Tangent structures and analytical mechanics

Maïdo Rahula

Abstract. We establish a link between the sector-forms of White [10] and the exterior forms of Cartan. We show that the Hamiltonian system on T^2M reduces to Lagrange's equations on the osculating bundle Osc$M$. The structures T^kM and $\mathrm{Osc}^{k-1}M$ are presented explicitly.

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1. Tangent bundles and osculators

The tangent functor T iterated k times associates to a smooth manifold M its k-fold tangent bundle T^kM (the kth *level* of M) and associates to a smooth map $\varphi : M_1 \to M_2$ the graded morphism $T^k\varphi : T^kM_1 \to T^kM_2$, the kth *derivative* of φ. The level T^kM has a multiple vector bundle structure with k projections onto $T^{k-1}M$

$$\rho_s \doteq T^{k-s}\pi_s : T^kM \to T^{k-1}M, \quad s = 1, 2, \dots, k,$$

where π_s is the natural projection $T^sM \to T^{s-1}M$.

Local coordinates in neighbourhoods

$$T^sU \subset T^sM, \ s = 1, 2, \dots, k, \quad \text{where } T^{s-1}U = \pi_s(T^sU),$$

are determined automatically by those in the neighbourhood $U \subset M$, the quantities (u^i) being regarded either as coordinate functions on U or as the coordinate components of the point $u \in U$:

U: $(u^i), \ i = 1, 2, \dots, n = \dim M,$
TU: $(u^i, u^i_1),$ with $u^i \doteq u^i \circ \pi_1, \ u^i_1 \doteq du^i,$
T^2U: $(u^i, u^i_1, u^i_2, u^i_{12}),$
with $u^i \doteq u^i \circ \pi_1\pi_2, \ u^i_1 \doteq du^i \circ \pi_2, \ u^i_2 \doteq d(u^i \circ \pi_1), \ u^i_{12} \doteq d^2u^i,$
etc.

We set up the following convention: *to introduce coordinates on T^kU we take the coordinates on $T^{k-1}U$ and repeat them with an additional index k* – so that a tangent vector is preceded by its point of origin. This indexing is convenient since

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the symbols with index s thereby become coordinates in the fibre of the projection $\rho_s,\ s=1,2,\ldots,k$.

Thus, for example, under the projections $\rho_s : T^3U \to T^2U,\ s=1,2,3$, the coordinates with index 1,2 and 3 are each suppressed in turn:

$$(u^i, u^i_1, u^i_2, u^i_{12}, u^i_3, u^i_{13}, u^i_{23}, u^i_{123})$$

$$\rho_1 \swarrow \qquad\qquad \rho_2 \downarrow \qquad\qquad \searrow \rho_3$$

$$(u^i, u^i_2, u^i_3, , u^i_{23}) \qquad (u^i, u^i_1, u^i_3, , u^i_{13}) \qquad (u^i, u^i_1, u^i_2, , u^i_{12}).$$

The level T^kM is a smooth manifold of dimension 2^kn and admits an important subspace of dimension $(k+1)n$ called the *osculating bundle* of M of order $k-1$ and denoted $\mathrm{Osc}^{k-1}M$. The bundle $\mathrm{Osc}^{k-1}M$ is determined by the equality of the projections

$$\rho_1 = \rho_2 = \ldots = \rho_k\,,$$

meaning that an element of T^kM belongs to the bundle $\mathrm{Osc}^{k-1}M$ precisely when all its k projections into $T^{k-1}M$ coincide. In this case all coordinates with the same number of lower indices coincide. For example, the first bundle $\mathrm{Osc}M$ is determined in $T^2U \subset T^2M$ by the equations $u^i_1 = u^i_2$, the second bundle Osc^2M in $T^3U \subset T^3M$ by $u^i_1 = u^i_2 = u^i_3,\ u^i_{12} = u^i_{13} = u^i_{23}$, etc. The coordinates in $\mathrm{Osc}^{k-1}M$ will be denoted by the derivatives of the coordinate functions on U, that is to say $(u^i, du^i, d^2u^i, \ldots, d^ku^i)$.

The immersion $\zeta : \mathrm{Osc}M \hookrightarrow T^2M$ and its derivative $T\zeta$ are determined in coordinates by matrix formulae:

$$\begin{pmatrix} u^i \\ u^i_1 \\ u^i_2 \\ u^i_{12} \end{pmatrix} \circ \zeta = \begin{pmatrix} u^i \\ du^i \\ du^i \\ d^2u^i \end{pmatrix}, \qquad \begin{pmatrix} u^i_3 \\ u^i_{13} \\ u^i_{23} \\ u^i_{123} \end{pmatrix} \circ T\zeta = \begin{pmatrix} du^i \\ d^2u^i \\ d^2u^i \\ d^3u^i \end{pmatrix},$$

$$T\zeta\Big(\frac{\partial}{\partial u^i}\,,\ \frac{\partial}{\partial(du^i)}\,,\ \frac{\partial}{\partial(d^2u^i)}\Big) = \Big(\frac{\partial}{\partial u^i}\,,\ \frac{\partial}{\partial u^i_1} + \frac{\partial}{\partial u^i_2}\,,\ \frac{\partial}{\partial u^i_{12}}\Big)\,.$$

The fibres of the bundle $\mathrm{Osc}M$ are the integral manifolds of the distribution

$$\langle\, \partial^1_i + \partial^2_i, \partial^{12}_i \rangle, \quad \text{with} \quad \partial^1_i + \partial^2_i \doteq \frac{\partial}{\partial u^i_1} + \frac{\partial}{\partial u^i_2}\,, \quad \partial^{12}_i \doteq \frac{\partial}{\partial u^i_{12}}.$$

The functions $(u^i_1 - u^i_2)$ vanish on $\mathrm{Osc}M$.

Historically, osculating bundles were introduced under various names long before the bundles T^kM. The systematic study begun 60 years ago by V.Vagner [9] culminated in recent times with Miron-Atanasiu theory [2]. Meanwhile the theme of levels T^kM remained unjustly neglected for the obvious reason that the multiple fibre bundle structure demands a whole new understanding and new approach: see [5], [7]. Attempts such as [10] and the so-called synthetic formulation of T^kM [3] made progress in that direction.

While an infinitesimal displacement of the point $u \in M$ is determined by a tangent vector u_1 to M, an infinitesimal displacement of the element $(u, u_1) \in TM$ is determined by the quantities (u_2, u_{12}), representing a tangent vector to TM, etc. This interpretation of the elements of T^kM allows us to develop the theory of higher order motion. Clearly the future belongs to these bundles.

White considers on the level T^kM or on a k-multiple vector bundle certain *sector-forms* which are functions simultaneously linear in all the fibres of k projections: see [10]. In particular the sector-forms on T^2U and T^3U can be written as

$$\Phi = \varphi_{ij} u_1^i u_2^j + \varphi_i u_{12}^i \,,$$
$$\Psi = \psi_{ijk} u_1^i u_2^j u_3^k + \psi_{ij}^1 u_1^i u_{23}^j + \psi_{ij}^2 u_2^i u_{13}^j + \psi_{ij}^3 u_3^i u_{12}^j + \psi_i u_{123}^i \,,$$

with coefficients in U. For example, in each term of Ψ we see the index 1 (or 2 or 3) appear exactly once. This means that the function Ψ is linear on the fibres of ρ_1 (and ρ_2 and ρ_3).

Any scalar function can be lifted from the level $T^{k-1}M$ to the level T^kM by k different projections $\rho_s : T^kM \to T^{k-1}M$. For example, for the sector form Φ above there are three possibilities of lifting to T^3M:

$$\Phi \circ \rho_1 = \varphi_{ij} u_2^i u_3^j + \varphi_i u_{23}^i \,, \quad \Phi \circ \rho_2 = \varphi_{ij} u_1^i u_3^j + \varphi_i u_{13}^i \,, \quad \Phi \circ \rho_3 = \varphi_{ij} u_1^i u_2^j + \varphi_i u_{12}^i \,.$$

Proposition. *Every Cartan k-form can be regarded as a sector-form in the sense of White, a scalar function on T^kM that is constant on the fibres of $Osc^{k-1}M$.*

Proof. The sector form Φ is constant on Osc$M$ if and only if its derivatives vanish on Osc$M$. Thus

$$\Phi = \varphi_{ij} u_1^i u_2^j + \varphi_i u_{12}^i \quad \Rightarrow$$
$$(\partial_i^1 + \partial_i^2)\Phi = \varphi_{ij} u_2^j + \varphi_{ji} u_1^j = (\varphi_{ij} + \varphi_{ji}) u_1^j - \varphi_{ij}(u_1^j - u_2^j),$$
$$\partial_i^{12}\Phi = \varphi_i \quad \Rightarrow \quad \varphi_{(ij)} = 0, \;\; \varphi_i = 0 \,.$$

By definition Φ is an antisymmetric bilinear form and can therefore be expressed in the coordinates (u^i, du^i) as a 2-form $\Phi = \varphi_{[ij]} du^i \wedge du^j$. Thus the sector-form Φ is constant on Osc$M$ if and only if it is a Cartan 2-form.

In the case $k = 3$ the fibres Osc^2M of dimension $3n$ are the integral manifolds of the distribution

$$\langle\, \partial_i^1 + \partial_i^2 + \partial_i^3, \;\partial_i^{23} + \partial_i^{13} + \partial_i^{12}, \;\partial_i^{123} \rangle.$$

For the sector-form Ψ (see above) we have

$$\Psi = \psi_{ijk} u_1^i u_2^j u_3^k + \psi_{ij}^1 u_1^i u_{23}^j + \psi_{ij}^2 u_2^i u_{13}^j + \psi_{ij}^3 u_3^i u_{12}^j + \psi_i u_{123}^i \;\Rightarrow$$
$$(\partial_i^1 + \partial_i^2 + \partial_i^3)\Psi = \;\psi_{ijk} u_2^j u_3^k + \;\psi_{jik} u_1^j u_3^k + \psi_{jki} u_1^j u_2^k + \psi_{ij}^1 u_{23}^j + \psi_{ij}^2 u_{13}^j + \psi_{ij}^3 u_{12}^j \,,$$
$$(\partial_i^{23} + \partial_i^{13} + \partial_i^{12})\Psi = \;\psi_{ji}^1 u_1^j + \psi_{ji}^2 u_2^j + \psi_{ji}^3 u_3^j \,,$$
$$\partial_i^{123}\Psi = \;\psi_i \,.$$

The derivatives vanish on the fibres Osc^2M when the following conditions hold:

$$\varphi_{(ijk)} = 0, \quad \psi_{ij}^1 + \psi_{ij}^2 + \psi_{ij}^3 = 0, \quad \psi_i = 0.$$

These conditions are necessary and sufficient for the sector-form Ψ to be constant on on Osc^2M, but not for Ψ to be a Cartan 3-form. However, every 3-form $\tilde\Psi =$

$\varphi_{ijk} du^i \wedge du^j \wedge du^k$ can be regarded as a homogeneous sector-form that is constant on $\mathrm{Osc}^2 M$.

The argument extends likewise to the cases $k > 3$. □

White's theory of sector-forms is much more extensive than that of Cartan exterior forms. In particular, exterior differentiation is an operation on the set of sector-forms that are constant on the osculating bundles.

There is, however, one inconvenience: sector-forms are represented in natural coordinates in terms which are not invariant. To get rid of this one can use affine connexions and adapted coordinates. In T^2U, for example, the 'bad' coordinates u^i_{12} can be replaced by adapted coordinates $U^i_{12} = \Gamma^i_{jk} u^j_1 u^k_2 + u^i_{12}$ using the coefficients Γ^i_{jk} of the affine connection. The sector-form Φ is represented by two invariant terms:

$$\Phi = (\varphi_{ij} - \varphi_k \Gamma^k_{ij}) u^i_1 u^j_2 + \varphi_i U^i_{12} \,.$$

In the parentheses we recognize the prototype of the covariant derivative. In fact, for the 1-form $\Theta = \theta_i u^i_1$ the ordinary differential can be written

$$d\Theta = \theta_{i,j} u^i_1 u^j_2 + \theta_i u^i_{12}, \quad \theta_{i,j} = \frac{\partial \theta_i}{\partial u^j} \,,$$

or $d\Theta = \nabla_j \theta_i u^i_1 u^j_2 + \theta_i U^i_{12}$ with the covariant derivative $\nabla_j \theta_i = \theta_{i,j} - \theta_k \Gamma^k_{ij}$.

The connexions play an important role here. The local forms appear in the unified and intrinsic structures

$$\Delta_h \oplus \Delta_v \text{ on } TM, \qquad \Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12} \text{ on } T^2M, \quad \text{etc.}$$

The theory extends by iteration to the levels T^kM: see [1], [8].

2. Hamilton, Lagrange, Legendre

The essential importance of the levels TM and T^2M for analytical mechanics was first emphasized by Godbillon [4].

Specifically, Hamiltonian geometry is built on the levels TM and T^2M. Associated to a function $H = H(u, u_1)$ (called the *Hamiltonian*) is the vector field X on TM where

$$X = \sum_i H_{u^i_1} \partial_i - \sum_i H_{u^i} \partial^1_i, \quad H_i \doteq \frac{\partial H}{\partial u^i} \,, \quad H_{u^i_1} \doteq \frac{\partial H}{\partial u^i_1} \,,$$

for which the flow $a_t = \exp tX$ is determined by the system of differential equations (*Hamiltonian system*)

$$\begin{cases} \dot{u}^i = H_{u^i_1} \\ \dot{u}^i_1 = -H_{u^i} \end{cases} , \qquad \dot{u}^i \doteq \frac{du^i}{dt} \,, \; \dot{u}^i_1 \doteq \frac{du^i_1}{dt} \,.$$

Under the correspondence

$$(u^i, u^i_1, u^i_2, u^i_{12}) \;\rightsquigarrow\; (u^i, u^i_1, \dot{u}^i, \dot{u}^i_1)$$

we see this as a section of the bundle $\pi_2 : T^2M \to TM$, of dimension $2n$. The function H and the symplectic form $\Omega = du^i \wedge du_1^i$ [6] are invariant with respect to the vector field X:

$$XH = 0, \quad \mathcal{L}_X\Omega = 0.$$

Theorem. *The Hamiltonian system reduces to Lagrange's equations on the osculating bundle* Osc$M$.

Proof. The passage from the Hamiltonian $H = H(u, u_1)$ to the Lagrangian $L = L(u, u_2)$ ought to be realized through the equation (*Legendre transformation*)

$$H(u, u_1) - \sum_i u_1^i u_2^i + L(u, u_2) = 0.$$

However, this equation, which should hold identically on T^2M, is contradictory:

$$d(H - \sum_i u_1^i u_2^i + L) \equiv 0 \;\Rightarrow\; H_{u^i} + L_{u^i} = 0,\; H_{u_1^i} = u_2^i,\; L_{u_2^i} = u_1^i.$$

On the other hand, on Osc$M$ where $u_1^i = u_2^i = \dot{u}$, the passage $H \rightsquigarrow L$ is well determined. On Osc$M$ the Hamiltonian system can be written in Lagrangian form:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{u}^i}\Big) - \frac{\partial L}{\partial u^i} = 0.$$

The Lagrangian system determines a section of the bundle Osc$M \to TM$, of the same dimension $2n$ as the Hamiltonian system on T^2M. □

The Hamiltonian geometry on the levels T^kM and the Lagrangian geometry on the osculating bundles Osc^{k-1}M for $k > 2$ are structured according to an iterative scheme.

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Author's address:

Maïdo Rahula
Tartu University, J.Liivi 2 Tartu, 50409 Estonia.
E-mail: rahula@ut.ee