Metric nonlinear connections on Lie algebroids

Liviu Popescu

Abstract. In this paper the problem of compatibility between a nonlinear connection and other geometric structures on Lie algebroids is studied. The notion of dynamical covariant derivative is introduced and a metric nonlinear connection is found in the more general case of Lie algebroids. We prove that the canonical nonlinear connection induced by a regular Lagrangian on a Lie algebroid is the unique connection which is metric and compatible with the symplectic structure.

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1 Introduction

The notions of nonlinear connection and metric structure are important tools in the differential geometry of the tangent bundle of a manifold. The metric compatibility of a nonlinear connection generalize the compatibility between a Riemannian metric and the linear connection and it is known as one of the Helmholtz conditions for the inverse problem of Lagrangian Mechanics (see for instance [2, 3, 5, 8, 11, 13]).

In this paper we generalize the metric compatibility of a nonlinear connection at the level of Lie algebroids. The notion of Lie algebroid and its prolongation over the vector bundle projection generalize the concept of tangent bundle. Mackenzie [10] has been achieved a unitary study of Lie algebroids and together with Higgins [6] have introduced the notion of prolongation of a Lie algebroid over a smooth map. Weinstein [22] shows that is possible to give a common description of the most interesting classical mechanical systems. He developed a generalized theory of Lagrangian Mechanics and obtained the equations of motions, using the Poisson structure [21] on the dual of a Lie algebroid and the Legendre transformation associated with a regular Lagrangian. In the last years the problems raised by Weinstein and related topics have been investigated by many authors (see for instance [1, 7, 9, 12, 15, 16, 17, 19]).

In the present paper we study the problem of compatibility between a nonlinear connection and some other geometric structures on Lie algebroid and its prolongation over the vector bundle projection. The paper is organized as follows. The second

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section contains the preliminary results on Lie algebroids. In the section three, the compatibility between a nonlinear connection and a pseudo-Riemannian metric is studied. The notion of dynamical covariant derivative on the prolongation of Lie algebroid over the vector bundle projection is introduced and its action on the Berwald basis is given. We find the expression of the Jacobi endomorphism on Lie algebroids and the relation with the curvature of the nonlinear connection. We prove that the canonical nonlinear connection induced by a regular Lagrangian is the unique connection which is metric and compatible with a symplectic structure. Also, using the notions of v-covariant derivative, dynamical covariant derivative and Jacobi endomorphism, we obtain the Helmholtz conditions in the framework of a Lie algebroid.

2 Preliminaries on Lie algebroids

Let M be a real, C^∞-differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. A *Lie algebroid* over a manifold M is a triple $(E, [\cdot,\cdot]_E, \sigma)$, where (E, π, M) is a vector bundle of rank m over M, which satisfies the following conditions:
a) the $C^\infty(M)$-module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]_E$.
b) $\sigma : E \to TM$ is a bundle map (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from the Lie algebra of sections $(\Gamma(E), [\cdot,\cdot]_E)$ to the Lie algebra of vector fields $(\chi(M), [\cdot,\cdot])$ satisfying the Leibniz rule

$$[s_1, fs_2]_E = f[s_1, s_2]_E + (\sigma(s_1)f)s_2,\ \forall s_1, s_2 \in \Gamma(E),\ f \in C^\infty(M).$$

From the above definition it results:
1° $[\cdot,\cdot]_E$ is a $\mathbb{R}$-bilinear operation,
2° $[\cdot,\cdot]_E$ is skew-symmetric, i.e. $[s_1, s_2]_E = -[s_2, s_1]_E, \quad \forall s_1, s_2 \in \Gamma(E)$,
3° $[\cdot,\cdot]_E$ verifies the Jacobi identity

$$[s_1, [s_2, s_3]_E]_E + [s_2, [s_3, s_1]_E]_E + [s_3, [s_1, s_2]_E]_E = 0,$$

and σ being a Lie algebra homomorphism, means that $\sigma[s_1, s_2]_E = [\sigma(s_1), \sigma(s_2)]$. For $\omega \in \bigwedge^k(E^*)$ the *exterior derivative* $d^E\omega \in \bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned} d^E\omega(s_1, ..., s_{k+1}) \quad &= \quad \sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1, ..., \hat{s}_i, ..., s_{k+1}) + \\ &\quad + \sum_{1\leq i<j\leq k+1}(-1)^{i+j}\omega([s_i, s_j]_E, s_1, ..., \hat{s}_i, ..., \hat{s}_j, ...s_{k+1}). \end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and it results that $(d^E)^2 = 0$. For $\xi \in \Gamma(E)$ the *Lie derivative* with respect to ξ is given by $\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi$, where i_ξ is the contraction with ξ.
If we take the local coordinates (x^i) on an open $U \subset M$, a local basis $\{s_\alpha\}$ of the sections of the bundle $\pi^{-1}(U) \to U$ generates local coordinates (x^i, y^α) on E. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha\frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta]_E = L^\gamma_{\alpha\beta}s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m},$$

are called the *structure functions* of the Lie algebroid and satisfy the *structure equations* on Lie algebroid

$$\sigma_\alpha^j \frac{\partial \sigma_\beta^i}{\partial x^j} - \sigma_\beta^j \frac{\partial \sigma_\alpha^i}{\partial x^j} = \sigma_\gamma^i L_{\alpha\beta}^\gamma, \qquad \sum_{(\alpha,\beta,\gamma)} \left(\sigma_\alpha^i \frac{\partial L_{\beta\gamma}^\delta}{\partial x^i} + L_{\alpha\eta}^\delta L_{\beta\gamma}^\eta \right) = 0,$$

2.1 The prolongation of the Lie algebroid over the vector bundle projection

Let (E, π, M) be a vector bundle. For the projection $\pi : E \to M$ we can construct the prolongation of E (see [6, 9, 12, 15]). The associated vector bundle is $(\mathcal{T}E, \pi_2, E)$ where

$$\mathcal{T}E = \underset{w\in E}{\cup} \mathcal{T}_w E, \quad \mathcal{T}_w E = \{(u_x, v_w) \in E_x \times T_w E \mid \sigma(u_x) = T_w\pi(v_w), \quad \pi(w) = x \in M\},$$

and the projection $\pi_2(u_x, v_w) = \pi_E(v_w) = w$, where $\pi_E : TE \to E$ is the tangent projection. We also have the canonical projection $\pi_1 : \mathcal{T}E \to E$ given by $\pi_1(u, v) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E \to TE$, $\sigma^1(u, v) = v$ will be the anchor of a new Lie algebroid over the manifold E. An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection π_1. We will denote $(V\mathcal{T}E, \pi_{2|_{V\mathcal{T}E}}, E)$ the vertical bundle of $(\mathcal{T}E, \pi_2, E)$ and $\sigma^1|_{V\mathcal{T}E} : V\mathcal{T}E \to VTE$ is an isomorphism. The local basis of $\Gamma(\mathcal{T}E)$ is given by $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, where [12]

$$\mathcal{X}_\alpha(u) = \left(s_\alpha(\pi(u)), \sigma_\alpha^i \frac{\partial}{\partial x^i}\bigg|_u \right), \quad \mathcal{V}_\alpha(u) = \left(0, \frac{\partial}{\partial y^\alpha}\bigg|_u \right),$$

and $(\partial/\partial x^i, \partial/\partial y^\alpha)$ is the local basis on TE. The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha},$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta]_E = L_{\alpha\beta}^\gamma \mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta]_E = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_E = 0.$$

The local expression of the differential of a function L on E is $d^E L = \sigma_\alpha^i \frac{\partial L}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha} \mathcal{V}^\alpha$, where $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ denotes the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ and $d^E x^i = \sigma_\alpha^i \mathcal{X}^\alpha$, $d^E y^\alpha = \mathcal{V}^\alpha$. The differential of sections of $(\mathcal{T}E)^*$ is determined by

$$d^E \mathcal{X}^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d^E \mathcal{V}^\alpha = 0.$$

The other canonical geometric objects (see [9]) are *Euler section* $C = y^\alpha \mathcal{V}_\alpha$ and the *vertical endomorphism* or *tangent structure* $J = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$. A section $\mathcal{S}$ of $\mathcal{T}E$ is called *semispray* (or *second order differential equation -SODE*) if $J(\mathcal{S}) = C$. In local coordinates a semispray has the expression $\mathcal{S}(x, y) = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha(x, y)\mathcal{V}_\alpha$. If we have the relation $[C, \mathcal{S}]_E = \mathcal{S}$, then $\mathcal{S}$ is called spray and the functions $\mathcal{S}^\alpha$ are homogeneous functions of degree 2 in y^α.

A nonlinear connection on $\mathcal{T}E$ is an almost product structure N on $\pi_2 : \mathcal{T}E \to E$ (i.e. a bundle morphism $N : \mathcal{T}E \to \mathcal{T}E$, such that $N^2 = id$) smooth on $\mathcal{T}E \backslash \{0\}$ such that $V\mathcal{T}E = \ker(id + N)$. If N is a connection on $\mathcal{T}E$ then $H\mathcal{T}E = \ker(id - N)$ is the horizontal subbundle associated to N and $\mathcal{T}E = V\mathcal{T}E \oplus H\mathcal{T}E$. The connection N on $\mathcal{T}E$ induces two projectors $\mathrm{h}, \mathrm{v} : \mathcal{T}E \to \mathcal{T}E$ such that $\mathrm{h}(\rho) = \rho^{\mathrm{h}}$ and $\mathrm{v}(\rho) = \rho^{\mathrm{v}}$ for every $\rho \in \Gamma(\mathcal{T}E)$, where $\mathrm{h} = \frac{1}{2}(id + N)$ and $\mathrm{v} = \frac{1}{2}(id - N)$. The sections

$$\delta_\alpha = (\mathcal{X}_\alpha)^{\mathrm{h}} = \mathcal{X}_\alpha - N_\alpha^\beta \mathcal{V}_\beta,$$

generate a local basis of $H\mathcal{T}E$. The frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\mathcal{T}E$ called *Berwald basis*. The dual basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha + N_\beta^\alpha \mathcal{X}^\beta$. The Lie brackets of the Berwald basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are [15]

$$[\delta_\alpha, \delta_\beta]_E = L_{\alpha\beta}^\gamma \delta_\gamma + \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta]_E = \frac{\partial N_\alpha^\gamma}{\partial y^\beta} \mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_E = 0,$$

where

$$\mathcal{R}_{\alpha\beta}^\gamma = \delta_\beta(N_\alpha^\gamma) - \delta_\alpha(N_\beta^\gamma) + L_{\alpha\beta}^\varepsilon N_\varepsilon^\gamma. \tag{2.1}$$

The curvature of the connection N is given by $\Omega = -\mathrm{N_h}$ where

$$\mathrm{N_h}(z, w) = [\mathrm{h}z, \mathrm{h}w]_E - \mathrm{h}[\mathrm{h}z, w]_E - \mathrm{h}[z, \mathrm{h}w]_E + \mathrm{h}^2[z, w]_E,$$

is the Nijenhuis tensor of h. In local coordinates we have

$$\Omega = -\frac{1}{2} \mathcal{R}_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma,$$

where $\mathcal{R}_{\alpha\beta}^\gamma$ are given by (2.1) and represent the local coordinates expression of the curvature.

3 Dynamical covariant derivative and metric non-linear connection

Definition 3.1. A map $\nabla : \mathfrak{T}(\mathcal{T}E\backslash\{0\}) \to \mathfrak{T}(\mathcal{T}E\backslash\{0\})$ is said to be a tensor derivation on $\mathcal{T}E\backslash\{0\}$ if the following conditions are satisfied:
i) ∇ is $\mathbb{R}$-linear
ii) ∇ is type preserving, i.e. $\nabla(\mathfrak{T}_s^r(\mathcal{T}E\backslash\{0\}) \subset \mathfrak{T}_s^r(\mathcal{T}E\backslash\{0\})$, for each $(r, s) \in \mathbb{N} \times \mathbb{N}$.
iii) ∇ obeys the Leibnitz rule $\nabla(P \otimes S) = \nabla P \otimes S + P \otimes \nabla S$, for any tensors P, S on $\mathcal{T}E\backslash\{0\}$.
iv) ∇ commutes with any contractions, where $\mathfrak{T}_\bullet^\bullet(\mathcal{T}E\backslash\{0\})$ is the space of tensors on $\mathcal{T}E\backslash\{0\}$.

For a semispray $\mathcal{S}$ we consider the $\mathbb{R}$-linear map $\nabla_0 : \Gamma(\mathcal{T}E\backslash\{0\}) \to \Gamma(\mathcal{T}E\backslash\{0\})$ given by

$$\nabla_0\rho = \mathrm{h}[\mathcal{S}, \mathrm{h}\rho]_E + \mathrm{v}[\mathcal{S}, \mathrm{v}\rho]_E, \quad \forall \rho \in \Gamma(\mathcal{T}E\backslash\{0\}).$$

It results that

$$\nabla_0(f\rho) = \mathcal{S}(f)\rho + f\nabla_0\rho, \quad \forall f \in C^\infty(E),\ \rho \in \Gamma(\mathcal{T}E\backslash\{0\}).$$

Any tensor derivation on $\mathcal{T}E\backslash\{0\}$ is completely determined by its actions on smooth functions and sections on $\mathcal{T}E\backslash\{0\}$ (see [20] generalized Willmore's theorem, p. 1217). Therefore there exists a unique tensor derivation ∇ on $\mathcal{T}E\backslash\{0\}$ such that

$$\nabla\mid_{C^\infty(E)}=\mathcal{S},\quad \nabla\mid_{\Gamma(\mathcal{T}E\backslash\{0\})}=\nabla_0.$$

We will call the tensor derivation ∇, the *dynamical covariant derivative* induced by the semispray $\mathcal{S}$ and a nonlinear connection N.

Proposition 3.1. *The following formulas hold*

$$[\mathcal{S},\mathcal{V}_\beta]_E=-\delta_\beta-\left(N_\beta^\alpha+\frac{\partial S^\alpha}{\partial y^\beta}\right)\mathcal{V}_\alpha$$

$$[\mathcal{S},\delta_\beta]_E=\left(N_\beta^\alpha-L_{\beta\varepsilon}^\alpha y^\varepsilon\right)\delta_\alpha+\mathcal{R}_\beta^\gamma\mathcal{V}_\gamma$$

where

$$\mathcal{R}_\beta^\gamma=-\sigma_\beta^i\frac{\partial S^\gamma}{\partial x^i}-\mathcal{S}(N_\beta^\gamma)+N_\beta^\alpha N_\alpha^\gamma+N_\beta^\alpha\frac{\partial S^\gamma}{\partial y^\alpha}+N_\varepsilon^\gamma L_{\alpha\beta}^\varepsilon y^\alpha.$$

The action of the dynamical covariant derivative on the Berwald basis is given by

$$\nabla\mathcal{V}_\beta=\mathrm{v}[\mathcal{S},\mathcal{V}_\beta]_E=-\left(N_\beta^\alpha+\frac{\partial S^\alpha}{\partial y^\beta}\right)\mathcal{V}_\alpha$$

$$\nabla\delta_\beta=\mathrm{h}[\mathcal{S},\delta_\beta]_E=\left(N_\beta^\alpha-L_{\beta\varepsilon}^\alpha y^\varepsilon\right)\delta_\alpha.$$

It is not difficult to extend the action of ∇ to the algebra of tensors by requiring for ∇ to preserve the tensor product. For a pseudo-Riemannian metric g on $V\mathcal{T}E$ (i.e. a $(2,0)$-type symmetric tensor $g=g_{\alpha\beta}(x,y)\mathcal{V}^\alpha\otimes\mathcal{V}^\beta$ of rank m on $V\mathcal{T}E$) we have

$$(\nabla g)(\rho_1,\rho_2)=\mathcal{S}(g(\rho_1,\rho_2))-g(\nabla\rho_1,\rho_2)-g(\rho_1,\nabla\rho_2),$$

and in local coordinates we get

$$g_{\alpha\beta/}:=(\nabla g)(\mathcal{V}_\alpha,\mathcal{V}_\beta)=\mathcal{S}(g_{\alpha\beta})+g_{\gamma\beta}\left(N_\alpha^\gamma+\frac{\partial S^\gamma}{\partial y^\alpha}\right)+g_{\gamma\alpha}\left(N_\beta^\gamma+\frac{\partial S^\gamma}{\partial y^\beta}\right). \tag{3.1}$$

Definition 3.2. The nonlinear connection N is called metric or compatible with the metric tensor g if $\bigtriangledown g=0$, that is

$$\mathcal{S}(g(\rho_1,\rho_2))=g(\nabla\rho_1,\rho_2)+g(\rho_1,\nabla\rho_2).$$

If $\mathcal{S}$ be a semispray, N a nonlinear connection and ∇ the dynamical covariant derivative induced by $(\mathcal{S},N)$, then we set:

Proposition 3.2. *The nonlinear connection $\widetilde{N}$ with the coefficients given by*

$$\widetilde{N}_\beta^\alpha=N_\beta^\alpha-\frac{1}{2}g^{\alpha\gamma}g_{\gamma\beta/},$$

is a metric nonlinear connection.

Proof. Since N^α_β are the coefficients of a nonlinear connection and $g^{\alpha\gamma}g_{\gamma\beta/}$, are the components of a tensor of type (1,1) it results that $\widetilde{N}^\alpha_\beta$ are also the coefficients of a nonlinear connection. We consider the dynamical covariant derivative induced by $(S,\widetilde{N})$ and we have

$$\begin{aligned}(\nabla g)(\mathcal{V}_\alpha,\mathcal{V}_\beta) &= \mathcal{S}(g_{\alpha\beta}) + g_{\gamma\beta}\left(\widetilde{N}^\gamma_\alpha + \frac{\partial S^\gamma}{\partial y^\alpha}\right) + g_{\gamma\alpha}\left(\widetilde{N}^\gamma_\beta + \frac{\partial S^\gamma}{\partial y^\beta}\right) \\ &= \mathcal{S}(g_{\alpha\beta}) + g_{\gamma\beta}\left(N^\gamma_\alpha + \frac{\partial S^\gamma}{\partial y^\alpha}\right) + g_{\gamma\alpha}\left(N^\gamma_\beta + \frac{\partial S^\gamma}{\partial y^\beta}\right) - g_{\gamma\beta}\frac{1}{2}g^{\gamma\varepsilon}g_{\varepsilon\alpha/} \\ &- g_{\gamma\alpha}\frac{1}{2}g^{\gamma\varepsilon}g_{\varepsilon\beta/} = g_{\alpha\beta/} - \frac{1}{2}g_{\alpha\beta/} - \frac{1}{2}g_{\alpha\beta/} = 0\end{aligned}$$

that is the connection $\widetilde{N}$ is metric. □

3.1 The case of SODE connection

A semispray (*SODE*) given by $\mathcal{S} = y^\alpha\mathcal{X}_\alpha + \mathcal{S}^\alpha\mathcal{V}_\alpha$ determines an associated nonlinear connection

$$N = -\mathcal{L}_\mathcal{S}J,$$

with local coefficients

$$N^\beta_\alpha = \frac{1}{2}\left(-\frac{\partial\mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L^\beta_{\alpha\varepsilon}\right). \tag{3.2}$$

Proposition 3.3. *The following equations hold*

$$[\mathcal{S},\mathcal{V}_\beta]_E = -\delta_\beta + \left(N^\alpha_\beta - L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\mathcal{V}_\alpha, \tag{3.3}$$

$$[\mathcal{S},\delta_\beta]_E = \left(N^\alpha_\beta - L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\delta_\alpha + \mathcal{R}^\alpha_\beta\mathcal{V}_\alpha, \tag{3.4}$$

where

$$\mathcal{R}^\alpha_\beta = -\sigma^i_\beta\frac{\partial\mathcal{S}^\alpha}{\partial x^i} - \mathcal{S}(N^\alpha_\beta) - N^\alpha_\gamma N^\gamma_\beta + (L^\gamma_{\varepsilon\beta}N^\alpha_\gamma + L^\alpha_{\gamma\varepsilon}N^\gamma_\beta)y^\varepsilon. \tag{3.5}$$

The dynamical covariant derivative induced by S and associated nonlinear connection is characterized by

$$\nabla\mathcal{V}_\beta = \mathrm{v}[\mathcal{S},\mathcal{V}_\beta]_E = \left(N^\alpha_\beta - L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\mathcal{V}_\alpha = -\frac{1}{2}\left(\frac{\partial S^\alpha}{\partial y^\beta} + L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\mathcal{V}_\alpha,$$

$$\nabla\delta_\beta = \mathrm{h}[\mathcal{S},\delta_\beta]_E = \left(N^\alpha_\beta - L^\alpha_{\beta\varepsilon}y^\varepsilon\right)\delta_\alpha.$$

$$g_{\alpha\beta/} := (\nabla g)(\mathcal{V}_\alpha,\mathcal{V}_\beta) = \mathcal{S}(g_{\alpha\beta}) - g_{\gamma\beta}N^\gamma_\alpha - g_{\gamma\alpha}N^\gamma_\beta + \left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon, \tag{3.6}$$

which is equivalent to

$$(\nabla g)(\mathcal{V}_\alpha,\mathcal{V}_\beta) = \mathcal{S}(g_{\alpha\beta}) + \frac{1}{2}\frac{\partial\mathcal{S}^\gamma}{\partial y^\alpha}g_{\gamma\beta} + \frac{1}{2}\frac{\partial\mathcal{S}^\gamma}{\partial y^\beta}g_{\gamma\alpha} + \frac{1}{2}\left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon. \tag{3.7}$$

Definition 3.3. The Jacobi endomorphism is given by $\Phi = \mathrm{v}[\mathcal{S}, \mathrm{h}\rho]_E$.

Locally, from (3.4) we obtain that $\Phi = \mathcal{R}^\alpha_\beta \mathcal{V}_\alpha \otimes \mathcal{X}^\beta$, where $\mathcal{R}^\alpha_\beta$ is given by (3.5) and represent the local coefficients of the Jacobi endomorphism.

Proposition 3.4. *The following result holds*

$$\Phi = i_\mathcal{S}\Omega + \mathrm{v}[\mathrm{v}\mathcal{S}, \mathrm{h}\rho]_E.$$

Proof. Indeed, $\Phi(\rho) = \mathrm{v}[\mathcal{S}, \mathrm{h}\rho]_E = \mathrm{v}[\mathrm{h}\mathcal{S}, \mathrm{h}\rho]_E + \mathrm{v}[\mathrm{v}\mathcal{S}, \mathrm{h}\rho]_E$ and $\Omega(\mathcal{S}, \rho) = \mathrm{v}[\mathrm{h}\mathcal{S}, \mathrm{h}\rho]_E$, which yields $\Phi(\rho) = \Omega(\mathcal{S}, \rho) + \mathrm{v}[\mathrm{v}\mathcal{S}, \mathrm{h}\rho]_E$. □

If $\mathcal{S}$ is a spray, then the coefficients $\mathcal{S}^\alpha$ are 2-homogeneous with respect to the variables y^β and it results

$$2\mathcal{S}^\alpha = \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} y^\beta = -2N^\alpha_\beta y^\beta + L^\alpha_{\beta\gamma} y^\beta y^\gamma = -2N^\alpha_\beta y^\beta.$$

$$\mathcal{S} = \mathrm{h}\mathcal{S} = y^\alpha \delta_\alpha, \quad \mathrm{v}\mathcal{S} = 0, \quad N^\alpha_\beta = \frac{\partial N^\alpha_\varepsilon}{\partial y^\beta} y^\varepsilon + L^\alpha_{\beta\varepsilon} y^\varepsilon,$$

which yields $\Phi = i_\mathcal{S}\Omega$, and locally we get $\mathcal{R}^\alpha_\beta = \mathcal{R}^\alpha_{\varepsilon\beta} y^\varepsilon$.

3.1.1 Lagrangian case

Let us consider a regular Lagrangian L on E, that is the matrix

$$g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta},$$

has constant rank m. The symplectic structure induced by the regular Lagrangian is [12]

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{V}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma^i_\alpha - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma^i_\beta - \frac{\partial L}{\partial y^\gamma} L^\gamma_{\alpha\beta} \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

Let us consider the energy function given by

$$E_L := y^\alpha \frac{\partial L}{\partial y^\alpha} - L,$$

then the symplectic equation

$$i_\mathcal{S}\omega_L = -d^E E_L, \quad \mathcal{S} \in \Gamma(\mathcal{T}E),$$

and the regularity condition of the Lagrangian determine the components of the semispray

$$\mathcal{S}^\varepsilon = g^{\varepsilon\beta}\left(\sigma^i_\beta \frac{\partial L}{\partial x^i} - \sigma^i_\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - L^\theta_{\beta\alpha} y^\alpha \frac{\partial L}{\partial y^\theta} \right), \tag{3.8}$$

where $g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$.
The connection N with the coefficients given by (3.2), determined by the semispray

(3.8) will be called the *canonical nonlinear connection* induced by a regular Lagrangian L. Its coefficients are given by

(3.9)
$$N^\alpha_\beta = \frac{1}{2}g^{\alpha\varepsilon}\left[S(g_{\varepsilon\beta}) + \sigma^i_\beta\frac{\partial^2 L}{\partial x^i\partial y^\varepsilon} - \sigma^i_\varepsilon\frac{\partial^2 L}{\partial x^i\partial y^\beta} - L^\gamma_{\beta\varepsilon}\frac{\partial L}{\partial y^\gamma} + \left(g_{\gamma\varepsilon}L^\gamma_{\beta\theta} + g_{\gamma\beta}L^\gamma_{\varepsilon\theta}\right)y^\theta\right]$$

Theorem 3.5. *The canonical nonlinear connection N induced by a regular Lagrangian L is a metric nonlinear connection.*

Proof. Introducing the expression of the semispray (3.8) into the equation (3.7) we obtain

$$\begin{aligned}(\nabla g)(\mathcal{V}_\alpha,\mathcal{V}_\beta) \quad = \quad & y^\varepsilon\sigma^i_\varepsilon\frac{\partial g_{\alpha\beta}}{\partial x^i} + g^{\varepsilon\gamma}\left(\sigma^i_\gamma\frac{\partial L}{\partial x^i} - \sigma^i_\theta\frac{\partial^2 L}{\partial x^i\partial y^\gamma}y^\theta - L^\theta_{\gamma\tau}y^\tau\frac{\partial L}{\partial y^\theta}\right)\frac{\partial g_{\alpha\beta}}{\partial y^\varepsilon}\\ & +\frac{1}{2}\left(g_{\gamma\beta}\frac{\partial g^{\gamma\varepsilon}}{\partial y^\alpha} + g_{\gamma\alpha}\frac{\partial g^{\gamma\varepsilon}}{\partial y^\beta}\right)\left(\sigma^i_\varepsilon\frac{\partial L}{\partial x^i} - \sigma^i_\theta\frac{\partial^2 L}{\partial x^i\partial y^\varepsilon}y^\theta - L^\theta_{\varepsilon\tau}y^\tau\frac{\partial L}{\partial y^\theta}\right)\\ & +\frac{1}{2}\left(\sigma^i_\beta\frac{\partial^2 L}{\partial x^i\partial y^\alpha} + \sigma^i_\alpha\frac{\partial^2 L}{\partial x^i\partial y^\beta}\right) - \sigma^i_\varepsilon\frac{\partial g_{\alpha\beta}}{\partial x^i}y^\varepsilon\\ & -\frac{1}{2}\left(\sigma^i_\alpha\frac{\partial^2 L}{\partial x^i\partial y^\beta} + \sigma^i_\beta\frac{\partial^2 L}{\partial x^i\partial y^\alpha}\right) - \frac{1}{2}\frac{\partial L}{\partial y^\varepsilon}\left(L^\varepsilon_{\beta\alpha} + L^\varepsilon_{\alpha\beta}\right)\\ & -\frac{1}{2}\left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon + \frac{1}{2}\left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon.\end{aligned}$$

By direct computation, using the equalities

$$g_{\gamma\beta}\frac{\partial g^{\gamma\varepsilon}}{\partial y^\alpha} = -g^{\gamma\varepsilon}\frac{\partial g_{\gamma\beta}}{\partial y^\alpha} = -g^{\gamma\varepsilon}\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}, \quad L^\theta_{\alpha\beta} = -L^\theta_{\beta\alpha}$$

it results $(\nabla g)(\mathcal{V}_\alpha,\mathcal{V}_\beta) = 0$, which ends the proof. □

Theorem 3.6. *The canonical nonlinear connection induced by a regular Lagrangian is a unique connection which is metric and compatible with the symplectic structure ω_L, that is*

$$\nabla g = 0, \tag{3.10}$$

$$\omega_L(\mathrm{h}\rho,\mathrm{h}\nu) = 0, \quad \forall\rho,\nu\in\Gamma(\mathcal{T}E\backslash\{0\}) \tag{3.11}$$

Proof. Using the equation $\mathcal{V}^\alpha = \delta\mathcal{V}^\alpha - N^\alpha_\beta\mathcal{X}^\beta$ it results

$$\omega_L = g_{\alpha\beta}(\delta\mathcal{V}^\beta - N^\beta_\gamma\mathcal{X}^\gamma)\wedge\mathcal{X}^\alpha + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i\partial y^\beta}\sigma^i_\alpha - \frac{\partial^2 L}{\partial x^i\partial y^\alpha}\sigma^i_\beta - \frac{\partial L}{\partial y^\varepsilon}L^\varepsilon_{\alpha\beta}\right)\mathcal{X}^\beta\wedge\mathcal{X}^\alpha$$

$$= g_{\alpha\beta}\delta\mathcal{V}^\beta\wedge\mathcal{X}^\alpha + \frac{1}{2}\left(g_{\alpha\gamma}N^\gamma_\beta - g_{\beta\gamma}N^\beta_\alpha + \frac{\partial^2 L}{\partial x^i\partial y^\beta}\sigma^i_\alpha - \frac{\partial^2 L}{\partial x^i\partial y^\alpha}\sigma^i_\beta - \frac{\partial L}{\partial y^\varepsilon}L^\varepsilon_{\alpha\beta}\right)\mathcal{X}^\beta\wedge\mathcal{X}^\alpha$$

$$= g_{\alpha\beta}\delta\mathcal{V}^\beta\wedge\mathcal{X}^\alpha + \frac{1}{2}\left(N_{\alpha\beta} - N_{\beta\alpha} + \frac{\partial^2 L}{\partial x^i\partial y^\beta}\sigma^i_\alpha - \frac{\partial^2 L}{\partial x^i\partial y^\alpha}\sigma^i_\beta - \frac{\partial L}{\partial y^\varepsilon}L^\varepsilon_{\alpha\beta}\right)\mathcal{X}^\beta\wedge\mathcal{X}^\alpha,$$

where $N_{\alpha\beta} := g_{\alpha\gamma}N^{\gamma}_{\beta}$. We have that $\omega_L(\mathrm{h}\rho, \mathrm{h}\nu) = 0$ if and only if the second part of the above relation vanishes, that is

$$N_{[\alpha\beta]} = \frac{1}{2}(N_{\alpha\beta} - N_{\beta\alpha}) = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha}\sigma^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta}\sigma^i_\alpha + \frac{\partial L}{\partial y^\varepsilon}L^\varepsilon_{\alpha\beta}\right).$$

It result that the skew symmetric part of $N_{\alpha\beta}$ is uniquely determined by the condition (3.11). The symmetric part of $N_{\alpha\beta}$ is completely determined by the metric condition (3.10). Indeed

$$\begin{aligned}
\mathcal{S}(g_{\alpha\beta}) &= g_{\gamma\beta}N^\gamma_\alpha + g_{\gamma\alpha}N^\gamma_\beta - \left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon \\
&= N_{\beta\alpha} + N_{\alpha\beta} - \left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon \\
&= 2N_{(\alpha\beta)} - \left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon.
\end{aligned}$$

that is

$$2N_{(\alpha\beta)} = \mathcal{S}(g_{\alpha\beta}) + \left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon.$$

The equations (3.10) and (3.11) uniquely determine the coefficients of the nonlinear connection

$$\begin{aligned}
N^\gamma_\beta &= g^{\gamma\alpha}N_{\alpha\beta} = g^{\gamma\alpha}(N_{(\alpha\beta)} + N_{[\alpha\beta]}) \\
&= \frac{1}{2}g^{\gamma\alpha}\left[\mathcal{S}(g_{\alpha\beta}) + \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\sigma^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta}\sigma^i_\alpha - \frac{\partial L}{\partial y^\varepsilon}L^\varepsilon_{\beta\alpha} + \left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon\right]
\end{aligned}$$

Conversely, introducing (3.8) into (3.2) we have (3.9) which ends the proof. □

From [18, 4] we have:

Definition 3.4. A linear connection on Lie algebroid is a map $\mathcal{D} : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ which satisfies the rules
i) $\mathcal{D}_{\rho+\omega}\eta = \mathcal{D}_\rho\eta + \mathcal{D}_\omega\eta$,
ii) $\mathcal{D}_\rho(\eta + \omega) = \mathcal{D}_\rho\eta + \mathcal{D}_\rho\omega$,
iii) $\mathcal{D}_{f\rho}\eta = f\mathcal{D}_\rho\eta$,
iv) $\mathcal{D}_\rho(f\eta) = (\sigma(\rho)f)\eta + f\mathcal{D}_\rho\eta$,
for any function $f \in C^\infty(M)$ and sections $\rho, \eta, \omega \in \Gamma(E)$.

For $\rho, \eta \in \Gamma(E)$, the section $\mathcal{D}_\rho\eta \in \Gamma(E)$ is called the *covariant derivative* of the section η with respect to the section ρ. Let N be a nonlinear connection, then a linear connection $\mathcal{D}$ on Lie algebroid $(E, [\cdot,\cdot]_E, \sigma)$ is called $N-$linear connection if [14]
i) $\mathcal{D}$ preserves by parallelism the horizontal distribution $H\mathcal{T}E$.
ii) The tangent structure J is absolute parallel with respect to $\mathcal{D}$, that is $\mathcal{D}J = 0$.
Consequently, the following properties hold:

$$(\mathcal{D}_\rho\eta^{\mathrm{h}})^{\mathrm{v}} = 0, \quad (\mathcal{D}_\rho\eta^{\mathrm{v}})^{\mathrm{h}} = 0, \quad \mathcal{D}_\rho\mathrm{h} = 0, \quad \mathcal{D}_\rho\mathrm{v} = 0,$$

$$\mathcal{D}_\rho(J\eta^{\mathrm{h}}) = J(\mathcal{D}_\rho\eta^{\mathrm{h}}), \quad \mathcal{D}_\rho(J\eta^{\mathrm{v}}) = J(\mathcal{D}_\rho\eta^{\mathrm{v}}).$$

If we denote $\mathcal{D}^{\mathrm{h}}_\rho\eta = \mathcal{D}_{\rho^{\mathrm{h}}}\eta$, $\mathcal{D}^{\mathrm{v}}_\rho\eta = \mathcal{D}_{\rho^{\mathrm{v}}}\eta$ then the following decomposition is obtained

$$\mathcal{D}_\rho = \mathcal{D}^{\mathrm{h}}_\rho + \mathcal{D}^{\mathrm{v}}_\rho, \quad \rho \in \Gamma(E)$$

We remark that $\mathcal{D}^{\mathrm{h}}$ and $\mathcal{D}^{\mathrm{v}}$ are not covariant derivative, because

$$\mathcal{D}^{\mathrm{h}}_{\rho} f = \sigma(\rho^{\mathrm{h}}) f \neq \sigma(\rho) f, \quad \mathcal{D}^{\mathrm{v}}_{\rho} f = \sigma(\rho^{\mathrm{v}}) f \neq \sigma(\rho) f$$

but, it still preserves many properties of $\mathcal{D}$. Indeed, $\mathcal{D}^{\mathrm{h}}$ and $\mathcal{D}^{\mathrm{v}}$ satisfy the Leibniz rule, and $\mathcal{D}^{\mathrm{h}}$ and $\mathcal{D}^{\mathrm{v}}$ will be called the h$-$*covariant derivation* and v$-$ *covariant derivation*, respectively.

Remark 3.7. *The invariant form of Helmholtz conditions on Lie algebroids is given by:*

$$\begin{aligned} &\mathcal{D}^{\mathrm{v}}_{\rho} g(\nu, \theta) = \mathcal{D}^{\mathrm{v}}_{\theta} g(\nu, \rho), \\ &\nabla g = 0, \\ &g(\Phi\rho, \nu) = g(\Phi\nu, \rho), \end{aligned}$$

for $\nu, \rho, \theta \in \Gamma(E)$*, which in local coordinates yield*

$$\frac{\partial g_{\alpha\beta}}{\partial y^{\varepsilon}} = \frac{\partial g_{\alpha\varepsilon}}{\partial y^{\beta}},$$

$$\mathcal{S}(g_{\alpha\beta}) - g_{\gamma\beta} N^{\gamma}_{\alpha} - g_{\gamma\alpha} N^{\gamma}_{\beta} = y^{\varepsilon} \left(g_{\gamma\beta} L^{\gamma}_{\varepsilon\alpha} + g_{\gamma\alpha} L^{\gamma}_{\varepsilon\beta} \right),$$

$$g_{\alpha\gamma} \left(\sigma^{i}_{\beta} \frac{\partial \mathcal{S}^{\gamma}}{\partial x^{i}} + \mathcal{S} N^{\gamma}_{\beta} + N^{\varepsilon}_{\beta} N^{\gamma}_{\varepsilon} - (L^{\delta}_{\varepsilon\beta} N^{\gamma}_{\delta} + L^{\gamma}_{\delta\varepsilon} N^{\delta}_{\beta}) y^{\varepsilon} \right) =$$

$$g_{\beta\gamma} \left(\sigma^{i}_{\alpha} \frac{\partial \mathcal{S}^{\gamma}}{\partial x^{i}} + \mathcal{S} N^{\gamma}_{\alpha} + N^{\varepsilon}_{\alpha} N^{\gamma}_{\varepsilon} - (L^{\delta}_{\varepsilon\alpha} N^{\gamma}_{\delta} + L^{\gamma}_{\delta\varepsilon} N^{\delta}_{\alpha}) y^{\varepsilon} \right).$$

In the case of standard Lie algebroid $(TM, [\cdot, \cdot], id)$ we obtain the classical Helmholtz conditions [11].

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Author's address:

Liviu Popescu
University of Craiova, Faculty of Economics,
Dept. of Mathematics and Statistics,
13 Al. I. Cuza st., RO-200585 Craiova, Romania.
e-mail: liviupopescu@central.ucv.ro , liviunew@yahoo.com