$deviation curvature $d-$tensor is given by

$$\overset{h}{P}{}^i_{j11} = \frac{1}{2}\frac{\partial^2 X^{(i)}_{(1)}}{\partial t \partial x^j} + \frac{1}{2}\frac{\partial^2 X^{(i)}_{(1)}}{\partial x^j \partial x^r}x^r_1 + \frac{1}{4}\frac{\partial X^{(i)}_{(1)}}{\partial x^r}\frac{\partial X^{(r)}_{(1)}}{\partial x^j} + \frac{1}{2}\frac{dH^1_{11}}{dt}\delta^i_j - \frac{1}{4}H^1_{11}H^1_{11}\delta^i_j.$$

Definition 2.9. The distinguished tensors

$$\overset{h}{R}{}^i_{jk1} = \frac{1}{3}\left[\frac{\partial \overset{h}{P}{}^i_{j11}}{\partial x^k_1} - \frac{\partial \overset{h}{P}{}^i_{k11}}{\partial x^j_1}\right], \qquad \overset{h}{B}{}^i_{jkm} = \frac{\partial \overset{h}{R}{}^i_{jk1}}{\partial x^m_1}$$

and

$$D^{i1}_{jkm} = \frac{\partial^3 F^{(i)}_{(1)1}}{\partial x^j_1 \partial x^k_1 \partial x^m_1}$$

are called the *third*, *fourth* and *fifth* $h-$*KCC-invariant* on the 1-jet vector bundle $J^1(\mathbb{R}, M)$ of the system (2.1).

Remark 2.10. Taking into account the transformation rules (2.2) of the components $F^{(i)}_{(1)1}$, we immediately deduce that the components D^{i1}_{jkm} behave like a $d-$tensor.

Example 2.11. For the first order jet rheonomic dynamical system (2.3) the third, fourth and fifth $h-$KCC-invariants are zero.

Theorem 2.12 (of characterization of the jet $h-$KCC-invariants). *All the five $h-$KCC-invariants of the system (2.1) cancel on $J^1(\mathbb{R}, M)$ if and only if there exists a flat symmetric linear connection $\Gamma^i_{jk}(x)$ on M such that*

$$F^{(i)}_{(1)1} = \Gamma^i_{pq}(x)x^p_1x^q_1 - H^1_{11}(t)x^i_1. \tag{2.5}$$

Proof. "$\Leftarrow$" By a direct calculation, we obtain

$$\overset{h}{\varepsilon}{}^{(i)}_{(1)1} = 0, \quad \overset{h}{P}{}^i_{j11} = -\mathfrak{R}^i_{pqj}x^p_1x^q_1 = 0 \text{ and } D^{i1}_{jkl} = 0,$$

where $\mathfrak{R}^i_{pqj} = 0$ are the components of the curvature of the flat symmetric linear connection $\Gamma^i_{jk}(x)$ on M.

"$\Rightarrow$" By integration, the relation

$$D^{i1}_{jkl} = \frac{\partial^3 F^{(i)}_{(1)1}}{\partial x^j_1 \partial x^k_1 \partial x^l_1} = 0$$

subsequently leads to

$$\begin{aligned}\frac{\partial^2 F^{(i)}_{(1)1}}{\partial x^j_1 \partial x^k_1} &= 2\Gamma^i_{jk}(t,x) \Rightarrow \frac{\partial F^{(i)}_{(1)1}}{\partial x^j_1} = 2\Gamma^i_{jp} x^p_1 + \mathcal{U}^{(i)}_{(1)j}(t,x) \Rightarrow \\ &\Rightarrow \quad F^{(i)}_{(1)1} = \Gamma^i_{pq} x^p_1 x^q_1 + \mathcal{U}^{(i)}_{(1)p} x^p_1 + \mathcal{V}^{(i)}_{(1)1}(t,x),\end{aligned}$$

where the local functions $\Gamma^i_{jk}(t,x)$ are symmetrical in the indices j and k.

The equality $\overset{h}{\varepsilon}{}^{(i)}_{(1)1} = 0$ on $J^1(\mathbb{R}, M)$ leads us to $\mathcal{V}^{(i)}_{(1)1} = 0$ and to $\mathcal{U}^{(i)}_{(1)j} = -H^1_{11}\delta^i_j$. Consequently, we have

$$\frac{\partial F^{(i)}_{(1)1}}{\partial x^j_1} = 2\Gamma^i_{jp} x^p_1 - H^1_{11}\delta^i_j \quad \text{and} \quad F^{(i)}_{(1)1} = \Gamma^i_{pq} x^p_1 x^q_1 - H^1_{11} x^i_1.$$

The condition $\overset{h}{P}{}^i_{j11} = 0$ on $J^1(\mathbb{R}, M)$ implies the equalities $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ and $\mathfrak{R}^i_{pqj} + \mathfrak{R}^i_{qpj} = 0$, where

$$\mathfrak{R}^i_{pqj} = \frac{\partial \Gamma^i_{pq}}{\partial x^j} - \frac{\partial \Gamma^i_{pj}}{\partial x^q} + \Gamma^r_{pq}\Gamma^i_{rj} - \Gamma^r_{pj}\Gamma^i_{rq}.$$

It is important to note that, taking into account the transformation laws (2.2), (1.3) and (1.1), we deduce that the local coefficients $\Gamma^i_{jk}(x)$ behave like a symmetric linear connection on M. Consequently, $\mathfrak{R}^i_{pqj}$ represent the curvature of this symmetric linear connection.

On the other hand, the equality $\overset{h}{R}{}^i_{jk1} = 0$ leads us to $\mathfrak{R}^i_{qjk} = 0$, which infers that the symmetric linear connection $\Gamma^i_{jk}(x)$ on M is flat. □

Acknowledgements. The present research was supported by the Romanian Academy Grant 4/2009.

References

[1] P.L. Antonelli, *Equivalence Problem for Systems of Second Order Ordinary Differential Equations*, Encyclopedia of Mathematics, Kluwer Academic, Dordrecht, 2000.

[2] P.L. Antonelli, L. Bevilacqua, S.F. Rutz, *Theories and models in symbiogenesis*, Nonlinear Analysis: Real World Applications **4** (2003), 743-753.

[3] P.L. Antonelli, I. Bucătaru, *New results about the geometric invariants in KCC-theory*, An. Şt. Univ. "Al.I.Cuza" Iaşi. Mat. N.S. 47 (2001), 405-420.

[4] P.L. Antonelli, R.S. Ingarden, M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Fundamental Theories of Physics, vol. 58, Kluwer Academic Publishers, Dordrecht, 1993.

[5] Gh. Atanasiu, V. Balan, N. Brînzei, M. Rahula, *The Differential-Geometric Structures: Tangent Bundles, Connections in Bundles and The Exponential Law in Jet-Spaces* (in Russian), LKI, Moscow, 2010.

[6] V. Balan, I.R. Nicola, *Berwald-Moor metrics and structural stability of conformally-deformed geodesic SODE*, Appl. Sci. 11 (2009), 19-34.

[7] V. Balan, I.R. Nicola, *Linear and structural stability of a cell division process model*, Hindawi Publishing Corporation, International Journal of Mathematics and Mathematical Sciences, vol. 2006, 1-15.

[8] I. Bucătaru, R. Miron, *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Romanian Academy Eds., 2007.

[9] E. Cartan, *Observations sur le mémoir précédent*, Mathematische Zeitschrift 37, 1 (1933), 619-622.

[10] S.S. Chern, *Sur la géométrie d'un système d'equations differentialles du second ordre*, Bulletin des Sciences Mathématiques 63 (1939), 206-212.

[11] D.D. Kosambi, *Parallelism and path-spaces*, Mathematische Zeitschrift 37, 1 (1933), 608-618.

[12] M. Neagu, *Riemann-Lagrange Geometry on 1-Jet Spaces*, Matrix Rom, Bucharest, 2005.

[13] M. Neagu, C. Udrişte, A. Oană, *Multi-time dependent sprays and h-traceless maps on $J^1(T, M)$*, Balkan J. Geom. Appl. 10, 2 (2005), 76-92.

[14] V.S. Sabău, *Systems biology and deviation curvature tensor*, Nonlinear Analysis. Real World Applications 6, 3 (2005), 563-587.

Authors' addresses:

Vladimir Balan
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics-Informatics I,
313 Splaiul Independentei, 060042 Bucharest, Romania.
E-mail: vladimir.balan@upb.ro
Website: http://www.mathem.pub.ro/dept/vbalan.htm

Mircea Neagu
University Transilvania of Braşov, Faculty of Mathematics and Informatics,
Department of Algebra, Geometry and Differential Equations,
B-dul Iuliu Maniu, Nr. 50, BV 500091, Braşov, Romania.
E-mail: mircea.neagu@unitbv.ro
Website: http://www.2collab.com/user:mirceaneagu