Efficiency and duality for multitime vector fractional variational problems on manifolds

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Abstract. We consider a multitime scalar variational problem (SVP), a multitime multiobjective variational problem (VVP) and a multitime vector fractional variational problem (VFP). For (SVP) we establish necessary optimality conditions. For the two vector variational problems (VVP) and (VFP), we define the notions of efficient solution and of normal efficient solution and using these notions we establish necessary efficiency conditions. Using the notion of (ρ, b)-quasiinvexity adapted for variational problems, we introduce a duality of Mond-Weir-Zalmai type for the fractional problem (VFP) through weak, direct and converse duality theorems.

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1 Introduction

In [9], Mititelu and Stancu-Minasian considered the following multiobjective fractional variational problem:

$$(\text{MSP}) \quad \begin{cases} \text{Maximize} \left(\dfrac{\displaystyle\int_a^b f_1(t,x,\dot{x})dt}{\displaystyle\int_a^b k_1(t,x,\dot{x})dt}, \dots, \dfrac{\displaystyle\int_a^b f_p(t,x,\dot{x})dt}{\displaystyle\int_a^b k_p(t,x,\dot{x})dt} \right) \\ \text{subject to } x(a) = a_0, \ x(b) = b_0, \\ \qquad\qquad g(t,x,\dot{x}) \leqq 0, \ h(t,x,\dot{x}) = 0, \ \forall t \in I, \end{cases}$$

where $I = [a, b]$ is interval, $x = (x_1, \dots, x_n) \colon I \to \mathbb{R}^n$ is piecewise smooth function on I with $\dot{x} = \dfrac{dx}{dt}$ its derivative, $f_1, k_1, \dots, f_p, k_p \colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g \colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ and $h \colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$ are functions of C^2-class.

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For this problem, they established necessary efficiency (Pareto minimum) conditions, and using generalized quasiinvex functions, developed a duality theory through weak, direct and converse duality theorems, see also [6].

In [14], Udrişte studied a control variational problem into multidimensional (multitime) framework, establishing some optimality conditions, that is he gave a multitime maximum principle, see also [15]. Pitea, Udrişte and Mititelu [11], [12] considered the multitime vector version of problem (MSP) within a geometrical framework, using curvilinear integrals, and they established necessary optimality conditions and developed a duality theory for this problem.

In this work, we study properties of a multitime multiobjective fractional variational problem, within a geometrical framework [11], [12] using multiple integrals.

Let (T,h) and (M,g) be two Riemannian manifolds of dimensions m and n, respectively. Denote by $t=(t^1,\dots,t^m)=(t^\alpha)$ the elements of a measurable set Ω in T and by $x=(x^1,\dots,x^n)=(x^k)\in\mathbb{R}^n$ the elements of M. Consider $J^1(T,M)$ the first order jet bundle associated to T and M and the functions

$$x\colon \Omega\to M,\quad X\colon J^1(T,M)\to\mathbb{R},\quad f=(f_1,\dots,f_p)\colon J^1(T,M)\to\mathbb{R}^p,$$
$$k=(k_1,\dots,k_p)\colon J^1(T,M)\to\mathbb{R}^p,\quad X^i_\alpha\colon J^1(T,M)\to\mathbb{R}^m,\quad Y_\beta\colon J^1(T,M)\to\mathbb{R}^q,$$

where $m,q\in\mathbb{N}^*$, $i=\overline{1,n}$, $\alpha=\overline{1,m}$ and $\beta=\overline{1,q}$.

The arguments of X, f, g, X^i_α, Y_β are $(t,x,x_\gamma)=(t,x(t),x_\gamma(t))$, where

$$x=x(t)=(x^1(t),\dots,x^n(t))=(x^k(t)),\quad t\in\Omega,$$
$$x_\gamma(t)=\frac{\partial x}{\partial t^\gamma}(t),\quad \gamma=\overline{1,m}.$$

We suppose that X, f, g, X^i_α, Y_β belong to $C^2(\Omega)$. We define the set of functions

$$\Phi=\{x\colon\Omega\to M\mid x \text{ is piecewise smooth on } \Omega\},$$

where Ω is a normed space with $\|x\|=\|x\|_\infty+\sum_{k=1}^n\|x^k_\gamma\|_\infty$.

Throughout in the paper, for two vectors $v=(v_1,\dots,v_n)$ and $w=(w_1,\dots,w_n)$ the relations of the form $v=w$, $v<w$, $v\leqq w$, $v\leq w$ are defined as follows:

$$v=w\Leftrightarrow v_i=w_i,\ i=\overline{1,n};\quad v<w\Leftrightarrow v_i<w_i,\ i=\overline{1,n};$$
$$v\leqq w\Leftrightarrow v_i\leqq w_i,\ i=\overline{1,n};\quad v\leq w\Leftrightarrow v\leqq w \text{ and } v\neq w.$$

The aim of present work is to establish necessary efficiency conditions and develop a duality theory for the following multitime fractional variational vector problem:

$$\text{(VFP)}\quad\begin{cases}\text{Maximize Pareto}\ \left(\dfrac{\displaystyle\int_\Omega f_1(t,x(t),x_\gamma(t))dv}{\displaystyle\int_\Omega k_1(t,x(t),x_\gamma(t))dv},\dots,\dfrac{\displaystyle\int_\Omega f_p(t,x(t),x_\gamma(t))dv}{\displaystyle\int_\Omega k_p(t,x(t),x_\gamma(t))dv}\right)\\ \text{subject to } X^i_\alpha(t,x(t),x_\gamma(t))=0,\ Y_\beta(t,x(t),x_\gamma(t))\leqq 0,\\ \qquad\qquad x(t)\big|_{\partial\Omega}=u(t)\ \text{(given)},\forall t\in\Omega,\end{cases}$$

where $dv=dt_1dt_2\cdots dt_n$.

First, there are established necessary optimality conditions for a multitime scalar variational problem (SVP) and after that necessary efficiency conditions for a multitime vector variational problem (VVP), both having the same constraints as (VFP).

2 Necessary optimality conditions for (SVP)

The first model studied in this paper is the scalar multitime variational problem

$$\text{(SVP)}\quad \begin{cases} \text{Minimize } I[x] = \displaystyle\int_\Omega X(t, x(t), x_\gamma(t))dv \\ \text{subject to } X^i_\alpha(t, x(t), x_\gamma(t)) = 0,\ Y_\beta(t, x(t), x_\gamma(t)) \leqq 0,\ t \in \Omega, \\ \qquad\qquad x(t)\big|_{\partial\Omega} = u(t). \end{cases}$$

The set of feasible solutions for this problem (the domain of (SVP)) is given by

$$\mathcal{D} = \{x \in \Phi \mid X^i_\alpha(t, x(t), x_\gamma(t)) = 0,\ Y_\beta(t, x(t), x_\gamma(t)) \leqq 0,\ x(t)\big|_{\partial\Omega} = u(t),\ \forall t \in \Omega\}.$$

Theorem 2.1 (Necessary optimality for (SVP)). *If $x = x(t) \in \mathcal{D}$ is an optimal solution for problem* (SVP)*, then there exist a real scalar τ and the piecewise smooth functions $\Lambda(t) = (\Lambda^\alpha_i(t)) \in \mathbb{R}^{nm}$ and $M(t) = (M^\beta(t)) \in \mathbb{R}^q$ which satisfy the following conditions:*

$$\text{(SFJ)}\quad \begin{cases} \tau\dfrac{\partial X}{\partial x^k} + \Lambda^\alpha_i(t)\dfrac{\partial X^i_\alpha}{\partial x^k} + M^\beta(t)\dfrac{\partial Y_\beta}{\partial x^k} - \\ \qquad -D_\gamma\left(\tau\dfrac{\partial X}{\partial x^k_\gamma} + \Lambda^\alpha_i(t)\dfrac{\partial X^i_\alpha}{\partial x^k_\gamma} + M^\beta(t)\dfrac{\partial Y_\beta}{\partial x^k_\gamma}\right) = 0, \\ M^\beta(t)Y_\beta(t, x(t), x_\gamma(t)) = 0,\ \beta = \overline{1, q}, \\ \tau \geqq 0,\ (M^\beta(t)) \geqq 0,\ t \in \Omega, \end{cases}$$

where $D_\gamma(\cdot) = \dfrac{d}{dt}(\cdot)$, $\dfrac{\partial X}{\partial x^k} := \dfrac{\partial X}{\partial x^k}(t, x(t), x_\gamma(t))$, $\dfrac{\partial X}{\partial x^k_\gamma} := \dfrac{\partial X}{\partial x^k_\gamma}(t, x(t), x_\gamma(t))$ *etc..*

Proof. Consider the following auxiliary multitime variational problem

$$\text{(AVP)}\quad \begin{cases} \text{Minimize } I[x] = \displaystyle\int_\Omega X(t, x(t), x_\gamma(t))dv \\ \text{s. t. } \displaystyle\int_\Omega \lambda^\alpha_i(t)X^i_\alpha(t, x(t), x_\gamma(t))dv = 0,\ \int_\Omega \mu^\beta(t)Y_\beta(t, x(t), x_\gamma(t))dv \leqq 0, \\ \qquad (\mu^\beta(t)) \geqq 0,\ x(t)\big|_{\partial\Omega} = u(t),\ t \in \Omega, \end{cases}$$

where λ^α_i and μ^β are piecewise smooth real functions on Ω.

Problems (SVP) and (AVP) have the same domain $\mathcal{D}$ and the same optimal solution $x(t)$.

Let $\varepsilon > 0$ be a scalar and let $p\colon \Omega \to \mathbb{R}^n$ be a function of $C^1(\Omega)$-class. Consider the next neighborhood of the optimal solution $x(t)$:

$$V_\varepsilon = \{\bar{x} \in X \mid \bar{x} = x(t) + \varepsilon p(t),\ p\big|_{\partial\mathcal{D}} = 0\}.$$

Then $x = x(t)$ is an optimal solution of problem (SVP) if $\varepsilon = 0$ is a minimal solution to the next nonlinear problem:

$$\text{(2.1)}\quad \begin{cases} \text{Minimize } F(\varepsilon) = \displaystyle\int_\Omega X(t, x(t) + \varepsilon p(t), x_\gamma(t) + \varepsilon\dot{p}(t))dv \\ \text{subject to} \\ G^i_\alpha(\varepsilon) = \displaystyle\int_\Omega \lambda^\alpha_i(t)X^i_\alpha(t, x(t) + \varepsilon p(t), x_\gamma(t) + \varepsilon\dot{p}(t))dv = 0,\ i = \overline{1, n},\ \alpha = \overline{1, m}, \\ H_\beta(\varepsilon) = \displaystyle\int_\Omega \mu^\beta(t)Y_\beta(t, x(t) + \varepsilon p(t), x_\gamma(t) + \varepsilon\dot{p}(t))dv \leqq 0,\ \beta = \overline{1, q}, \\ (\mu^\beta(t)) \geqq 0,\ u\big|_{\partial\Omega} = u,\ p\big|_{\partial\Omega} = 0,\ t \in \Omega. \end{cases}$$

The Fritz John conditions for (2.1) at $\varepsilon = 0$ are the following: there exists a scalar $\tau \in \mathbb{R}$, a matrix $B = (B_i^\alpha) \in \mathbb{R}^{nm}$ and a vector $C = (C^\beta) \in \mathbb{R}^q$, $C \geqq 0$, such that:

$$\text{(FJ)} \quad \begin{cases} \tau \nabla F(0) + B_i^\alpha \nabla G_\alpha^i(0) + C^\beta \nabla H_\beta(0) = 0 \\ C^\beta H_\beta(0) = 0 \\ \tau \geqq 0, \ \mu^\beta \geqq 0, \end{cases}$$

with $\nabla F(0) = \displaystyle\int_\Omega \left(\frac{\partial X}{\partial x^k} p^k + \frac{\partial X}{\partial x^k_\gamma} p^k_\gamma \right) dv$; $\nabla G^i_\alpha(0) = \displaystyle\int_\Omega \lambda_i^\alpha(t) \left(\frac{\partial X^i_\alpha}{\partial x^k} p^k + \frac{\partial X^i_\alpha}{\partial x^k_\gamma} p^k_\gamma \right) dv$; and $\nabla H_\beta(0) = \displaystyle\int_\Omega \mu^\beta(t) \left(\frac{\partial Y_\beta}{\partial x^k} p^k + \frac{\partial h}{\partial x^k_\gamma} p^k_\gamma \right) dv$.

Consequently, the first condition of (FJ) becomes

$$\tau \int_\Omega \left(\frac{\partial X}{\partial x^k} p^k + \frac{\partial X}{\partial x^k_\gamma} p^k_\gamma \right) dv + B_i^\alpha \int_\Omega \lambda_i^\alpha(t) \left(\frac{\partial X^i_\alpha}{\partial x^k} p^k + \frac{\partial X^i_\alpha}{\partial x^k_\gamma} p^k_\gamma \right) dv +$$
$$+ C^\beta \int_\Omega \mu^\beta(t) \left(\frac{\partial Y_\beta}{\partial x^k} p^k + \frac{\partial Y_\beta}{\partial x^k_\gamma} p^k_\gamma \right) dv = 0,$$

$$\text{(2.2)} \quad \begin{aligned} &\int_\Omega \left[\tau \frac{\partial X}{\partial x^k} + \lambda_i^\alpha(t) \frac{\partial X^i_\alpha}{\partial x^k} + \mu^\beta(t) \frac{\partial Y_\beta}{\partial x^k} \right] p^k dv + \\ &+ \int_\Omega \left[\tau \frac{\partial X}{\partial x^k_\gamma} + \lambda_i^\alpha(t) \frac{\partial X^i_\alpha}{\partial x^k_\gamma} + \mu^\beta(t) \frac{\partial Y_\beta}{\partial x^k_\gamma} \right] p^k_\lambda dv = 0, \end{aligned}$$

where we denoted $\Lambda_i^\alpha(t) = B_i^\alpha \lambda_i^\alpha(t)$ and $M^\beta(t) = C^\beta \mu^\beta(t)$.

We denote $E(t, x, x_\gamma) = \tau X(t, x, x_\gamma) + \Lambda_i^\alpha(t) X^i_\alpha(t, x, x_\gamma) + M^\beta(t) Y_\beta(t, x, x_\gamma)$ and (2.2) can be written

$$\text{(2.3)} \quad \int_\Omega \frac{\partial E}{\partial x^k} p^k dv + \int_\Omega \frac{\partial E}{\partial x^k_\gamma} p^k_\gamma dv = 0.$$

For an useful version of the second integral in (2.3) we have

$$D_\gamma \left(\frac{\partial E}{\partial x^k_\gamma} p^k \right) = \frac{\partial E}{\partial x^k_\gamma} p^k_\gamma + D_\gamma \left(\frac{\partial E}{\partial x^k_\gamma} \right) p^k$$

and integrating, we obtain

$$\int_\Omega \frac{\partial E}{\partial x^k_\gamma} p^k_\gamma dv = \int_\Omega D_\gamma \left(\frac{\partial E}{\partial x^k_\gamma} p^k \right) dv - \int_\Omega D_\gamma \left(\frac{\partial E}{\partial x^k_\gamma} \right) p^k dv.$$

But using the divergence formula, we have

$$\int_\Omega D_\gamma \left(\frac{\partial E}{\partial x^k_\gamma} p^k \right) dv = \int_{\partial\Omega} \left[\frac{\partial E}{\partial x^k_\gamma} p^k \right] \vec{n} d\sigma = 0,$$

where $\vec{n} = (n_1, \ldots, n_m)$ is the unit normal vector to $\partial\Omega$ at the current point of $\partial\Omega$ and $p^k\big|_{\partial\Omega} = 0$, $k = \overline{1, n}$.

Then we obtain

$$\int_\Omega \frac{\partial E}{\partial x^k_\gamma} p^k_\gamma dv = -\int_\Omega D_\gamma \left(\frac{\partial E}{\partial x^k_\gamma}\right) p^k dv,$$

therefore(2.3) becomes

$$\int_\Omega \left[\frac{\partial E}{\partial x^k} - D_\gamma\left(\frac{\partial E}{\partial x^k_\gamma}\right)\right] p^k dv = 0. \tag{2.4}$$

By the fundamental lemma of variational calculus, from (2.4) it follows the Euler-Lagrange equation $\frac{\partial E}{\partial x^k} - D_\gamma\left(\frac{\partial E}{\partial x^k_\gamma}\right) = 0$, that is the first relation of (SFJ).

The second condition of (FJ) becomes

$$M^\beta(t)Y_\beta(t, x(t), x_\gamma(t)) = 0.$$

Definition 2.1 ([4]). A point $x^* \in \mathcal{D}$ is called *normal optimal solution* of problem (SVP) if $\tau \neq 0$.

3 Necessary efficiency conditions for (VVP)

•Efficiency for multitime vector variational problems. In the framework of problem (SVP), we consider the vector functional

$$I[x] = \int_\Omega f(t, x(t), x_\gamma(t))dv,$$

which can be written on components as

$$I[x] = (I_1[x], \dots, I_p[x]) = \left(\int_\Omega f_1(t, x(t), x_\gamma(t))dv, \dots, \int_\Omega f_p(t, x(t), x_\gamma(t))dv\right).$$

Consider now the multitime variational vector problem

$$\text{(VVP)} \quad \begin{cases} \text{Minimize Pareto } I[x] = \displaystyle\int_\Omega f(t, x(t), x_\gamma(t))dv \\ \text{subject to } X^i_\alpha(t, x(t), x_\gamma(t)) = 0,\ Y_\beta(t, x(t), x_\gamma(t)) \leqq 0, \\ \qquad\qquad x\big|_{\partial\Omega} = u(t),\ \forall t \in \Omega. \end{cases}$$

The domain of (VVP) is also $\mathcal{D}$.

Definition 3.1 ([2]). A point $x^* \in \mathcal{D}$ is an *efficient solution (Pareto minimum)* of problem (VVP) if there exist no $x \in \mathcal{D}$ such that $I[x] \leq I[x^*]$.

Theorem 3.1 (Necessary efficiency of (VVP)). *Consider the vector multitime variational problem* (VVP) *and let* $x = x(t) \in \mathcal{D}$ *be an efficient solution of* (VVP).

Then there are a vector $\tau \in \mathbb{R}^p$, and the piecewise smooth functions $\lambda(t)$ in $\mathbb{R}^{nm}$ and $\mu(t)$ in $\mathbb{R}^q$, defined on Ω, which satisfy the conditions

$$\text{(VFJ)} \quad \begin{cases} \tau^r \dfrac{\partial f_r}{\partial x^k} + \lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x^k} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x^k} - \\ \qquad -D_\gamma\left(\tau^r \dfrac{\partial f_r}{\partial x_\gamma^k} + \lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x_\gamma^k} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x_\gamma^k}\right) = 0, \\ \mu^\beta(t) Y_\beta(t, x(t), x_\gamma(t)) = 0, \ \beta = \overline{1,q}, \\ \tau \geqq 0, \ (\mu^\beta(t)) \geqq 0, \ t \in \Omega. \end{cases}$$

Proof. If $x = x(t) \in \mathcal{D}$ is an efficient solution of (VVP), the inequality $I[\bar{x}] \leq I[x]$, $\forall \bar{x} \in \mathcal{D}$, is false. Then there exists $r \in \{1, \ldots, p\}$ and a neighborhood N_r in $\mathcal{D}$ of the point x such that $I_r[\bar{x}] \geqq I_r[x]$, $\forall \bar{x} \in N_r$. Therefore, x is an optimal solution to following scalar variational problem

$$\text{(SVP)}_r \quad \begin{cases} \text{Minimize } I_r[x] = \displaystyle\int_\Omega f_r(t, x(t), x_\gamma(t))dv \\ \text{subject to } X_\alpha^i(t, x(t), x_\gamma(t)) = 0, \ Y_\beta(t, x(t), x_\gamma(t)) \leqq 0, \\ \qquad x \in \Phi, \ x\big|_{\partial\Omega} = u(t), \ \forall t \in \Omega. \end{cases}$$

Then, according to Theorem 2.1, there are scalars $\nu_r \in \mathbb{R}$ and the piecewise smooth real functions $\lambda_{i,r}^\alpha$ and μ_r^β, such that the next conditions are true:

$$(3.1) \quad \begin{cases} \nu_r \dfrac{\partial f_r}{\partial x^k} + \lambda_{i,r}^\alpha(t)^T \dfrac{\partial X_\alpha^i}{\partial x^k} + \mu_r^\beta(t)^T \dfrac{\partial Y_\beta}{\partial x^k} - \\ \qquad -D_\gamma\left(\nu_r \dfrac{\partial f_r}{\partial x_\gamma^k} + \lambda_{i,r}^\alpha(t)^T \dfrac{\partial X_\alpha^i}{\partial x_\gamma^k} + \mu_r^\beta(t)^T \dfrac{\partial Y_\beta}{\partial x_\gamma^k}\right) = 0, \\ \mu_r^\beta(t)^T Y_\beta(t, x(t), x_\gamma(t)) = 0, \\ \nu_r \geqq 0, \ \mu_r^\beta(t) \geqq 0, \ t \in \Omega, \end{cases}$$

where $\dfrac{\partial f_r}{\partial x} := \dfrac{\partial f_r}{\partial x}(t, x(t), x_\gamma(t))$, while i, α, β are variables.

We denote by $S = \displaystyle\sum_{r=1}^p \nu_r$ and $\tau^r = \begin{cases} \dfrac{\nu_r}{S}, & \text{when } I_r[\bar{x}] \geqq I_r[x] \\ 0, & \text{when } I_r[\bar{x}] < I_r[x]. \end{cases}$

Also we denote by

$$(3.2) \qquad \tau = (\tau^1, \ldots, \tau^p), \quad \lambda_i^\alpha(t) = \frac{\lambda_{i,r}^\alpha(t)}{S}, \quad \mu^\beta(t) = \frac{\mu_r^\beta(t)}{S}.$$

Taking into account relations (3.2) in (3.1), we find (VFJ).

Definition 3.2 ([4]). The point $x^0 \in \mathcal{D}$ is a *normal efficient solution* of (VVP) if within the conditions (VFJ) there exist $\tau \geq 0$ with $e'\tau = 1$.

•Efficiency for multitime fractional vector variational problems. First of all, we recall some definitions and auxiliary results which will be needed later in our discussion about efficiency conditions for problem (VFP), defined by the following multitime fractional variational vector problem

$$\text{(VFP)}\quad \begin{cases} \text{Minimize } J[x]=\left(\dfrac{\displaystyle\int_\Omega f_1(t,x(t),x_\gamma(t))dv}{\displaystyle\int_\Omega k_1(t,x(t),x_\gamma(t))dv},\dots,\dfrac{\displaystyle\int_\Omega f_p(t,x(t),x_\gamma(t))dv}{\displaystyle\int_\Omega k_p(t,x(t),x_\gamma(t))dv}\right) \\ \text{subject to } X^i_\alpha(t,x(t),x_\gamma(t))=0,\ Y_\beta(t,x(t),x_\gamma(t))\leqq 0, \\ \qquad x(t)\big|_{\partial\Omega}=u(t), \forall t\in\Omega. \end{cases}$$

From now on, we assume that $\displaystyle\int_\Omega k_r(t,x(t),x_\gamma(t))dv>0$ for all $r=\overline{1,p}$. The domain of (VFP) is the same set $\mathcal{D}$.

Definition 3.3 ([2]). A point $x^0\in\mathcal{D}$ is said to be an *efficient solution* of (VFP) if there is no $x\in\mathcal{D}$, $x\neq x^0$, such that $J[x]\leq J[x^0]$.

We present now the necessary efficiency conditions for (VFP). Let $x^0(t)$ be an efficient solution of (FVP). Consider the problem

$$\text{(FP)}_r(x^0)\quad \begin{cases} \underset{x}{\text{Minimize}}\ \dfrac{\displaystyle\int_\Omega f_r(t,x(t),x_\gamma(t))dv}{\displaystyle\int_\Omega k_r(t,x(t),x_\gamma(t))dv} \\ \text{subject to } x(t)\big|_{\partial\Omega}=u(t), \\ \qquad X^i_\alpha(t,x(t),x_\gamma(t))=0,\ Y_\beta(t,x(t),x_\gamma(t))\leqq 0, \\ \qquad \dfrac{\displaystyle\int_\Omega f_j(t,x(t),x_\gamma(t))dv}{\displaystyle\int_\Omega k_j(t,x(t),x_\gamma(t))dv}\leqq\dfrac{\displaystyle\int_\Omega f_j(t,x^0(t),x^0_\gamma(t))dv}{\displaystyle\int_\Omega k_j(t,x^0(t),x^0_\gamma(t))dv},\ j=\overline{1,p},\ j\neq r. \end{cases}$$

Definition 3.4. The efficient solution $x^0\in\mathcal{D}$ is a *normal efficient solution* of (VFP) if x^0 is optimal point to at least one of scalar problems (FP)_r, $r=\overline{1,p}$.

Theorem 3.2 (Necessary efficiency in (FVP)). *Let $x=x(t)\in\mathcal{D}$ be a normal efficient solution of problem* (FVP). *Then there exist a vector $\tau=(\tau^r)\in\mathbb{R}^p$ and piecewise smooth functions $\lambda=(\lambda^\alpha_i(t))\in\mathbb{R}^{nm}$ and $\mu=(\mu^\beta(t))\in\mathbb{R}^q$, defined on Ω, which satisfy the conditions*

$$\text{(MFJ)}\quad \begin{cases} \tau^r\left[\dfrac{\partial f_r}{\partial x^k}-R^0_r\dfrac{\partial k_r}{\partial x^k}\right]+\lambda^\alpha_i(t)\dfrac{\partial X^i_\alpha}{\partial x^k}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x^k}- \\ \qquad -D_\gamma\left(\tau^r\left[\dfrac{\partial f_r}{\partial x^k_\gamma}-R^0_r\dfrac{\partial k_r}{\partial x^k_\gamma}\right]+\lambda^\alpha_i(t)\dfrac{\partial X^i_\alpha}{\partial x^k_\gamma}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x^k_\gamma}\right)=0, \\ \mu^\beta(t)Y_\beta(t,x(t),x_\gamma(t))=0,\ \beta=\overline{1,q}, \\ \tau\geq 0,\ e_r\tau^r=1,\ (\mu^\beta(t))\geqq 0,\ t\in\Omega. \end{cases}$$

Proof. It is similar to that of Theorem 4.1 from [9].

Put $F_r(x)=\displaystyle\int_\Omega f_r(t,x,x_\gamma)dv$, $\quad K_r(x)=\displaystyle\int_\Omega k_r(t,x,x_\gamma)dv$. Then, $R^0_r=\dfrac{F_r(x)}{K_r(x)}$, $r=\overline{1,p}$, and Theorem 3.2 becomes

Theorem 3.3 (Necessary efficiency in (VFP)). *Let $x = x(t) \in \mathcal{D}$ be a normal efficient solution of problem* (VFP). *Then there exist vector $\tau = (\tau^r) \in \mathbb{R}^p$ and piecewise smooth functions $\lambda = (\lambda_i^\alpha(t)) \in \mathbb{R}^{pm}$ and $\mu = (\mu^\beta(t)) \in \mathbb{R}^q$, defined on Ω, which satisfy the conditions*

$$(\mathrm{MFJ})_0 \quad \begin{cases} \tau^r \left[K_r(x)\dfrac{\partial f_r}{\partial x^k} - F_r(x)\dfrac{\partial k_r}{\partial x^k}\right] + \lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x^k} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x^k} - \\ \quad -D_\gamma\left[\tau^r\left(K_r(x)\dfrac{\partial f_r}{\partial x_\gamma^k} - F_r(x)\dfrac{\partial k_r}{\partial x_\gamma^k}\right) + \lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x_\gamma^k} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x_\gamma^k}\right] = 0, \\ \mu^\beta(t)Y_\beta(t, x(t), x_\gamma(t)) = 0,\ \beta = \overline{1,q}, \\ \tau \geq 0,\ e_r\tau^r = 1,\ (\mu^\beta(t)) \geqq 0,\ t \in \Omega. \end{cases}$$

In relations $(\mathrm{MFJ})_0$, we put $\lambda_i^\alpha(t) := K_r(x)\lambda_i^\alpha(t)$, $\mu^\beta(t) := K_r(x)\mu^\beta(t)$.

Definition 3.5. If conditions (MFJ) or $(\mathrm{MFJ})_0$ hold with $\tau \geq 0$, $e'\tau = 1$, then the point $x^0 \in \mathcal{D}$ is called *normal efficient solution* of (VFP).

Let $\rho \in \mathbb{R}$ and $b\colon \Phi \times \Phi \to [0, \infty)$ a function. Consider $H(x) = \displaystyle\int_\Omega h(t, x(t), x_\gamma(t))dv$.

Definition 3.6 ([14]). The function H is *called (strictly)* (ρ, b)*-quasiinvex* at x^0 if there exist vector functions $\eta(t) = (\eta_1(t), \dots, \eta_n(t)) \in \mathbb{R}^n$ of C^1-class with $\eta(t)\big|_{\partial\Omega} = 0$ and $\theta(x, x^0) \in \mathbb{R}^n$ such that for any x $(x \neq x^0)$,

$$\begin{aligned} &H(x) \leqq H(x^0) \quad \Rightarrow \\ &b(x, x^0)\int_\Omega \left\{\eta_i \frac{\partial h}{\partial x^i}(t, x^0(t), x_\gamma^0(t)) + (D_\gamma \eta_i)\frac{\partial h}{\partial x_\gamma^i}(t, x^0(t), x_\gamma^0(t)\right\} dv (<) \\ &\leqq -\rho b(x, x^0)\|\theta(x, x^0)\|^2. \end{aligned}$$

For a specialized study of invexity, we address the reader to [5].

4 Mond-Weir-Zalmai type duality for (VFP)

Consider a function $y \in \Phi$ and we associate to (VFP) the following multitime fractional variational vector dual problem

$$(\mathrm{WFD}) \quad \begin{cases} \text{Maximize Pareto } \left(\dfrac{\int_\Omega f_1(t, y, y_\gamma)dv}{\int_\Omega k_1(t, y, y_\gamma)dv}, \dots, \dfrac{\int_\Omega f_p(t, y, y_\gamma)dv}{\int_\Omega k_p(t, y, y_\gamma)dv}\right) \\ \text{subject to } \tau^r \left[K_r(y)\dfrac{\partial f_r}{\partial x^k} - F_r(y)\dfrac{\partial k_r}{\partial x^k}\right] + \lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x^k} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x^k} - \\ \quad -D_\gamma\left(\tau^r\left[K_r(x)\dfrac{\partial f_r}{\partial x_\gamma^k} - F_r(x)\dfrac{\partial k_r}{\partial x_\gamma^k}\right] + \lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x_\gamma^k} + \mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x_\gamma^k}\right) = 0, \\ \lambda_i^\alpha(t)X_\alpha^i(t, y, y_\gamma) + \mu^\beta(t)Y_\beta(t, y, y_\gamma) \geqq 0,\ i = \overline{1,p},\ \alpha = \overline{1,m},\ \beta = \overline{1,q}, \\ \tau = (\tau^r) \geqq 0,\ e_r\tau^r = 1,\ (\mu^\beta(t)) \geqq 0,\ y(t)\big|_{\partial\Omega} = u(t),\ t \in \Omega. \end{cases}$$

Denote by $\pi(x)$ the value of problem (VFP) at $x \in \mathcal{D}$ and $\delta(y, \lambda, \eta, \nu)$ be the value of dual (MFD) at $(y, \lambda, \eta, \nu) \in \Delta$, where Δ is the domain of (WFD). In what follows, we develop a duality theory between (VFP) and (WFD), [10], [13] and [17].

Theorem 4.1 (Weak duality). *Let $x \in \mathcal{D}$ and $(y,\lambda,\mu,\nu) \in \Delta$ be feasible points of problems* (VFP) *and* (WFD). *Assume satisfied the conditions:*

a) *for each* $r=\overline{1,p}$, *the integral* $\int_\Omega [K_r(y)f_r(t,x(t),x_\gamma(t)) - F_r(y)k_r(t,x(t),x_\gamma(t))]dv$ *is* (ρ_r', b)*-quasiinvex at* y *with respect to* η *and* θ;

b) $\int_\Omega [\lambda^i_\alpha(t)X^\alpha_i(t,x,x_\gamma) + \mu^\beta(t)Y_\beta(t,x,x_\gamma)]dv$ *is* (ρ,b)*-quasiinvex at* y *with respect to* η *and* θ;

c) *one of the functions of* a)-b) *is strictly* (ρ,b)*-quasiinvex at* y *with respect to* η *and* θ;

d) $\tau^r\rho_r' + \rho \geqq 0$.

Then $\pi(x) \leq \delta(y,\lambda,\mu,\nu)$ *is false.*

Proof. By reductio ad absurdum, suppose $\pi(x) \leq \delta(y,\lambda,\mu,\nu)$, or componentwise

$$\frac{\int_\Omega f_r(t,x,x_\gamma)dv}{\int_\Omega k_r(t,x,x_\gamma)dv} \leqq \frac{\int_\Omega f_r(t,y,y_\gamma)dv}{\int_\Omega k_r(t,y,y_\gamma)dv}, \quad r=\overline{1,p}.$$

This relation can be written

$$\begin{aligned}\int_\Omega [K_r(y)f_r(t,x,x_\gamma) - F_r(y)k_r(t,x,x_\gamma)]dv \leqq 0 \\ \left[= \int_\Omega [K_r(y)f_r(t,y,y_\gamma) - F_r(y)k_r(t,y,y_\gamma)]dv\right].\end{aligned} \tag{4.1}$$

According to hypothesis a), (4.1) implies

$$\begin{aligned} b(x,y)\int_\Omega \Big\{ \eta_s \left[K_r(y)\frac{\partial f_r}{\partial y^s} - F_r(y)\frac{\partial k_r}{\partial y^s}\right] + \\ +(D_\gamma \eta_s)' \left[K_r(y)\frac{\partial f_r}{\partial y^s_\gamma} - F_r(y)\frac{\partial k_r}{\partial y^s_\gamma}\right]\Big\} dv \leqq -\rho_r' b(x,y)\|\theta(x,y)\|^2.\end{aligned} \tag{4.2}$$

Multiplying (4.1) and (4.2) by $\tau^r \geqq 0$, and summing over $r=\overline{1,p}$, it results

$$\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y)] \leqq 0 \quad \Rightarrow \tag{4.3}$$

$$\begin{aligned} b(x,y)\int_\Omega \Big\{\eta_s\tau^r \left[K_r(y)\frac{\partial f_r}{\partial y^s} - F_r(y)\frac{\partial k_r}{\partial y^s}\right] + (D_\gamma\eta_s)\tau^r\left[K_r(y)\frac{\partial f_r}{\partial y^s_\gamma} - F_r(y)\frac{\partial k_r}{\partial y^s_\gamma}\right]\Big\} dv \\ \leqq -\tau^r\rho_r' b(x,y)\|\theta(x,y)\|^2.\end{aligned}$$

From the domains $\mathcal{D}$ and Δ it follows

$$\begin{aligned}\int_\Omega [\lambda^\alpha_i(t)X^i_\alpha(t,x,x_\gamma) + \mu^\beta(t)Y_\beta(t,x,x_\gamma)]dv \leqq \\ \leqq \int_\Omega [\lambda^\alpha_i(t)X^i_\alpha(t,y,y_\gamma) + \mu^\beta(t)Y_\beta(t,y,y_\gamma)]dv\end{aligned} \tag{4.4}$$

and according to c) from (4.4) we find

$$
\begin{aligned}
(4.5)\qquad & b(x,y)\int_\Omega \left\{\eta_s\left[\lambda_i^\alpha(t)'\frac{\partial X_\alpha^i}{\partial y^s}+\mu^\beta(t)\frac{\partial Y_\beta}{\partial y^s}\right]+\right.\\
&\left.+(D_\gamma\eta_s)\left[\lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial y_\gamma^s}+\mu^\beta(t)\frac{\partial Y_\beta}{\partial y_\gamma^s}\right]\right\}dv \leqq \rho b(x,y)\|\theta\|^2.
\end{aligned}
$$

Summing now side by side implications (4.3) and (4.4) and taking into account c), we obtain

$$
\begin{aligned}
(4.6)\qquad & \tau^r[F_r(x)K_r(y)-K_r(x)F_r(y)]+\int_\Omega[\lambda_i^\alpha(t)X_\alpha^i(t,x,x_\gamma)+\mu^\beta(t)Y_\beta(t,x,x_\gamma)]dv-\\
&-\int_\Omega[\lambda_i^\alpha(t)X_\alpha^i(t,y,y_\gamma)+\mu^\beta(t)Y_\beta(t,y,y_\gamma)]dv\leqq 0
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow b(x,y)\int_\Omega\left\{\eta_s\left\{\tau^r\left[K_r(y)\frac{\partial f_r}{\partial y^s}-F_r(y)\frac{\partial k_r}{\partial y^s}\right]+\lambda_i^\alpha(t)'\frac{\partial X_\alpha^i}{\partial y^s}+\mu^\beta(t)\frac{\partial Y_\beta}{\partial y^s}\right\}+\right.\\
&\left.+(D_\gamma\eta_s)\left\{\tau^r\left[K_r(y)\frac{\partial f_r}{\partial y_\gamma^s}-F_r(y)\frac{\partial k_r}{\partial y_\gamma^s}\right]+\lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial y_\gamma^s}+\mu^\beta(t)\frac{\partial Y_\beta}{\partial y_\gamma^s}\right\}\right\}dv\\
&< -b(x,y)\|\theta(x,y)\|^2\{\tau^r\rho_r+\rho\}.
\end{aligned}
$$

From the second implication of (4.6), we get $b(x,y)>0$, which can be written

$$
(4.7)\qquad \int_\Omega\left[\eta_s\frac{\partial V}{\partial y^s}+(D_\gamma\eta_s)\frac{\partial V}{\partial y_\gamma^s}\right]dt< -\|\theta(x,y)\|^2\{\tau^r\rho_r+\rho\},
$$

where $V=\tau^r[K_r(y)f_r(t,y,v)-F_r(y)k_r(t,y,v)]+\lambda_i^\alpha(t)X_\alpha^i(t,y,v)+\mu^\beta(t)Y_\beta(t,y,v)$.

Since $(D_\gamma\eta_s)\dfrac{\partial V}{\partial y_\gamma^s}=D_\gamma\left(\eta_s\dfrac{\partial V}{\partial y_\gamma^s}\right)-\eta_sD_\gamma\left(\dfrac{\partial V}{\partial y_\gamma^s}\right)$, it follows

$$
\int_\Omega(D_\gamma\eta_s)\frac{\partial V}{\partial y_\gamma^s}dv=\int_\Omega D_\gamma\left(\eta_s\frac{\partial V}{\partial y_\gamma^s}\right)dv-\int_\Omega\eta_sD_\gamma\left(\frac{\partial V}{\partial y_\gamma^s}\right)dv.
$$

Using the divergence formula, we have

$$
\int_\Omega D_\gamma\left(\eta_s\frac{\partial V}{\partial y_\gamma^s}\right)dv=\int_{\partial\Omega}\left(\eta_s\frac{\partial V}{\partial y_\gamma^s}\right)\vec{\eta}(t)d\sigma=0,
$$

where $\vec{\eta}(t)$ is unit vector to surface $\partial\Omega$ at the current point, and $\eta_s(t)\big|_{\partial\Omega}=0$.

Then relation (4.7) becomes

$$
(4.8)\qquad \int_a^b\eta_s\left[\frac{\partial V}{\partial y^s}-D_\gamma\left(\frac{\partial V}{\partial y_\gamma^s}\right)\right]dv< -\|\theta(x,y)\|^2\{\tau^r\rho_r+\rho\}.
$$

Taking into account the first constraint of problem (WFD), we have

$$
\frac{\partial V}{\partial y^s}-D_\gamma\left(\frac{\partial V}{\partial y_\gamma^s}\right)=0,
$$

and relation (4.8) becomes $0< -\|\theta(x,y)\|^2\{\tau^r\rho_r+\rho\}$.

According to the hypothesis d) of the theorem, this inequality becomes $0 < 0$, which is false. Then, from (4.6)

$$\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y)] + \int_\Omega [\lambda_i^\alpha(t)X_\alpha^i(t,x,x_\gamma) + \mu^\beta(t)Y_\beta(t,x,x_\gamma)]dv - \\ - \int_\Omega [\lambda_i^\alpha(t)X_\alpha^i(t,y,y_\gamma) + \mu^\beta(t)Y_\beta(t,y,y_\gamma)]dv > 0. \tag{4.9}$$

Taking into account relation (4.4), from (4.9) it results $\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y)] > 0$, which contradict relation (4.3). Therefore $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Theorem 4.2 (Direct duality). *Let x^0 be a normal efficient solution of the primal* (VFP) *and suppose satisfied the hypotheses of Theorem* 4.1. *Then there are vector $\tau^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda^0 = (\lambda_i^\alpha)^0 \colon \Omega \to \mathbb{R}^{nm}$ and $\mu^0 = (\mu^0) \colon \Omega \to \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual* (MWFD) *and $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Proof. Because x^0 is a normal efficient solution to (VFP), according to Theorem 3.3 there are vector $\tau^0 = (\tau^r)^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda^0 = (\lambda_i^\alpha)^0 \colon \Omega \to \mathbb{R}^{nm}$ and $\mu^0 = (\mu^\beta)^0 \colon \Omega \to \mathbb{R}^q$ which satisfy relations $(\mathrm{MFJ})_0$. It follows that $(x^0, \tau^0, \lambda^0, \mu^0) \in \Delta$ and $\pi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

Theorem 4.3 (Converse duality). *Let $(x^0, \tau^0, \lambda^0, \mu^0) \in \Delta$ be an efficient solution of dual and* (MWFD) *and assume satisfied the next conditions*

i) *$\bar{x}$ is a normal efficient solution of primal* (VFP)*;*

ii) *the hypotheses of Theorem* 4.1 *are satisfied with $(y, \tau, \lambda, \mu) = (x^0, \tau^0, \lambda^0, \mu^0)$.*

Then $\bar{x} = x^0$ and $\pi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

Proof. On the contrary, suppose that $\bar{x} \neq x^0$ and we will obtain a contradiction. According to Theorem 3.3, because $\bar{x}$ is normal efficient solution of (VFP), then there are vector $\bar{\tau} \in \mathbb{R}^p$ and vector functions $\bar{\lambda} = (\bar{\lambda}_i^\alpha) \colon \Omega \to \mathbb{R}^{nm}$ and $\bar{\mu} = (\bar{\mu}^\beta) \colon \Omega \to \mathbb{R}^q$ that satisfy conditions $(\mathrm{MFJ})_0$. It results $\bar{\lambda}_i^\alpha(t)X_\alpha^i(t, \bar{x}, \bar{x}_\gamma) + \bar{\mu}^\beta(t)Y_\beta(t, \bar{x}, \bar{x}_\gamma) = 0$, therefore $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu}) \in \Delta$. Moreover, $\pi(\bar{x}) = \delta(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu})$. According to Theorem 4.1 relation $\pi(\bar{x}) \leq \delta(x^0, \tau^0, \lambda^0, \mu^0)$ is false. Then the relation $\delta(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu}) \leq \delta(x^0, \tau^0, \lambda^0, \mu^0)$ is false. Therefore, the maximal efficiency of $(x^0, \tau^0, \lambda^0, \mu^0)$ is contradicted. Then, it results that the assumption $\bar{x} \neq x^0$, above made, is false. It follows $\bar{x} = x^0$ and also $\bar{\tau} = \tau^0$, $\bar{\lambda} = \lambda^0$, $\bar{\mu} = \mu^0$. Finally, we have $\pi(\bar{x}) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

For other advances on this field, we address the reader to [1], [3], [7], [8], [16].

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