Totally singular Lagrangians and affine Hamiltonians of higher order

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Abstract. A higher order Lagrangian or an affine Hamiltonian is totally singular if its vertical Hessian vanishes. A natural duality relation between totally singular Lagrangians and affine Hamiltonians is studied in the paper. We prove that the energy of a totally singular affine Hamiltonian has as a dual a suitable first order totally singular Lagrangian. Relations between the solutions of Euler and Hamilton equations of dual objects are studied by mean of semi-sprays. In order to generate examples for $k > 1$, some natural lift procedures are constructed.

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1 Introduction

Some results and constructions from [14] are extended in this paper from the case $k = 2$ to the general case, $k \geq 2$. Some physical and mathematical aspects that motivate the study of totally singular Lagrangians in the second order case can be found also in [1, 8, 7] and the references therein.

For hyperregular Lagrangians of higher order, the Legendre duality between Lagrangians and Hamiltonians was studied in various papers (see [16] for recent results and references). But the class of hyperregular Lagrangians and Hamiltonians is too restrictive. We study Lagrangians and Hamiltonians of higher order that have null vertical Hessians, called in the paper as *totally singular*; they are the ,,most singular" Lagrangians and Hamiltonians. We consider in the paper that a totally singular Lagrangian of order k is allowed if it is in duality with a totally singular Hamiltonian of order k. An allowed totally singular Lagrangian has a dual allowed totally singular Hamiltonian; for the converse situation, Theorem 2.1 asserts that, assuming some conditions, an allowed totally singular Hamiltonian of order k has a dual allowed totally singular Lagrangian of order k and both can be related to ordinary dual (allowed totally singular) Lagrangians and Hamiltonians of first order on $T^{k-1}M$.

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In order to have consistent examples of totally singular Lagrangians and Hamiltonians of higher order, lifting procedures are given in the last section. In this way, certain examples considered in [14, 9, 4] can be lifted to totally singular Lagrangians and Hamiltonians of higher order. Following [17], in an analogous manner one can study the time-dependent case. Further investigations on general jet spaces, complex spaces or using linear frames can be made following approaches in [6], [13] and [3] respectively.

2 Higher order Hamiltonians and Lagrangians

Let M be a differentiable manifold. We use a coordinate construction of T^kM, $k \geq 2$, as in [11], [12] or [16]. The fibered manifold $(T^kM, \pi_k, T^{k-1}M)$ is an affine bundle, for $k \geq 2$. A section $S : T^kM \to T^{k+1}M$ of the affine bundle $(T^{k+1}M, \pi_{k+1}, T^kM)$ is called a *semi-spray* of order k on M; S can be seen as well as a vector field on the manifold T^kM. A *Lagrangian of order* k on M is a differentiable function $L : T^kM \to I\!R$ or $L : W \to I\!R$, where $W \subset T^kM$ is an open fibered submanifold. For example, in [11] and [12] $W = \widetilde{T^kM} = T^kM \backslash \{0\}$ (where $\{0\}$ is the image of the ,,null" section, i.e. the section of null velocities) and $L : T^kM \to I\!R$ is continuous.

The totally singular Hamiltonians of order $k \geq 2$, studied in our paper, are affine Hamiltonians as in [16]. Let us consider the affine bundle $T^kM \stackrel{\pi_k}{\to} T^{k-1}M$ and $u \in T^{k-1}M$. The fiber $T^k_uM = \pi_k^{-1}(u) \subset T^kM$ is a real affine space, modelled on the real vector space $T_{\pi(u)}M$. The *vectorial dual* of the affine space T^k_uM is $T^{k\dagger}_uM = Aff(T^k_uM, I\!R)$, where Aff denotes affine morphisms. Denoting by $T^{k\dagger}M = \bigcup_{u \in T^{k-1}M} T^{k\dagger}_uM$ and $\pi^\dagger : T^{k\dagger}M \to T^{k-1}M$ the canonical projection, then $(T^{k\dagger}M, \pi^\dagger, T^{k-1}M)$ is a vector bundle. There is a canonical vector bundle morphism $\Pi : T^{k\dagger}M \to T^{k*}M$, over the base $T^{k-1}M$. This projection is also a canonical projection of an affine bundle with type fiber $I\!R$. An *affine Hamiltonian* of order k on M is a section $h : T^{k*}M \to T^{k\dagger}M$ of this affine bundle (or of an open fibered submanifold $W \subset T^{k*}M$), i.e. $\Pi \circ h = 1_{T^{k*}M}$ (or $\Pi \circ h = 1_W$ respectively). Thus an affine Hamiltonian is not a real function, but a section in an affine bundle with a one dimensional fiber.

Considering some local coordinates (x^i) on M, $(x^i, y^{(1)i}, \ldots, y^{(k-1)i})$ on $T^{k-1}M$, and $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, T)$ on $T^{k\dagger}M$, then the coordinates p_i and T change according to the rules

$$p_{i'} = \frac{\partial x^i}{\partial x^{i'}} p_i; \; T' = T + \frac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\frac{\partial x^i}{\partial x^{i'}} p_i,$$

where

$$\Gamma_U^{(k-1)} = y^{(1)i}\frac{\partial}{\partial x^i} + \cdots + (k-1)y^{(k-1)i}\frac{\partial}{\partial y^{(k-2)i}}.$$

The the affine Hamiltonian $h : \widetilde{T^{k*}M} \to \widetilde{T^{k\dagger}M}$ has the local form

$$h(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, \ldots, y^{(k-1)i}, p_i, H_0(x^i, \ldots, y^{(k-1)i}, p_i)) \tag{2.1}$$

and the local function H_0 changes according to the rule

$$H_0'(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}, p_{i'}) = H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) + \frac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\frac{\partial x^i}{\partial x^{i'}} p_i.$$

There is a *co-Legendre map* $\mathcal{H} : T^{k*}M \to T^kM$, locally given by

$$\begin{aligned}\mathcal{H}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) &= (x^i, y^{(1)i}, \ldots, y^{(k-1)i},\\ \mathcal{H}^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) &= \frac{\partial H_0}{\partial p_i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)).\end{aligned}$$

Since $\frac{\partial^2 H_0'}{\partial p_{i'} \partial p_{j'}} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial^2 H_0}{\partial p_i \partial p_j}$, it follows that $h^{ij} = \frac{\partial^2 H_0}{\partial p_i \partial p_j}$ defines a symmetric bilinear d-form on $T^{k-1}M$, called the *vertical Hessian* of h.

For an affine Hamiltonian h of order k ($k \geq 2$) and the local domain of coordinates U, one can consider the local functions on $T^*T^{k-1}M$:

$$\mathcal{E}_U = p_{(0)i} y^{(1)i} + \cdots + (k-1) p_{(k-1)i} y^{(k)i} + k H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_{(1)i}). \tag{2.2}$$

It can be easily proved (as in [16]) that the local functions $\mathcal{E}_U$ glue together to a global function $\mathcal{E}_0 : \widetilde{T^*T^{k-1}M} \to I\!R$, called the *energy* of h.

We say that an affine Hamiltonian is *totally singular* if its vertical Hessian is null. Notice that the difference of two affine Hamiltonians of order k is a vectorial Hamiltonian of order k (i.e. a real function on $T^{k*}M$, see [16]) and every affine Hamiltonian of order k is a sum of an affine totally singular Hamiltonian of order k and a vectorial Hamiltonian of order k. If a totally singular affine Hamiltonian h of order k on M has a local form (2.1), then the local function H_0 has the form

$$\begin{aligned}H_0(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_i) &= p_i S^i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}) +\\ &\quad f(x^j, y^{(1)j}, \ldots, y^{(k-1)j}),\end{aligned} \tag{2.3}$$

where (S^i) defines an affine section $S : T^{k-1}M \to T^kM$ given locally by

$$(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) \xrightarrow{S} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, S^i) \text{ and } f \in \mathcal{F}(T^{k-1}M).$$

It defines a semi-spray $\Gamma_0 \in \mathcal{X}(T^{k-1}M)$, called the *associated semi-spray* of h:

$$\Gamma_0 = y^{(1)i} \frac{\partial}{\partial x^i} + \cdots + (k-1) y^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}} + k S^i \frac{\partial}{\partial y^{(k-1)i}}. \tag{2.4}$$

Let us consider some examples.

1°. Let Γ_0, given by formula (2.4), be the local form of a semi-spray. Then the formula $H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = S^i p_i$ defines a totally singular affine Hamiltonian of order k.

2°. Let $L_0 : T^{k-1}M \to I\!R$ be a regular Lagrangian of order $k-1$ and let Γ_0 be the semi-spray defined by L_0 (see [11]). Then, using the above example, a totally singular affine Hamiltonian of order k is obtained.

Let $L : TM \to I\!R$ and $H : T^*M \to I\!R$ be a totally singular Lagrangian and Hamiltonian respectively, of first order, having local forms

$$L(x^i, y^i) = \alpha_i(x^j) y^i + \beta(x^j),\ H(x^i, p_i) = p_i \varphi^i(x^j) + \gamma(x^j),$$

where $\alpha = \alpha_i dx^i \in \mathcal{X}^*(M)$, $\varphi = \varphi^i \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$ and $\beta, \gamma \in \mathcal{F}(M)$ (see [14]). According also to [14], L and H are in duality if $i_\varphi d\alpha = d\beta$ and $L(x, \varphi) + H(x, \alpha) - \alpha(\varphi) = const$.

The energy of a totally singular affine Hamiltonian h given by (2.3) is

$$\mathcal{E} = p_{(0)i}y^{(1)i} + \ldots + (k-1)p_{(k-2)i}y^{(k-1)i} + kp_{(k-1)i}S^i + kf.$$

We can view $\mathcal{E}$ as a totally singular Hamiltonian $\mathcal{E} : T^*T^{k-1}M \to I\!R$. Let us look in that follows for a totally singular Lagrangian on $T^{k-1}M$, that is in duality with $\mathcal{E}$.

Proposition 2.1. *Locally, there is a (first order) totally singular Lagrangian* $L : \bar{U} \subset TT^{k-1}M \to I\!R$, *which is dual to the (first order) totally singular Hamiltonian* $\mathcal{E} : TT^{k-1}M \to I\!R$.

Proof. The vector field on $T^{k-1}M$, that corresponds to $\mathcal{E}$ is $\varphi = \Gamma_0$, the semi-spray defined by h, given by formula (2.4). We denote $\varphi^{(0)i} = y^{(1)i}, \ldots, \varphi^{(k-2)i} = (k-1)y^{(k-1)i}$, $\varphi^{(k-1)i} = kS^i(x^j,\, y^{(1)j}, \ldots,\, y^{(k-1)j})$ and $\gamma = f$. Let us take $\alpha := \bar{\alpha}$, with

$$\bar{\alpha} = \alpha_{(0)i}dx^i + \alpha_{(1)i}dy^{(1)i} + \cdots + \alpha_{(k-1)i}dy^{(k-1)i} \tag{2.5}$$

and denote

$$H_0(x^i, y^{(1)j}, \ldots, , y^{(k-1)j}, p_{(k-1)j}) = S^i p_{(k-1)i} + f.$$

The equality $i_\varphi d\alpha = d\beta$ gives the following system of partial differential equations:
(2.6)

$$\begin{cases} k\frac{\partial H_0}{\partial x^i}(x^i, y^{(1)j}, \ldots, , y^{(k-1)j}, \alpha_{(k-1)j}) + \Gamma_0(\alpha_{(0)j}) = 0, \\ \alpha_{(0)i} + k\frac{\partial H_0}{\partial y^{(1)i}} + \Gamma_0(\alpha_{(1)j}) = 0, \quad \ldots \quad , \alpha_{(k-2)i} + k\frac{\partial H_0}{\partial y^{(k-1)i}} + \Gamma_0(\alpha_{(k-1)j}) = 0. \end{cases}$$

Eliminating successively $\alpha_{(k-2)i}, \ldots,\ \alpha_{(0)i}$ in the last $k-1$ equations, the first equation becomes:

$$\begin{aligned} \Gamma_0^{k-1}(\alpha_{(k-1)i}) \;=\; & (-1)^k k\frac{\partial H_0}{\partial x^i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \alpha_{(k-1)i}) + \\ & (-1)^{k-1}k\Gamma_0\left(\frac{\partial H_0}{\partial y^{(1)i}}\right) + \cdots - k\Gamma_0^{k-1}\left(\frac{\partial H_0}{\partial y^{(k-1)i}}\right). \end{aligned} \tag{2.7}$$

Let us denote by $F_i \in \mathcal{F}(T^{k*}M)$ the right side of this equation. Let $\{z^\alpha\}_{\alpha=\overline{1,(k+1)m}}$ be a system of local coordinates on the manifold $T^{k*}M$ such that $\Gamma_0 = \frac{\partial}{\partial z^1}$. Then the local form of the differential equation (2.7) is $\frac{\partial^{k-1}\alpha_{(k-1)i}}{\partial (z^1)^{k-1}} = F_i(z^\alpha, \alpha_{(k-1)i})$. Since this differential equation has local solutions, the conclusion follows. □

Let us consider the canonical projections $T^{k\dagger}M \xrightarrow{\Pi} T^{k*}M = T^{k-1}M \times_M T^*M \xrightarrow{p_1} T^{k-1}M$. For a d-form $\alpha = (\alpha_i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}))$ on $T^{k-1}M$, we denote by $\alpha' : T^{k-1}M \to T^{k*}M = T^{k-1}M \times_M T^*M$ the map defined by $\alpha'(z) = (z, \alpha_z)$. We say that a map $h_\alpha : T^{k-1}M \to T^{k\dagger}M$ is an *α-Hamiltonian* if $\Pi \circ h_\alpha = \alpha'$. Using local coordinates, the local form of h_α is

$$\begin{aligned} &(x^j, y^{(1)j}, \ldots, y^{(k-1)j}) \xrightarrow{h} (x^j, y^{(1)j}, \ldots, y^{(k-1)j}, \alpha_i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}), \\ &-h_0(x^j, y^{(1)j}, \ldots, y^{(k-1)j})) \end{aligned}$$

and the local functions h_0 change on the intersection of two coordinate charts according to the rule

$$kh'_0(x^{j'}, y^{(1)j'}, \dots, y^{(k-1)j'}) = kh_0(x^j, y^{(1)j}, \dots, y^{(k-1)j}) + \Gamma_U(y^{(k-1)i'})\alpha_{i'}.$$

For example, if $\chi : T^{k*}M \to T^{k\dagger}M$ is an affine Hamiltonian and $\alpha : T^{k-1}M \to T^*M$ is a d-form on $T^{k-1}M$, then $h_\alpha = \chi \circ \alpha'$ is an α-Hamiltonian.

Proposition 2.2. *Let $\alpha = (\alpha_i)$ be a d-form on $T^{k-1}M$, h_0 be the local function of an α-Hamiltonian h_α and*

$$L(x^j, y^{(1)j}, \dots, y^{(k)j}) = ky^{(k)i}\alpha_i(x^j, y^{(1)j}, \dots, y^{(k-1)j}) - kh_0(x^j, y^{(1)j}, \dots, y^{(k-1)j}). \tag{2.8}$$

Then $L \in \mathcal{F}(T^kM)$.

Proof. Indeed, we have: $L(x^{j'}, y^{(1)j'}, \dots, y^{(k-1)j'}) = ky^{(k)i'}\alpha_{i'} - kh'_0 = k\frac{\partial x^{i'}}{\partial x^i}y^{(k)i}\alpha_{i'} + \Gamma_U(y^{(k-1)I'})\alpha_{i'} - kh_0 - \Gamma_U(y^{(k-1)I'})\alpha_{i'} = ky^{(k)i}\alpha_i - kh_0 = L(x^j, y^{(1)j}, \dots, y^{(k)j})$. □

For a curve $\gamma : I \to M$, $t \to (\gamma^i(t))$, its *k-tangent lift* is a curve $\gamma^{(k)} : I \to T^kM$ that has the local form

$$t \to (\gamma^i(t), \frac{d\gamma^i}{dt}(t), \dots, \frac{1}{k!}\frac{d^k\gamma^i}{dt^k}(t)).$$

Let $L : T^kM \to I\!R$ be a Lagrangian of order k. The critical curves $\gamma : [0,1] \to M$, $t \overset{\gamma}{\to} (x^i(t))$, of its integral action

$$I(\gamma) = \int_0^1 L\left(x^i, \frac{dx^i}{dt}, \dots, \frac{1}{k!}\frac{d^kx^i}{dt^k}\right) dt,$$

are solutions of the Lagrange equation

$$\frac{\partial L}{\partial x^i} - \frac{1}{1!}\frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \dots + (-1)^k\frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial L}{\partial y^{(k)i}} = 0. \tag{2.9}$$

The *integral action* of the affine Hamiltonian h along a curve $\gamma : [0,1] \to T^*M$, $t \overset{\gamma}{\to} (x^i(t), p_i(t))$, is defined in [16] by the formula:

$$I(\gamma) = \int_0^1 \left[p_i\frac{1}{(k-1)!}\frac{d^kx^i}{dt^k} - kH_0\left(x^i, \frac{dx^i}{dt}, \dots, \frac{1}{(k-1)!}\frac{d^{k-1}x^i}{dt^{k-1}}, p_i\right)\right] dt. \tag{2.10}$$

The critical condition (or Fermat condition in the case of an extremum) for γ, gives the Hamilton equation for h in the condensed form:

$$\begin{cases} \dfrac{(-1)^k}{k!}\dfrac{d^kp_i}{dt^k} - \dfrac{\partial H_0}{\partial x^i} + \dfrac{d}{dt}\dfrac{\partial H_0}{\partial y^{(1)i}} - \dots + \dfrac{(-1)^{k-1}}{(k-1)!}\dfrac{d^{k-1}}{dt^{k-1}}\dfrac{\partial H_0}{\partial y^{(k-1)i}} = 0, \\ \dfrac{1}{k!}\dfrac{d^kx^i}{dt^k} - \dfrac{\partial H_0}{\partial p_i} = 0. \end{cases} \tag{2.11}$$

Let h and L be totally singular of order k, having the local forms (2.3) and (2.8) respectively. Then a d-form $\alpha = (\alpha_i(x^j, y^{(1)j}, \dots, y^{(k-1)j}))$ on $T^{k-1}M$ corresponds to L and a semi-spray Γ_0 of order $k-1$, given by (2.4), corresponds to h. We say that L is in *duality* with h if the formula (2.7) holds, with $\alpha_{(k-1)i} = \alpha_i$ and $h_\alpha = h \circ \alpha'$ (i.e. the α-Hamiltonian h_α corresponds to h and α).

Lemma 2.1. *If L is in duality with h, then for $a=\overline{1,k-1}$ one have*

$$\begin{aligned}\frac{\partial H_0}{\partial x^i}(x^i,y^{(1)i},\dots,y^{(k-1)i},\alpha_i) &= -\frac{\partial L}{\partial x^i}(x^i,y^{(1)i},\dots,y^{(k-1)i},S^i),\\ \frac{\partial H_0}{\partial y^{(a)i}}(x^i,y^{(1)i},\dots,y^{(k-1)i},\alpha_i) &= -\frac{\partial L}{\partial y^{(a)i}}(x^i,y^{(1)i},\dots,y^{(k-1)i},S^i).\end{aligned}$$

Proof. It suffices to prove only the first relation, since the proof of each of the other relations follows the same idea. Using relations $h_0=h\circ\alpha'$ and (2.8), we obtain $L(x^j,y^{(1)j},\dots,y^{(k)j})=k\alpha_j(y^{(k)j}-S^i)-f$. Thus the first relation holds. □

We say also that a totally singular Lagrangian of order k is *allowed* if there is a semi-spray Γ_0, of order $k-1$, and a d-form $\alpha=(\alpha_i)$ such that the following formula holds:

$$\begin{aligned}\Gamma_0^{k-1}(\alpha_i) &= (-1)^{k-1}k\frac{\partial L}{\partial x^i}(x^i,y^{(1)i},\dots,y^{(k-1)i},S^i)+(-1)^{k-2}k\Gamma_0\left(\frac{\partial L}{\partial y^{(1)i}}\right)+\\ &\quad\cdots-k\Gamma_0^{k-1}\left(\frac{\partial L}{\partial y^{(k-1)i}}\right).\end{aligned}$$

It is easy to see that a totally singular Lagrangian of order k is allowed if it is in duality with a totally singular Hamiltonian of order k. Thus a local dual of a totally singular Hamiltonian of order k is allowed. The following result can be proved by a straightforward verification, using local coordinates.

Proposition 2.3. *Let $\alpha=(\alpha_i(x^j,y^{(1)j},\dots,y^{(k-1)j}))$ be a d-form on $T^{k-1}M$ and $h: T^{k-1}M\to T^{k\dagger}M$ be an α-*Hamiltonian *such that there is a 1-form $\bar\alpha\in\mathcal{X}^*(T^{k-1}M)$ where α is the top component of $\bar\alpha$, i.e. $\bar\alpha=\alpha_{(0)i}dx^i+\alpha_{(1)i}dy^{(1)i}+\dots+\alpha_{(k-1)i}dy^{(k-1)i}$, with $\alpha_{(k-1)i}=\alpha_i$. Then the formula*

$$\begin{aligned}L(x^j,y^{(1)j},\dots,y^{(k-1)j},Y^{(0)i},Y^{(1)i},\dots,Y^{(k-1)i}) &= (Y^{(0)i}-y^{(1)i})\alpha_{(0)i}+\\ \cdots+(Y^{(k-2)i}-y^{(k-1)i})\alpha_{(k-2)i}&+Y^{(k-1)i}\alpha_{(k-1)i}-h\end{aligned}\tag{2.12}$$

defines a totally singular Lagrangian on $T^{k-1}M$.

The restriction of L to T^kM has the form $L_0(x^j,\dots,y^{(k)j})=y^{(k)i}\alpha_i-h$. Thus if a totally singular Lagrangian L_0 on T^kM has the property that $\alpha=(\alpha_i)$ is the top component of a 1-form α' on $T^{k-1}M$, then L_0 is the restriction to T^kM of a totally singular Lagrangian L on $T^{k-1}M$ (since $T^kM\subset TT^{k-1}M$).

Let h be a totally singular Hamiltonian of order k on M, having the corresponding local function $H_0(x^j,y^{(1)j},\dots,y^{(k-1)j},p_i)=p_iS^i(x^j,y^{(1)j},\dots,y^{(k-1)j})+f(x^j,y^{(1)j},\dots,y^{(k-1)j})$. We can consider the local 1-form $\bar\alpha=\alpha_{(0)i}dx^i+\alpha_{(1)i}dy^{(1)i}+\cdots+\alpha_{(k-1)i}dy^{(k-1)i}$ that is a solution of the system (2.6). Considering the d-form α on $T^{k-1}M$, defined by its top component, we can construct a totally singular Lagrangian of order k on M.

Theorem 2.1. *Let h be a totally singular affine Hamiltonian of order k. If the system (2.7) has a d-form $\alpha=(\alpha_{(k-1)i})$ on $T^{k-1}M$ as a global solution, then there is an allowed totally singular Lagrangian, $L: TT^{k-1}M\to I\!R$ (on $T^{k-1}M$), such that:*

1. *The energy $\mathcal{E}$ of h is a dual Hamiltonian of L.*

2. *The restriction of L to $T^kM \subset TT^{k-1}M$ is an allowed totally singular Lagrangian $L_1 : T^kM \to I\!R$ (of order k on M).*

3. *The pairs (h, L_1) and $(\mathcal{E}, L)$ are each dual pairs.*

Proof. Using $\alpha_{(k-1)i}$ in (2.6), one obtain a 1-form $\bar{\alpha} \in \mathcal{X}^*(T^{k-1}M)$ given by (2.5). Using Proposition 2.3, one obtains L. The definitions of $\mathcal{E}$ in 2.2 and L from 2.12, prove the first statement. One has $L_1(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, y^{(k)i}) = y^{(k)i}\alpha_{(k-1)i} - h_{\bar{\alpha}}$, where α is the d-form on $T^{k-1}M$ defined by $(\alpha_{(k-1)i})$ and the α-Hamiltonian is $h_\alpha = h \circ \alpha'$, Using Proposition 2.2 one obtain the second statement. The construction of $\bar{\alpha}$ shows that L_1 is in duality with h; the last statement follows using 1. □

We notice that Theorem 2.1 can be adapted in the case when the d-form $\alpha = (\alpha_{(k-1)i})$ is a solution of the system (2.7) on an open fibered submanifold of $T^{k*}M \to T^{k-1}M$.

Proposition 2.4. *Let $t \stackrel{\gamma_1}{\to} (\gamma_1^i(t), \gamma_1^{(1)i}(t), \ldots, \gamma_1^{(k-1)i}(t))$ be an integral curve of the semi-spray Γ_0. Then:*

1. *the curve γ_1 is the $(k-1)$-tangent lift of a curve $t \stackrel{\gamma}{\to} (\gamma^i(t))$, i.e. $\gamma_1 = \gamma^{(k-1)}$;*

2. *the curve $t \stackrel{\gamma_2}{\to} (\gamma^i, \omega_i)$ in T^*M, where*

$$\omega_i(t) = \alpha_i(\gamma^i(t), \frac{d\gamma}{dt}(t), \ldots, \frac{1}{(k-1)!}\frac{d^{k-1}\gamma}{dt^{k-1}}(t))$$

   is a solution of the Hamilton equation of h;

3. *the curve γ is a solution of the Euler equation of L.*

Proof. The first assertion follows using that Γ_0 is a semi-spray. Along the curve $\gamma^{(k-1)}$ one have $\frac{d}{dt} = \Gamma_0$. The conclusion of the second statement follows using relation (2.7). In order to prove the third statement one use Lemma 2.1 and 2. □

According to [14], not all the solutions of the Lagrange equation of a totally singular Lagrangian of order 2 come from the integral curves of a semi-spray of order 1. More precisely, let $L : T^2M \to I\!R$,

$$L(x^i, y^{(1)i}, y^{(2)i}) = 2y^{(2)i}\alpha_i(x^j, y^{(1)j}) - 2\beta(x^i, y^{(1)i})$$

be a totally singular Lagrangian of second order. In [14] it is proved that the solutions of its Lagrange equations are the integral curves of a second order semi-spray on M, provided that the skew symmetric d-tensor $\bar{\alpha}$ given by $\bar{\alpha}_{ij} = \frac{\partial \alpha_j}{\partial y^{(1)i}} - \frac{\partial \alpha_i}{\partial y^{(1)j}}$ is non-degenerate.

3 Lifting procedures

In order to have consistent examples of totally singular Lagrangians and affine Hamiltonians of order $k \geq 2$, we give in this section some algorithms that allow to lift an totally singular Lagrangian of order $k \geq 1$, that is s-non-degenerated, to an allowed non-singular Lagrangian of order $k+1$, also s-non-degenerated.

We recall that if $\bar{\alpha} \in \mathcal{X}^*(T^{k-1}M)$ has the local expression $\bar{\alpha} = \alpha_{(0)i}dx^i + \alpha_{(1)i}dy^{(1)i} + \cdots + \alpha_{(k-1)i}dy^{(k-1)i}$, then the d-form α defined by $(\alpha_{(k-1)i})$ is called its *top component*. We say that the d-form α on $T^{k-1}M$ is *non-degenerated* if the matrix

$$\left(\alpha_{ij} = \frac{\partial \alpha_i}{\partial y^{(k-1)j}}\right)_{i,j=\overline{1,m}}$$

is non-degenerate in every point of $T^{k-1}M$ of coordinates $(x^j, y^{(1)j}, \ldots, y^{(k-1)j})$. We denote $(\alpha^{ij}) = (\alpha_{ij})^{-1}$. Notice that the condition does not depend on coordinates.

We say that the d-form α on $T^{k-1}M$ is *s-non-degenerated* (the initial *s* comes from skew-symmetric) if the matrix

$$\left(\tilde{\alpha}_{ij} = \frac{\partial \alpha_i}{\partial y^{(k-1)j}} - \frac{\partial \alpha_j}{\partial y^{(k-1)i}}\right)_{i,j=\overline{1,m}} \tag{3.1}$$

is non-degenerate in every point of $T^{k-1}M$ of coordinates $(x^j, y^{(1)j}, \ldots, y^{(k-1)j})$. We denote $(\tilde{\alpha}^{ij}) = (\tilde{\alpha}_{ij})^{-1}$. Notice also that this condition does not depend on coordinates.

Let L be a totally singular Lagrangian of order $k \geq 2$ having the form (2.8), such that α is non-degenerated. Let us consider the local functions

$$t^i = \tilde{\alpha}^{ij}\left(\Gamma_U^{(n-1)}(\alpha_j) + \frac{\partial h_0}{\partial y^{(k-1)j}}\right). \tag{3.2}$$

The following result can be proved by a straightforward verification using local coordinates.

Proposition 3.1. *There is a global affine section $t : T^{k-1}M \to T^kM$,*

$$t = \Gamma_U^{(k-1)} + t^i \frac{\partial}{\partial y^{(k)i}}.$$

We can consider the affine Hamiltonian h of order k given by

$$\begin{aligned} kH_0(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j) &= p_i t^i - L(x^i, \ldots, y^{(k-1)i}, t^i) = \\ p_i t^i - k t^i \alpha_i + k h_0 &= t^i p_i + k(h_0 - t^i \alpha_i). \end{aligned}$$

A d-form on $T^{k-1}M$ can be viewed as a section $\omega : T^kM \to \pi_k^* T^*M$ of the vector bundle $\pi_k^*T^*M \to T^kM$, where $\pi_k : T^kM \to M$ is the canonical projection of a fibered manifold. A section $\tilde{\omega} : T^kM \to \tilde{\pi}_k^*T^*TM$ of the vector bundle $\tilde{\pi}_k^*T^*TM \to T^kM$ is called a *second d-form* on $T^{k-1}M$, where $\tilde{\pi}_k : T^kM \to TM$ is the canonical projection of a fibered manifold. Notice that $\pi_k^*T^*M = T^kM \times_M T^*M$ and $\tilde{\pi}_k^*T^*TM = T^kM \times_{TM} T^*TM$, as fibered products. There is a canonical epimorphism (i.e. a surjection on fibers) of vector bundles $f_1 : T^*TM \to T^*M$ (of cotangent vector bundles $T^*TM \to TM$ and $T^*M \to M$, over the canonical base map TM). (It can be also obtained as a composition $T^*TM \to TT^*M \to T^*M$, where $T^*TM \to TT^*M$ is the canonical flip and $TT^*M \to T^*M$ is the canonical projection.) Using local coordinates, f_1 is given by $(x^i, y^j, p_i, q_j) \xrightarrow{f_1} (x^i, p_i)$. Then there is an induced vector bundle epimorphism $\tilde{f}_1 : \tilde{\pi}_k^*T^*TM \to \pi_k^*T^*M$. A second d-form $\tilde{\omega} : T^kM \to \tilde{\pi}_k^* T^*TM$ induces a d-form $\omega = \tilde{\omega} \circ \tilde{f}_1$; we say that ω is the d-form *associated* with $\tilde{\omega}$. We say that a second d-form is *non-degenerated* if its associated d-form is non-degenerated.

As an example, a form $\omega : T^kM \to T^*T^kM$ on T^kM defines canonically a second d-form $\tilde{\omega} : T^kM \to \tilde{\pi}_k^* T^*TM$ by the formula $\tilde{\omega} = f_2^* \circ \omega$, where $f_2^* : T^*T^kM \to \tilde{\pi}_k^* T^*TM$ is induced by the map $f_2 : T^*T^kM \to T^*TM$. Using local coordinates, we have:

$(x^i, y^{(1)i}, \ldots, y^{(k)i}, p_{(0)i}, \ldots, p_{(k)i}) \stackrel{f_2}{\to} (x^i, y^{(1)i}, \ldots, y^{(k)i}, p_{(k-1)i}, p_{(k)i})$,
$(x^i, y^{(1)i}, \ldots, y^{(k)i}) \stackrel{\omega}{\to} (x^i, y^{(1)i}, \ldots, y^{(k)i}, \omega_{(0)i}, \ldots, \omega_{(k)i})$,
$(x^i, y^{(1)i}, \ldots, y^{(k)i}) \stackrel{\tilde{\omega}}{\to} (x^i, y^{(1)i}, \ldots, y^{(k)i}, \omega_{(k-1)i}, \omega_{(k)i})$.

If $\tilde{\omega}$ is a non-degenerate second d-form that has the local expression
$(x^i, y^{(1)i}, \ldots, y^{(k)i}) \stackrel{\tilde{\omega}}{\to} (x^i, y^{(1)i}, \ldots, y^{(k)i}, \beta_i(x^i, \ldots, y^{(k)i}), \alpha_i(x^i, \ldots, y^{(k)i}))$,
we can construct a semi-spray $S : T^kM \to T^{k+1}M$ using the formula

$$(k+1)S^i = \alpha^{ij} \left(\Gamma^{(k)} (\alpha_j) - \beta_j \right). \tag{3.3}$$

The fact that S is a semi-spray can be proved by a straightforward calculation, using that the change rule of local functions $\{\alpha_i, \beta_j\}$ is

$$\alpha_i = \frac{\partial x^{i'}}{\partial x^i} \alpha_{i'}, \beta_i = \frac{\partial y^{(1)i'}}{\partial x^i} \alpha_{i'} + \frac{\partial x^{i'}}{\partial x^i} \beta_{i'}, \Gamma^{(k)\prime} = \Gamma^{(k)} - \Gamma^{(k)}(y^{(k)i'}) \frac{\partial}{\partial y^{(k)i'}}.$$

We have seen that a second d-form on $T^{k-1}M$ defines a d-form on $T^{k-1}M$. It can be easily proved that any d-form on $T^{k-1}M$ is the top d-form of a form on $T^{k-1}M$, thus it is associated with the corresponding second d-form; these associations are not unique. But there are situations when if a d-form is given, one can construct in a canonical way a second d-form associated with. For example, if the d-form s is exact, i.e. there is a global function $L \in \mathcal{F}(T^kM)$ such that, using coordinates, $\omega_i = \frac{\partial L}{\partial y^{(k)i}}$, then ω is the top form of the differential dL and ω is associated with the second d-form $\tilde{\omega}$ given locally by $(\frac{\partial L}{\partial y^{(k-1)i}}, \frac{\partial L}{\partial y^{(k)i}})$. Below we consider a less trivial situation.

Let ω be a bilinear d-form on $T^{k-1}M$ and $t : T^{k-1}M \to T^kM$ be an affine section (or a semi-spray of order $k-1$). We consider the d-vector field of order k, $z : T^kM \to \pi_k^* TM$, given by

$$z^i(x^i, y^{(1)i}, \ldots, y^{(k)i}) = y^{(k)i} - t^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}).$$

Then $\bar{\omega} = i_z \omega$ is a d-form on T^kM, having the local form $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \stackrel{\bar{\omega}}{\to} (x^i, y^{(1)i}, \ldots, y^{(k)i}, z^j \omega_{ji})$.

Proposition 3.2. *If $k > 1$, let us suppose that ω is a skew-symmetric bilinear d-form on $T^{k-1}M$ and $t : T^{k-1}M \to T^kM$ is an affine section. Then there is a canonical non-degenerate second d-form $\tilde{\omega}$ on $T^{k-1}M$, associated with ω and t.*

Proof. We use local coordinates. We denote $\bar{\omega}_i = z^j \omega_{ji}, \bar{\theta}_i = -\frac{\partial \bar{\omega}_j}{\partial y^{(k-1)i}} z^j$, where $z^j = y^{(k)j} - t^j$ We have to prove that $(\bar{\omega}_i, \bar{\theta}_i)$ defines a second d-form $\tilde{\omega}$, as claimed in the Proposition. The condition that $(\bar{\omega}_i, \bar{\theta}_i)$ comes from a second d-form is that

$$\begin{cases} \bar{\omega}_i = \frac{\partial x^{i'}}{\partial x^i} \bar{\omega}_{i'}, \\ \bar{\theta}_i = \frac{\partial x^{i'}}{\partial x^i} \bar{\theta}_{i'} + \frac{\partial y^{(1)i'}}{\partial x^i} \bar{\omega}_{i'}, \end{cases} \tag{3.4}$$

if the coordinates change. The first relation is obviously fulfilled. Since $\frac{\partial}{\partial y^{(k-1)i}} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial y^{(k-1)i'}} + \frac{\partial y^{(1)i'}}{\partial x^i}\frac{\partial}{\partial y^{(k)i'}}$, then, for $k>1$, we have $\bar{\theta}_i = -\frac{\partial \bar{\omega}_j}{\partial y^{(k-1)i}} z^j = -\frac{\partial x^{i'}}{\partial x^i}\frac{\partial \bar{\omega}_{j'}}{\partial y^{(k-1)i'}}\frac{\partial x^{j'}}{\partial x^j} z^j - \frac{\partial y^{(1)i'}}{\partial x^i}\frac{\partial \bar{\omega}_{j'}}{\partial y^{(k)i'}}\frac{\partial x^{j'}}{\partial x^j} z^j = -\frac{\partial x^{i'}}{\partial x^i}\frac{\partial \bar{\omega}_{j'}}{\partial y^{(k-1)i'}} z^{j'} - \frac{\partial y^{(1)i'}}{\partial x^i}\omega_{i'j'} z^{j'} = \frac{\partial x^{i'}}{\partial x^i}\bar{\theta}_{i'} + \frac{\partial y^{(1)i'}}{\partial x^i}\bar{\omega}_{i'}$, thus the second relation also holds. □

Notice that if ω is a symmetric bilinear d-form on $T^{k-1}M$, then denoting by $\bar{\omega}_i = z^j\omega_{ji}$, $\bar{\theta}_i = \frac{\partial \bar{\omega}_j}{\partial y^{(k-1)i}} z^j$ we obtain in a similar way a canonical non-degenerate second d-form $\tilde{\omega}$, associated with ω and an affine section t.

Let L be a totally singular Lagrangian of order $k \geq 2$ having the form (2.8). Considering the section $t : T^{k-1}M \to T^kM$ defined by the formula (3.2) and the skew symmetric and non-degenerate bilinear form $\tilde{\alpha}$ on T^kM defined by formula (3.1), we can construct a non-degenerate second d-form of order k and a section $S : T^kM \to T^{k+1}M$, as above. Taking $\bar{\alpha}_i = \alpha_{ij}\cdot(y^{(k)j} - t^j(x^j, y^{(1)j}, \ldots, y^{(k-1)j}))$, then we define a new Lagrangian $\bar{L}$ of order $k+1$, using the formula

$$\bar{L}(x^j, y^{(1)j}, \ldots, y^{(k+1)j}) = (k+1)(y^{(k+1)i} - S^i)\bar{\alpha}_i(x^j, y^{(1)j}, \ldots, y^{(k)j}). \tag{3.5}$$

Then $\bar{\alpha} = (\bar{\alpha}_i)$ is an s-non-degenerate d-form on T^kM, since $\frac{\partial \bar{\alpha}_i}{\partial y^{(k)j}} - \frac{\partial \bar{\alpha}_j}{\partial y^{(k)i}} = 2\alpha_{ij}$ is a non-degenerated bilinear form. Notice that the $\bar{\alpha}$-Hamiltonian of $\bar{L}$ is defined by the local functions $\bar{h}_0 = S^i\bar{\alpha}_i$. We call $\bar{L}$ as the *lift* of L; it is easy to see that $\bar{L}$ is also s-non-degenerated (i.e. $\bar{\alpha}$ is s-non-degenerated).

In the case $k=1$, let $L : TM \to I\!R$, $L(x^i, y^{(1)i}) = \alpha_i(x^j)y^{(1)i} + \beta(x^j)$ be a totally singular Lagrangian, where $\alpha \in \mathcal{X}^*(M)$, $\beta \in \mathcal{F}(M)$. Then the formula

$$\bar{L}(x^i, y^{(1)i}, y^{(2)i}) = 2(y^{(2)i} - S^i(x^j, y^{(1)j}))\bar{\omega}_i + \beta(x^j),$$

where $\bar{\omega}_i = y^{(1)j}\alpha_{ij}$, $\alpha_{ij} = (d\alpha)_{ij} = \frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i}$, defines a Lagrangian of second order on M that has a null Hessian. If L is non-degenerated, then $\bar{L}$ is s-non-degenerated, since $\frac{\partial \bar{\omega}_i}{\partial y^{(1)j}} - \frac{\partial \bar{\omega}_j}{\partial y^{(1)i}} = 2\alpha_{ij}$.

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