On quarter-symmetric metric connections on pseudo-Riemannian manifolds

Iulia Elena Hirică and Liviu Nicolescu

Abstract. The geometric significance of semi-symmetric connections was originally studied by K. Yano ([13]). The notion was extended to quarter symmetric connections by S. Golab ([3]).

In the present paper the theory is extended and it is shown that the Golab algebra associated to a quarter symmetric metric connection is essential in order to characterize the geometry of a pseudo-Riemannian manifold.

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Introduction

Throughout this paper one considers M a connected paracompact, smooth manifold of dimension n. Let $\mathcal{X}(M)$ be the Lie algebra of vector fields on M, T_pM the vector space of tangent vectors in a point $p \in M$, $\mathcal{T}^{(r,s)}(M)$ the $\mathcal{C}^\infty(M)$-module of tensor fields of type (r,s) on M, $\Lambda^p(M)$ the $\mathcal{C}^\infty(M)-$module of $p-$forms on M.

Let A be a $(1,2)-$tensor field on M. The $\mathcal{C}^\infty(M)-$modul $\mathcal{X}(M)$ becomes a $\mathcal{C}^\infty(M)-$algebra if we consider the multiplication rule given by $X \circ Y = A(X,Y)$, $\forall X, Y \in \mathcal{X}(M)$. This algebra is denoted by $\mathcal{U}(M,A)$ and it is called the algebra associated to A. If ∇ and ∇' are two linear connections on M and $A = \nabla' - \nabla$, then $\mathcal{U}(M,A)$ is called the deformation algebra defined by the pair (∇, ∇') ([10]).

In the present paper we continue and develop the study of [4], generalizing the notion of quarter-symmetric metric connections along the line of symmetric connections on pseudo-Riemann manifolds. Interesting properties of semi-symmetric connections or quarter-symmetric connections can be obtained on manifolds endowed with special structures ([1], [6], [7]) and extensive literature with applications can be mentioned ([2], [12]).

The aim of this work is to characterize the F-principal vector fields in the deformation algebra of two linear connections. It is illustrated the close ties between

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certain algebraic properties of the Golab algebras and the geometric properties of the manifold. It is proven that the Golab algebra is associative is equivalent with the fact that the curvature tensor of the quarter-symmetric metric connection R coincides with the curvature tensor of the Levi-Civita connection $\overset{\circ}{R}$, when $\overset{\circ}{R}_p$ is a surjective mapping. This invariance is also studied on Einstein spaces. In the last section Golab connections are extended.

1 F -principal vector fields in the algebra associated to a $(1,2)$-tensor field

Definition 1.1 Let $F \in \mathcal{T}^{(1,1)}(M)$ and $A \in \mathcal{T}^{(1,2)}(M)$. Let m be a positive integer. An element $X \in \mathcal{U}(M,A)$ is called a (m,F) -principal vector field if there exists a 1-form $\omega \in \Lambda^1(M)$ such that

$$A(Z,X^{(m)}) = \omega(Z)F(X), \forall Z \in \mathcal{X}(M), X^{(m)} = X^{(m-1)} \circ X, X^1 = X. \tag{1.1}$$

Remark 1.1 Almost (m,F)-principal vector fields were studied in ([3]). In the present paper one considers (m,F)-principal vector fields, with $m=1$, called F-principal vector fields.

Proposition 1.1 *Let $F \in \mathcal{T}^{(1,1)}(M)$ and $A \in \mathcal{T}^{(1,2)}(M)$.*
The following assertions are equivalent:
i) All the elements of the algebra $\mathcal{U}(M,A)$ are F- principal vector fields.
ii) There exists a 1-form $\omega \in \Lambda^1(M)$ such that

$$A = \omega \otimes F. \tag{1.2}$$

Proposition 1.2 *If the algebra $\mathcal{U}(M,A)$ is commutative and $rank(F)=n$, the following assertions are equivalent:*
i) All the elements of the algebra $\mathcal{U}(M,A)$ are F- principal vector fields.
ii) $A=0$.
Proof. i) $\Rightarrow$ ii) In local coordinates (1.2) becomes

$$A^r_{kj} = \omega_k F^r_j.$$

From $A^i_{jk} = A^i_{kj}$, one has $\omega_k F^r_j = \omega_j F^r_k$. Therefore $(\omega_k \delta^s_j - \omega_j \delta^s_k) F^s_s = 0$.
Since $rank(F)=n$, the previous relation implies

$$\omega_k \delta^s_j - \omega_j \delta^s_k = 0.$$

We take $s=j$, we summ and get $(n-1)\omega_k = 0$. Hence $\omega = 0$ and $A=0$.
ii) $\Rightarrow$ *i*). Obvious. □

Theorem 1.1 *Let (M,g) be a 2 -dimensional Riemann space such that the Ricci tensor is nondegenerate. Let ∇, respectively $\overline{\nabla}$ be the Levi-Civita connection associated to g, respectively Ric and $A = \overline{\nabla} - \nabla$. We consider $F \in \mathcal{T}^{(1,1)}(M)$ defined by $g(F(X),Y) = Ric(X,Y), \forall X,Y \in \mathcal{X}(M)$.*
The following assertions are equivalent:

i) (M, g) *is a space of (nonvanishing) constant curvature.*
ii) All the elements of the algebra $\mathcal{U}(M, A)$ *are* F*- principal vector fields.*
iii) ∇ *and* $\overline{\nabla}$ *have the same geodesics.*
iv) $\nabla = \overline{\nabla}$.
Proof. i) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv) ([10])
ii) $\Leftrightarrow$ iv) We use proposition 1.2. □

Theorem 1.2. *Let* $A \in \mathcal{T}^{(1,2)}(M)$. *If* $\mathcal{U}(M, A)$ *is a commutative algebra and* $F \in \mathcal{T}^{(1,1)}(M)$ *such that* $F^2 = \epsilon I$, *where* $\epsilon \in \{-1, 1\}$ *and* I *is the identity tensor field, then the following assertions are equivalent:*
i) All the elements of the algebra $\mathcal{U}(M, A)$ *are* F*- principal vector fields.*
ii) $A = 0$.
Proof. One uses Proposition 1.2. □

The geometric significance of the F-principal vector fields for hypersurfaces in the Euclidean space is given by the following results:
Theorem 1.3 *Let* $M \subset \mathbf{R}^{n+1}$ *be a hypersurface in the Euclidean space,* $n \geq 2$. *Let* g, *respectively* b *be the first, respectively the second fundamental form. Let* ∇, *respectively* $\overline{\nabla}$ *be the Levi-Civita connection associated to* g, *respectively* b. *We consider* $A = \overline{\nabla} - \nabla$ *and* F *the shape operator.*
The following assertions are equivalent:
i) the $\overline{\nabla}$ *-geodesics are the* ∇*-geodesics.*
ii) the ∇ *-geodesics are the* $\overline{\nabla}$*-geodesics.*
iii) $\nabla_X b = 0, \forall X \in \mathcal{X}(M)$.
iv) $\nabla = \overline{\nabla}$.
v) M *is a spheric hypersurface.*
vi) All the elements of the algebra $\mathcal{U}(M, A)$ *are* F*- principal vector fields.*
Proof. i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv) $\Leftrightarrow$ v) We use Theorem D ([11]).
iv) $\Leftrightarrow$ vi) From proposition 1.2. □

2 Quarter-symmetric metric connections on pseudo-Riemannian manifolds

Let (M, g) be an n-dimensional pseudo-Riemannian manifold, $\theta \in \Lambda^1(M)$ and $F \in \mathcal{T}^{(1,1)}(M)$.

Definition 2.1 A linear connection ∇ on M is called a *quarter-symmetric metric connection* or *Golab connection* associated to the pair (θ, F) if

$$\nabla_X g = 0, \nabla_X Y - \nabla_Y X - [X, Y] = \theta(Y)F(X) - \theta(X)F(Y),$$

$\forall X, Y \in \mathcal{X}(M)$.

Remark 2.1 For a given pair $(\theta, F), \theta \in \Lambda^1(M), F \in \mathcal{T}^{(1,1)}(M)$ on a pseudo-Riemannain manifold (M, g), there exists an unique Golab connection associated to (θ, F).

If one denotes by $\overset{\circ}{\nabla}$ the Levi-Civita connection associated to g, then the quarter-symmetric metric connection associated to (θ, F) is given by the formula

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \theta(Y)F(X) - S(X, Y)P, \forall X, Y \in \mathcal{X}(M), \tag{2.1}$$

where $g(P,Z)=\theta(Z), S(X,Y)=g(F(X),Y), \forall X,Y,Z \in \mathcal{X}(M)$.

The deformation algebra $\mathcal{U}\left(M, \nabla - \overset{\circ}{\nabla}\right)$ is called the Golab algebra associated to the pair (θ, F).

We denote by $\overline{\nabla}$ the transposed connection of ∇, i.e.,

$$\overline{\nabla}_X Y = \nabla_Y X + [X,Y].$$

The relation (2.1) leads to

$$\overline{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \theta(X)F(Y) - S(X,Y)P. \tag{2.2}$$

Let us denote by $\overset{s}{\nabla}$ the symmetric connection associated to ∇ i.e.
$\overset{s}{\nabla} = \frac{1}{2}(\nabla + \overline{\nabla})$. Hence

$$\overset{s}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \frac{1}{2}\theta(X)F(Y) + \frac{1}{2}\theta(Y)F(X) - \frac{1}{2}\{S(X,Y)+S(Y,X)\}P. \tag{2.3}$$

Let $R, \overset{\circ}{R}, \overline{R}$ and $Ric, \overset{\circ}{Ric}, \overline{Ric}$ be the curvature, respectively the Ricci tensors associated to $\nabla, \overset{\circ}{\nabla}, \overline{\nabla}$.

One denotes by $A = \nabla - \overset{\circ}{\nabla}, \overline{A} = \overline{\nabla} - \overset{\circ}{\nabla}, \overset{s}{A} = \overset{s}{\nabla} - \overset{\circ}{\nabla}$ and therefore

$$A(X,Y) = \theta(Y)F(X) - S(X,Y)P, \tag{2.1$'$}$$

$$\overline{A}(X,Y) = \theta(X)F(Y) - S(Y,X)P, \tag{2.2$'$}$$

$$\overset{s}{A}(X,Y) = \frac{1}{2}\{\theta(X)F(Y)+\theta(Y)F(X)\} - \frac{1}{2}\{S(X,Y)+S(Y,X)\}P. \tag{2.3$'$}$$

Theorem 2.1 *Let (M,g) be an n-dimensional $(n>3)$ pseudo-Riemannian manifold. Let θ be a 1-form on M and $F = fI \in \mathcal{T}^{(1,1)}(M)$, where $f \in \mathcal{F}(M), f(p) \neq 0, \forall p \in M$ and I is the identity tensor field. Let ∇ be the Golab connection associated to the pair (θ, F).*

If the mapping $\overset{\circ}{R}_p$: $T_pM \times T_pM \times T_pM \longrightarrow T_pM$ is surjective, for each $p \in M$, then the following assertions are equivalent:

i) $\theta = 0$.

ii) $R = \overset{\circ}{R}$.

iii) $Ric = \overset{\circ}{Ric}$.

iv) $\overline{R} = \overset{\circ}{R}$.

v) $\overset{s}{R} = \overset{\circ}{R}$, for $n \neq 4$.

vi) The Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is commutative.

vii) The Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is associative.

viii) All the elements of the Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ are F- principal vector fields.

Proof. i)⇒ii), i) ⇒iii), i)⇒iv), i)⇒v), i)⇔vi), i)⇒vii) are obvious.

ii)⇒i). From ii) we get $\nabla_X R = \nabla_X \overset{\circ}{R}, \forall X \in \mathcal{X}(M)$. Also

$$(\nabla_X R)(Y,Z,V) = (\overset{\circ}{\nabla}_X \overset{\circ}{R})(Y,Z,V) + A(X, \overset{\circ}{R}(Y,Z)V) - \\ - \overset{\circ}{R}(A(X,Y),Z)V - \overset{\circ}{R}(Y, A(X,Z))V - \overset{\circ}{R}(Y,Z)A(X,V). \tag{2.4}$$

Using Bianchi identities, (2.4) and (2.1)', one has in coordinates

$$f[(\delta^r_i \overset{\circ}{R}{}^{\,q}_{ljk} + \delta^r_j \overset{\circ}{R}{}^{\,q}_{lki} + \delta^r_k \overset{\circ}{R}{}^{\,q}_{lij})\theta_q + (g_{il} \overset{\circ}{R}{}^{\,r}_{qjk} + g_{jl} \overset{\circ}{R}{}^{\,r}_{qki} + g_{kl} \overset{\circ}{R}{}^{\,r}_{qij})\theta^q] = 0, \tag{2.5}$$

where $\theta^q = g^{iq}\theta_i$. Contracting $r = i$ and summing, (2.5) implies

$$[(n-3) \overset{\circ}{R}_{rljk} + g_{kl} \overset{\circ}{Ric}_{rj} - g_{jl} \overset{\circ}{Ric}_{rk}]\theta^r = 0. \tag{2.6}$$

Multiplying with g^{jl} in (2.6) and summing, one gets

$$(n-2) \overset{\circ}{Ric}_{qk}\theta^q = 0. \tag{2.7}$$

From (2.6) and (2.7) we get $(n-3) \overset{\circ}{R}_{qljk} \theta^q = 0$. Since $n > 3$, one has $\theta \circ \overset{\circ}{R} = 0$. Moreover $\overset{\circ}{R}_p$ is surjective and then $\forall p \in M, \theta_p(T_pM) = 0$ and $\theta_p = 0$. Therefore $\theta = 0$.

iii) ⇒ ii) From (2.1) one gets

$$R^i_{jkl} = \overset{\circ}{R}{}^{\,i}_{jkl} - \delta^i_k(\pi_{j,l} - \pi_i\pi_l) + \delta^i_l(\pi_{j,k} - \pi_j\pi_k) + \\ + g_{jk}g^{iq}(\pi_{q,l} - \pi_q\pi_l) - g_{jl}g^{iq}(\pi_{q,k} - \pi_q\pi_k) - \pi^q\pi_q(\delta^i_k g_{jl} - \delta^i_l g_{jk}), \tag{2.8}$$

where $\pi = f\theta$. Using iii), (2.8) becomes

$$\pi_{j,l} - \pi_i\pi_l = \frac{1}{n-2} g_{jl}\{(1-n)\pi_q\pi^q - g^{rq}(\pi_{r,q} - \pi_r\pi_q)\}. \tag{2.9}$$

Multiplying with g^{ji} and summing, one has

$$g^{rq}(\pi_{r,q} - \pi_r\pi_q) = -\frac{n}{2}\pi^q\pi_q. \tag{2.10}$$

Replacing (2.10) in (2.9), we find

$$\pi_{j,l} - \pi_j\pi_l = -\frac{1}{2} g_{jl}\pi^q\pi_q. \tag{2.11}$$

From (2.11) and (2.8) one has $R = \overset{\circ}{R}$.

iv) ⇒ i) From $\overline{R} = \overset{\circ}{R}$, one gets $\overline{\nabla}_X \overline{R} = \overline{\nabla}_X \overset{\circ}{R}, \forall X \in \mathcal{X}(M)$. Hence, using Bianchi identities, one has

$$\overline{A}^r_{il} \overset{\circ}{R}{}^{\,q}_{rjk} + \overline{A}^r_{jl} \overset{\circ}{R}{}^{\,q}_{rki} + \overline{A}^r_{kl} \overset{\circ}{R}{}^{\,q}_{rij} = \overline{A}^q_{ir} \overset{\circ}{R}{}^{\,r}_{ljk} + \overline{A}^q_{jr} \overset{\circ}{R}{}^{\,r}_{lki} + \overline{A}^q_{kr} \overset{\circ}{R}{}^{\,r}_{lij}. \tag{2.12}$$

Using $\overline{A}^k_{ij} = f(\theta_i\delta^k_j - g_{ij}\theta^k)$, we get

$$(g_{il} \overset{\circ}{R}{}^{\,q}_{rjk} + g_{jl} \overset{\circ}{R}{}^{\,q}_{rki} + g_{kl} \overset{\circ}{R}{}^{\,q}_{rij})\pi^r = 0, \tag{2.13}$$

where $\overline{A}^k_{ij}$ are the components of $\overline{A}$. Multiplying with g^{il} and summing, we obtain $(n-2)\overset{\circ}{R}{}^{q}_{jkl}\pi_q = 0$. Since $n > 2$, we get $f\theta = 0$ and therefore $\theta = 0$.

vii) $\Rightarrow$ i) The associativity condition

$X \circ (Y \circ Z) = (X \circ Y) \circ Z$, $\forall X, Y, Z \in \mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$

becomes

$$(2.14)\qquad [g(Y,Z)\pi(X) + g(X,Z)\pi(Y) - g(X,Y)\pi(Z)]P - g(Z,Y)\pi(P)X = 0,$$

where $\pi = f\theta$. For $Z = Y$ we get $\pi(X)P = \pi(P)X$. Then $\pi_i\pi^r = \pi_q\pi^q\delta^r_i$.

Taking $i = r$ and summing, we obtain $f\theta = 0$ and therefore $\theta = 0$.

v) $\Rightarrow$ i) From $\overset{s}{\nabla}_X \overset{s}{R} = \overset{s}{\nabla}_X \overset{\circ}{R}$, using Bianchi identities, one has

$$(2.15)\quad 2\pi^r(g_{ih}\overset{\circ}{R}{}^{l}_{rjk} + g_{jh}\overset{\circ}{R}{}^{l}_{rki} + g_{kh}\overset{\circ}{R}{}^{l}_{rij}) + \pi_r(\delta^l_i\overset{\circ}{R}{}^{r}_{hjk} + \delta^l_j\overset{\circ}{R}{}^{r}_{hki} + \delta^l_k\overset{\circ}{R}{}^{r}_{hij}) = 0,$$

where $\pi = f\theta$. Contracting $l = i$ in (2.15), one has

$$(2.16)\qquad 2\pi^r(g_{kh}\overset{\circ}{Ric}_{rj} - g_{jh}\overset{\circ}{Ric}_{rk}) + (n-4)\overset{\circ}{R}{}^{r}_{hjk}\pi_r = 0.$$

Multiplying with g^{jh} and summing, we get $(n-2)\overset{\circ}{Ric}_{rk}\pi^r = 0$.

Formula (2.16) implies $(n-4)\overset{\circ}{R}{}^{r}_{hjk}\pi_r = 0$. From $\theta \circ \overset{\circ}{R} = 0$, we get $\theta = 0$, since $\overset{\circ}{R}_p$ is surjective, $\forall p \in M$.

i) $\Leftrightarrow$ viii) One uses the Proposition 2.1. □

A more general characterization of the associativity condition can be given:

Theorem 2.2 *The same hypothesis of the Theorem 2.1.*

i) If $R = \lambda\overset{\circ}{R}$, where λ is a nonvanishing constant, then the Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is associative.

ii) If $\overline{R} = \lambda\overset{\circ}{R}$, where λ is a nonvanishing constant, then the deformation algebra $\mathcal{U}(M, \overline{\nabla} - \overset{\circ}{\nabla})$ is associative.

Proof. i). From $R = \lambda\overset{\circ}{R}$ one gets

$$\begin{aligned}(\nabla_X R)(Y,Z,V) = \lambda\{(\overset{\circ}{\nabla}_X\overset{\circ}{R})(Y,Z,V) + A(X,\overset{\circ}{R}(Y,Z)V) -\\ -\overset{\circ}{R}(A(X,Y),Z)V - \overset{\circ}{R}(Y,A(X,Z))V - \overset{\circ}{R}(Y,Z)A(X,V)\}.\end{aligned}$$

Using Bianchi identities and the fact that $\overset{\circ}{R}_p$ is a surjective mapping, one has $\theta = 0$. Theorem 2.1 implies that $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is associative.

ii) One uses a similar argument. □

3 Quarter-symmetric metric connections on Einstein spaces

In the sequel we consider $F \in \mathcal{T}^{(1,1)}(M)$, given by $g(F(X), Y) = Ric(X, Y)$ and θ an arbitrary 1-form on M.

In the case of an Einstein space, certain algebraic properties of some properly chosen deformation algebras are translated into geometric ones.

Theorem 3.1 *Let (M, g) be an n-dimensional $(n > 3)$, Einstein space.*

If ∇ is the quarter-symmetric metric connection associated to the pair (θ, F), $\overline{\nabla}$ is its transposed connection and $\overset{s}{\nabla}$ is its associated symmetric connection, then the following assertions are equivalent:

i) The Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is associative.

ii) The Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is commutative.

iii) The deformation algebra $\mathcal{U}(M, \overline{\nabla} - \overset{\circ}{\nabla})$ is associative.

iv) The deformation algebra $\mathcal{U}(M, \overline{\nabla} - \overset{\circ}{\nabla})$ commutative.

v) The deformation algebra $\mathcal{U}(M, \overset{s}{\nabla} - \overset{\circ}{\nabla})$ is associative.

vi) $\theta = 0$.

vii) All the elements of the Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ are F- principal vector fields.

Proof. vi) $\Rightarrow$i), vi) $\Rightarrow$ii), vi) $\Rightarrow$iii), vi) $\Rightarrow$iv), vi) $\Rightarrow$v), iv) $\Rightarrow$vi) are obvious.

i) $\Rightarrow$vi) Since the Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is associative, we find

$$A^r_{jk} A^q_{ir} = A^r_{ij} A^q_{rk}. \tag{3.1}$$

(M, g) is an Einstein space and then $Ric = \alpha g, \alpha$ being a non vanishing constant. Therefore $F^i_j = \alpha \delta^i_j$ and (3.1) becomes

$$\delta^q_i g_{jk} \theta^r \theta_r - g_{ij} \theta_k \theta^q + g_{jk} \theta_i \theta^q + g_{ik} \theta_j \theta^q = 0. \tag{3.2}$$

Taking $q = i$ and summing, we obtain $\theta_q \theta^q = 0$. Formula (3.2) implies

$$g_{ij} \theta_k \theta^q - g_{jk} \theta_i \theta^q - g_{ik} \theta_j \theta^q = 0. \tag{3.3}$$

Multiplying with g^{jk} and summing, (3.3) implies $(n-2)\theta_i \theta^q = 0, \forall i, q \in \{1, \ldots, n\}$ and then $\theta = 0$.

ii) $\Rightarrow$vi) Since the Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is commutative, one gets $\delta^k_i \theta_j - g_{ij} \theta^k = \delta^k_j \theta_i - g_{ij} \theta^k$. Taking $k = i$ and summing, we obtain $\theta = 0$.

iii) $\Rightarrow$vi) The algebra $\mathcal{U}(M, \overline{\nabla} - \overset{\circ}{\nabla})$ is associative and then

$$\overline{A}^r_{jk} \overline{A}^q_{ir} = \overline{A}^r_{ij} \overline{A}^q_{rk}. \tag{3.4}$$

Using $\overline{A}^k_{ij} = \alpha(\theta_i \delta^k_j - g_{ij} \theta^k)$, one has $(n-2)\theta_i \theta^q = 0, \forall i, q \in \{1, \ldots, n\}$ and then $\theta = 0$.

v) $\Rightarrow$ vi)

Since $\overset{s}{A}{}^k_{ij} = \frac{\alpha}{2}(\theta_i \delta^k_j + \theta_j \delta^k_i - 2 g_{ij} \theta^k)$,

the associativity condition of the algebra $\mathcal{U}(M, \overset{s}{\nabla} - \overset{\circ}{\nabla})$ implies

$$\begin{gathered} \delta^l_i \theta_j \theta_k - 2\delta^l_i g_{jk} \theta_r \theta^r - 2 g_{ij} \theta_k \theta^l + \\ + 6 g_{jk} \theta_i \theta^l - \delta^l_k \theta_i \theta_j + 2\delta^l_k g_{ij} \theta^r \theta_r = 0. \end{gathered} \tag{3.5}$$

Taking $l = i$ and summing, then (3.5) becomes

$$(n-3)\theta_j\theta_k - 2(n-4)g_{jk}\theta^r\theta_r = 0. \tag{3.6}$$

Multiplying with g^{jk} and summing, one has $\theta^r\theta_r = 0$. The relation (3.6) implies $(n-3)\theta_j\theta_k = 0$ and then $\theta = 0$.

vi) $\Leftrightarrow$ vii) One uses the Proposition 1.2. □

The invariance of the curvature tensor field or the Ricci tensor field is one of the central concepts of Riemannian geometry and it can be studied from different points of view. We illustrate the close ties that exist between this invariance and the algebraic properties of the Golab deformation algebra.

Theorem 3.2 *Under the same hypothesis as the previous theorem, one has:*

i) If $\overline{R} = \overset{\circ}{R}$ *(or* $\overset{s}{R} = \overset{\circ}{R}$*, for* $n \neq 4$*). then the Golab algebra* $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ *is associative.*

ii) If $Ric = \overset{\circ}{Ric}$ *. then the Golab algebra* $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ *is associative.*

Proof. i) If $\overline{R} = \overset{\circ}{R}$, using $\overline{\nabla}_X \overset{\circ}{R} = \overline{\nabla}_X \overline{R}$ and the Bianchi identity, one has

$(n-2)\overset{\circ}{R}{}^{\,i}_{jkl}\beta_i = 0$, where $\beta = \alpha\theta$.

Hence $\theta = 0$. Therefore the Golab algebra $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ is associative.

If $\overset{s}{R} = \overset{\circ}{R}$, for $n \neq 4$, we get $\overset{s}{\nabla}_X \overset{s}{R} = \overset{s}{\nabla}_X \overset{\circ}{R}$. It follows

$$(n-4)\overset{\circ}{R}{}^{\,r}_{hjk}\beta_r + 2\beta^r(g_{hk}\overset{\circ}{R}_{rj} - g_{jh}\overset{\circ}{R}_{rh}) = 0.$$

We multiply by g^{jh} and summ. One gets

$$(n-2)\overset{\circ}{R}_{qk}\,\beta^q = 0.$$

Therefore

$$(n-4)\overset{\circ}{R}{}^{\,r}_{hjk}\beta_r = 0.$$

Hence $\beta = 0$. The result follows from $\theta = 0$.

ii) $Ric = \overset{\circ}{Ric}$ implies

$$\beta_{i,j} - \beta_i\beta_j = -\frac{1}{2}g_{ij}\beta^k\beta_k.$$

Therefore, by a direct computation we get $R = \overset{\circ}{R}$ and then we use the idea of i). □

4 F-principal Golab connections

The aim of the last section is to extend the notion of quarter-symmetric metric connections.

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Let $\theta \in \Lambda^1(M)$, $F \in \mathcal{T}^{(1,1)}(M)$ and ∇ be the Golab connection associated to (θ, F).

Definition 4.1 A linear connection $\tilde{\nabla}$ on M is called a *F-principal Golab connection* if all the elements of the algebra $\mathcal{U}(M, \tilde{\nabla} - \nabla)$ are F-principal vector fields.

Theorem 4.1 *Let (M, g) be an n-dimensional pseudo-Riemannian manifold, $\theta \in \Lambda^1(M)$ and $F \in \mathcal{T}^{(1,1)}(M)$.*

If ∇ is the quarter-symmetric metric connection associated to (θ, F) and $\tilde{\nabla}$ is a F-principal connection, then the deformation algebras $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ and $\mathcal{U}(M, \tilde{\nabla} - \overset{\circ}{\nabla})$ have the same F- principal vector fields.

Proof. The proposition 1.1 implies that $\tilde{\nabla}$ is a F-principal connection if and only if there exists the 1-form $\omega \in \Lambda^1(M)$ such that

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(X)F(Y), \forall X, Y \in \mathcal{X}(M).$$

The previous relation implies

$$\tilde{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \omega(X)F(Y) + \theta(Y)F(X) - S(X, Y)P,$$

where $g(P, Z) = \theta(Z), S(X, Y) = g(F(X), Y), \forall X, Y, Z \in \mathcal{X}(M)$.

We denote $A = \nabla - \overset{\circ}{\nabla}, \tilde{A} = \tilde{\nabla} - \overset{\circ}{\nabla}$. Therefore

$$A(X, Y) = \theta(Y)F(X) - S(X, Y)P,$$

$$\tilde{A}(X, Y) = \omega(X)F(Y) + \theta(Y)F(X) - S(X, Y)P.$$

Hence $\tilde{A}(X, Y) - A(X, Y) = \omega(X)F(Y), \forall X, Y \in \mathcal{X}(M)$.

If $W \in \mathcal{U}(M, A)$ is a F-principal vector field, there exists $\sigma \in \Lambda^1(M)$ such that $A(Z, W) = \sigma(Z)F(W), \forall Z \in \mathcal{X}(M)$.

Hence $\tilde{A}(Z, W) = (\sigma + \omega)(Z)F(W), \forall Z \in \mathcal{X}(M)$. Therefore W is a F-principal vector field in the algebra $\mathcal{U}(M, \tilde{A})$.

The converse is also true. This implies that the deformation algebras $\mathcal{U}(M, \nabla - \overset{\circ}{\nabla})$ and $\mathcal{U}(M, \tilde{\nabla} - \overset{\circ}{\nabla})$ have the same F- principal vector fields. □

Example 4.1

Let (M, g) be a pseudo-Riemannian manifold, $\overset{\circ}{\nabla}$ be the Levi-Civita associated to g, Ric be the Ricci tensor field and K be the Ricci invariant. One considers the 1-form $\theta \in \Lambda^1(M)$ defined by $\theta(X) = \overset{\circ}{\nabla}_X K, \forall X \in \mathcal{X}(M)$ and let $F \in \mathcal{T}^{(1,1)}(M)$, given by $g(F(X), Y) = Ric(X, Y), \forall X, Y \in \mathcal{X}(M)$.

The quarter-symmetric metric connection ∇ associated to the pair (θ, F) is given by the formula

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \theta(Y)F(X) - Ric(X, Y)P, \forall X, Y \in \mathcal{X}(M),$$

where $g(P, Z) = \theta(Z), \forall Z \in \mathcal{X}(M)$.

Let $\omega \in \Lambda^1(M)$ be an arbitrary 1-form. Therefore the linear connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \omega(X)F(Y) + \theta(Y)F(X) - Ric(X, Y)P, \forall X, Y \in \mathcal{X}(M),$$

is a F-principal Golab connection.

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Authors' address:

Iulia Elena Hirică, Liviu Nicolescu
University of Bucharest, Faculty of Mathematics and Informatics,
Str. Academiei nr.14, RO-010014 Bucharest, Romania.
E-mail: ihirica@fmi.unibuc.ro , lnicol@geometry.math.unibuc.ro