Polyharmonic submanifolds in Euclidean spaces

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Abstract. B.Y. Chen introduced biharmonic submanifolds in Euclidean spaces and raised the conjecture "Any biharmonic submanifold is minimal". In this article, we show some affirmative partial answers of generalized Chen's conjecture. Especially, we show that the triharmonic hypersurfaces with constant mean curvature are minimal.

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1 Introduction

Let $\mathbf{x} : (M^n, g) \to \mathbb{E}^N$ be an isometric immersion from an n-dimensional Riemannian manifold into an N-dimensional Euclidean space, where g denotes the induced Riemannian metric on M. Then it is well known that

$$\Delta \mathbf{x} = n\mathbf{H}, \tag{1.1}$$

where Δ is the non-positive Laplace operator, and $\mathbf{H}$ the mean curvature vector of M, respectively. From equation (1.1), M is minimal if and only if $\mathbf{x} : (M, g) \to \mathbb{E}^N$ is a harmonic map. B.Y. Chen introduced *biharmonic submanifolds*:

Definition 1.1. A submanifold $\mathbf{x} : M \to \mathbb{E}^N$ is said to be a biharmonic submanifold if

$$\triangle \mathbf{H} = \frac{1}{n} \triangle^2 \mathbf{x} = 0.$$

We also note that M is biharmonic if and only if $\mathbf{x}$ is a biharmonic map. Furthermore, B.Y. Chen raised the interesting conjecture [2]:

Conjecture 1 Any biharmonic submanifold in $\mathbb{E}^N$ is minimal.

There are many affirmative partial answers to Conjecture 1. In particular, there are some complete affirmative answers if M is one of the following:
(a) a curve [7],
(b) a surface in $\mathbb{E}^3$ [2],

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(c) a hypersurface in $\mathbb{E}^4$ [5, 10].
On the other hand, I. Dimitric showed that any biharmonic submanifold with constant mean curvature is minimal [7].

We introduce polyharmonic submanifolds for generalized notion of biharmonic submanifold.

Definition 1.2. A submanifold $\mathbf{x} : M \to \mathbb{E}^N$ is said to be a polyharmonic submanifold of order k if

$$\triangle^{k-1}\mathbf{H} = \frac{1}{n} \triangle^k \mathbf{x} = 0,$$

where k is positive integer.

For polyharmonic submanifold of order k, we consider the following problem:

Conjecture 2 Any polyharmonic submanifold of order k is minimal.

Remark 1.3. Polyharmonic submanifolds of order s are automatically polyharmonic submanifolds of order t $(s < t)$. Especially, biharmonic submanifolds are polyharmonic submanifolds of order k $(k > 2)$ (polyharmonic submanifolds of order 2 are called biharmonic submanifolds). Thus, Conjecture 2 is generalized conjecture of Conjecture 1.

The author gave a complete affirmative answer for the following case:

Theorem 1.1 ([13]). *Any polyharmonic curve parametrized by arc length is straight line.*

In this article we give some affirmative partial answers to Conjecture 2.

The biharmonicity equation is a special case of the following condition:

$$\triangle\mathbf{H} = \lambda\mathbf{H}, \quad \lambda \in \mathbb{R}.$$

The study of Euclidean submanifolds with $\triangle\mathbf{H} = \lambda\mathbf{H}$ was initiated by Chen in 1988 [3]. It is known that submanifolds in $\mathbb{E}^N$ with $\triangle\mathbf{H} = \lambda\mathbf{H}$ are either biharmonic $(\lambda = 0)$, of 1-type or null 2-type. In particular, all surfaces in $\mathbb{E}^3$ with $\triangle\mathbf{H} = \lambda\mathbf{H}$ are of constant mean curvature. Moreover a surface in $\mathbb{E}^3$ satisfies $\triangle\mathbf{H} = \lambda\mathbf{H}$ if and only if it is minimal, an open portion of a totally umbilical sphere or an open portion of a circular cylinder. All hypersurfaces of $\mathbb{E}^4$ with $\triangle\mathbf{H} = \lambda\mathbf{H}$ are of constant mean curvature [6].

In Section 2, we introduce notation and fundamental formulas. In Section 3, we give the necessary and sufficient condition for polyharmonic submanifolds of order k and show that any triharmonic hypersurface with constant mean curvature is minimal. Moreover, we show that "biharmonic submanifolds" and "polyharmonic submanifolds with $\triangle\mathbf{H} = \lambda\mathbf{H}$" are equivalent. In Section 4, we consider polyharmonic psudoumbilical submanifolds and give several results. One of them recover Dimitric's result.

2 Preliminaries

First of all, we recall useful formulas. Let ∇ and D be the Levi-Civita connections of (M, g) and $\mathbb{E}^N = (\mathbb{R}^N, \langle\ ,\ \rangle)$, respectively. For any vector fields $X, Y \in \mathfrak{X}(M)$, the Gauss formula

$$D_X Y = \nabla_X Y + h(X, Y),$$

where h stands for the second fundamental form of M in $\mathbb{E}^N$. For any normal vector field ξ, the Weingarten map A_ξ with respect to ξ is given by

$$D_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where $\nabla^\perp$ stands for the normal connection of the normal bundle of M in $\mathbb{E}^N$. It is well known that h and A are related by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

For any $x \in M$, let $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_N\}$ be an orthonormal basis of $\mathbb{E}^N$ at x such that $\{e_1, \cdots, e_n\}$ is an orthonormal basis of T_xM. Then, h is decomposed at x

$$h(X, Y) = \Sigma_{\alpha=n+1}^N h_\alpha(X, Y) e_\alpha.$$

The mean curvature vector $\mathbf{H}$ of M at x is also given by

$$\mathbf{H}(x) = \Sigma_{\alpha=n+1}^N H_\alpha(x) e_\alpha,$$

where $H_\alpha(x) := \frac{1}{n}\Sigma_{i=1}^n h_\alpha(e_i, e_i)$. It is well known that the necessary and sufficient condition for M in $\mathbb{E}^N$ to be biharmonic, namely, $\boldsymbol{\Delta}\mathbf{H} = 0$, is the following ([2, 4]):

$$\text{(2.1)}\qquad \begin{cases} \Delta^\perp \mathbf{H} - \Sigma_{i=1}^n h(A_{\mathbf{H}} e_i, e_i) = 0, \\ n\ \nabla |\mathbf{H}|^2 + 4 \text{ trace } A_{\nabla^\perp \mathbf{H}} = 0, \end{cases}$$

where $\Delta^\perp$ is the non-positive Laplace operator associated with the normal connection $\nabla^\perp$. Similarly, we obtain that the necessary and sufficient condition for M in $\mathbb{E}^N$ to satisfy $\triangle\mathbf{H} = \lambda\mathbf{H}$ is the following:

$$\text{(2.2)}\qquad \begin{cases} \Delta^\perp \mathbf{H} - \Sigma_{i=1}^n h(A_{\mathbf{H}} e_i, e_i) = \lambda\mathbf{H}, \\ n\ \nabla |\mathbf{H}|^2 + 4 \text{ trace } A_{\nabla^\perp \mathbf{H}} = 0. \end{cases}$$

3 Polyharmonic hypersurfaces in Euclidean spaces

In this section, we give affirmative partial answers generalized Chen's conjecture.

First, we give the necessary and sufficient condition for M in $\mathbb{E}^N$ to be polyharmonic submanifolds.

Lemma 3.1. *The necessary and sufficient condition for M in $\mathbb{E}^N$ to be polyharmonic submanifolds of order k, namely, $\triangle^{k-1}\mathbf{H} = T_k + N_k = 0$ (T_k is tangental part and*

N_k is normal part), are the following:

$$\begin{aligned}T_k =& \triangle^T T_{k-1} - \sum_{j=1}^{n} A_{h(T_{k-1},e_j)}e_j - \sum_{j=1}^{n}(\nabla_{e_j} A_{N_{k-1}})(e_j) - \sum_{j=1}^{n} A_{\nabla^{\perp}_{e_j} N_{k-1}}e_j\\ =&0,\\ N_k =& \sum_{j=1}^{n} h(\nabla_{e_j} T_{k-1}, e_j) + \sum_{j=1}^{n} \nabla^{\perp}_{e_j} h(T_{k-1}, e_j) - \sum_{j=1}^{n} h(T_{k-1}, \nabla_{e_j} e_j)\\ &+ \triangle^{\perp} N_{k-1} - \sum_{j=1}^{n} h(A_{N_{k-1}} e_j, e_j) = 0,\end{aligned}$$

where, $T_0 = 0, N_0 = \mathbf{H}, \triangle^T$ is the non-positive Laplace operator associated with the connection ∇ and $\triangle^{\perp}$ is the non-positive Laplace operator associated with the normal connection $\nabla^{\perp}$, respectively.

Proof.

$$\begin{aligned}\triangle T_{k-1} =& \triangle^T T_{k-1} - \sum_{j=1}^{n} A_{h(T_{k-1},e_j)}e_j + \sum_{j=1}^{n} h(\nabla_{e_j} T_{k-1}, e_j)\\ &+ \sum_{j=1}^{n} \nabla^{\perp}_{e_j} h(T_{k-1}, e_j) - \sum_{j=1}^{n} h(T_{k-1}, \nabla_{e_j} e_j),\\ \triangle N_{k-1} =& -\sum_{j=1}^{n}(\nabla_{e_j} A_{N_{k-1}})(e_j) - \sum_{j=1}^{n} A_{\nabla^{\perp}_{e_j} N_{k-1}}e_j\\ &+ \triangle^{\perp} N_{k-1} - \sum_{j=1}^{n} h(A_{N_{k-1}} e_j, e_j).\end{aligned}$$

□

Especially, the necessary and sufficient condition for M in $\mathbb{E}^N$ to be polyharmonic submanifolds of order 3 (triharmonic submanifolds):

Lemma 3.2. *The necessary and sufficient condition for M in $\mathbb{E}^N$ to be triharmonic, namely, $\triangle^2\mathbf{H} = T_3 + N_3 = 0$, is the following:*

(3.1)

$$\begin{cases} T_3 = & -\sum_{j=1}^{n}(\nabla_{e_j} A_{\triangle^{\perp}\mathbf{H}})(e_j) - \sum_{j=1}^{n} A_{\nabla^{\perp}_{e_j}\triangle^{\perp}\mathbf{H}}e_j\\ & -\sum_{j=1}^{n} \triangle^T\{(\nabla_{e_j} A_{\mathbf{H}})(e_j)\} + \sum_{i,j=1}^{n} A_{h(e_j,\{(\nabla_{e_i} A_{\mathbf{H}})(e_i)\})}e_j\\ & -\sum_{j=1}^{n} \triangle^T A_{\nabla^{\perp}_{e_j}\mathbf{H}}e_j + \sum_{i,j=1}^{n} A_{h(e_j, A_{\nabla^{\perp}_{e_i}\mathbf{H}}e_i)}e_j\\ & +\sum_{i,j=1}^{n}(\nabla_{e_j} A_{h(A_{\mathbf{H}}e_i,e_i)})(e_j) + \sum_{i,j=1}^{n} A_{\nabla^{\perp}_{e_j} h(A_{\mathbf{H}}e_i,e_i)}e_j = 0.\\ N_3 = & \triangle^{\perp}\triangle^{\perp}\mathbf{H} - \sum_{j=1}^{n} h(A_{\triangle^{\perp}\mathbf{H}}e_j, e_j) - \sum_{i,j=1}^{n} h(e_j, \nabla_{e_j}\{(\nabla_{e_i} A_{\mathbf{H}})(e_i)\})\\ & -\sum_{i,j=1}^{n} \nabla^{\perp}_{e_j} h(e_j, \{(\nabla_{e_i} A_{\mathbf{H}})(e_i)\}) + \sum_{i,j=1}^{n} h(\nabla_{e_j} e_j, \{(\nabla_{e_i} A_{\mathbf{H}})(e_i)\})\\ & -\sum_{i,j=1}^{n} h(e_j, \nabla_{e_j}\{A_{\nabla^{\perp}_{e_i}\mathbf{H}}e_i\}) - \sum_{i,j=1}^{n} \nabla^{\perp}_{e_j} h(e_j, A_{\nabla^{\perp}_{e_i}\mathbf{H}}e_i)\\ & +\sum_{i,j=1}^{n} h(\nabla_{e_j} e_j, A_{\nabla^{\perp}_{e_i}\mathbf{H}}e_i) - \sum_{j=1}^{n} \triangle^{\perp} h(A_{\mathbf{H}}e_j, e_j)\\ & +\sum_{i,j=1}^{n} h(A_{h(A_{\mathbf{H}}e_i,e_i)}e_j, e_j) = 0.\end{cases}$$

Proof. Using Lemma 3.1, we obtain (3.1). □

If M is a hypersurface of $\mathbb{E}^{n+1}$ with constant mean curvature, then equations (3.1) can be denoted simple forms.

Lemma 3.3. *The necessary and sufficient condition for M^n in $\mathbb{E}^{n+1}$ with constant mean curvature to be triharmonic is the following:*

$$\begin{cases} H_{n+1}A_{e_{n+1}}(\nabla|A_{e_{n+1}}|^2) = 0, \\ H_{n+1}(-\triangle|A_{e_{n+1}}|^2 + |A_{e_{n+1}}|^4) = 0. \end{cases} \tag{3.2}$$

Proof. First, we calculate the tangential part T_3. Computation shows that

$$\sum_{i=1}^{m} h(A_{\mathbf{H}}e_i, e_i) = H_{n+1}|A_{e_{n+1}}|^2 e_{n+1}.$$

Thus, we have

$$T_3 = H_{n+1}\sum_{j=1}^{m}\{2e_j(|A_{e_{n+1}}|^2)A_{e_{n+1}}e_j + |A_{e_{n+1}}|^2(\nabla_{e_j}A_{e_{n+1}})(e_j)\}.$$

Because of

$$\sum_{j=1}^{m}(\nabla_{e_j}A_{e_{n+1}})(e_j) = n\nabla H_{n+1} = 0,$$

we obtain that

$$T_3 = 2H_{n+1}A_{e_{n+1}}(\nabla|A_{e_{n+1}}|^2).$$

Next, we calculate the normal part N_3. Computation shows that

$$N_3 = H_{n+1}\left\{-\triangle^{\perp}(|A_{e_{n+1}}|^2 e_{n+1}) + |A_{e_{n+1}}|^4 e_{n+1}\right\}.$$

Because of

$$\triangle^{\perp}(|A_{e_{n+1}}|^2 e_{n+1}) = \triangle(|A_{e_{n+1}}|^2)e_{n+1},$$

we obtain that

$$N_3 = H_{n+1}(-\triangle|A_{e_{n+1}}|^2 + |A_{e_{n+1}}|^4)e_{n+1}.$$ □

Using this lemma, we give an affirmative partial answer of generalized Chen's conjecture.

Theorem 3.4. *Any triharmonic hypersurface of $\mathbb{E}^{n+1}$ with constant mean curvature is minimal.*

Proof. From the first equation of (3.2), if $H_{n+1} = 0$, then M is minimal. Thus, we have $A_{e_{n+1}}(\nabla|A_{e_{n+1}}|^2) = 0$. So, we obtain $\nabla|A_{e_{n+1}}|^2 = 0$. Therefore, $|A_{e_{n+1}}|^2$ is constant. From the second equation of (3.2), $|A_{e_{n+1}}|^2 = 0$. Since we can denote $A_{e_{n+1}}e_i = \mu_i e_i$, we have $\mathbf{H} = \frac{1}{n}\sum_{i=1}^{n}\mu_i$, and

$$|A_{e_{n+1}}|^2 = \sum_{i=1}^{n}\langle A_{e_{n+1}}e_i, A_{e_{n+1}}e_i\rangle = \sum_{i=1}^{n}\mu_i^2 = 0.$$

Thus, $\mu_i = 0$ for all $i = 1, \cdots, n$. Therefore, M is minimal. □

From Theorem 1.1 and Theorem 3.4, we obtain the following:

Corollary 3.5. *Any triharmonic submanifold of $\mathbb{E}^3$ with constant mean curvature is minimal.*

As for polyharmonic submanifolds, it is easily seen that the polyharmonic submanifolds with $\triangle\mathbf{H} = \lambda\mathbf{H}$ and the biharmonic submanifolds are equivalent

Proposition 3.6. *The following properties are equivalent:*
1) *polyharmonic submanifolds with $\triangle\mathbf{H} = \lambda\mathbf{H}$.*
2) *biharmonic submanifolds.*

Proof. It is obvious. □

Remark 3.1. For polyharmonic submanifolds with $\triangle\mathbf{H} = \lambda\mathbf{H}$, a surface in $\mathbb{E}^3$ [2], a hypersurface in $\mathbb{E}^4$ [5, 10] are minimal.

4 Polyharmonic psudoumbilical submanifolds

In this section, we consider psudoumbilical submanifolds. First, we recall Dimitric's result:

Theorem 4.1 ([7]). *Let $\mathbf{x} : M^n \to \mathbb{E}^N$ be a psudoumbilical submanifold, that is, $A_{\mathbf{H}}$ is proportional to the identity. If $\triangle\mathbf{H} = 0$ and $n \neq 4$, then M is minimal.*

We shall consider psudoumbilical submanifolds with $\triangle\mathbf{H} = \lambda\mathbf{H}$.

Proposition 4.2. *Let M be a psudoumbilical hypersurface of $\mathbb{E}^{n+1}$. Then M satisfies $\triangle\mathbf{H} = \lambda\mathbf{H}$ if and only if it is minimal or a submanifold with constant mean curvature satisfying $\lambda = -n|\mathbf{H}|^2$.*

Proof. Psudoumbilicity implies $A_{e_{n+1}} = H_{n+1}I$. From the second equation of (2.2), we have

$$\nabla|\mathbf{H}|^2 = 0.$$

So $|\mathbf{H}|^2$ is constant. From the first equation of (2.2), we have

$$H_{n+1}(n|\mathbf{H}|^2 + \lambda) = 0.$$

Therefore, we have that M is minimal or a submanifold with constant mean curvature satisfying $\lambda = -n|\mathbf{H}|^2$. □

Proposition 4.2 implies the following:

Corollary 4.3. *Any biharmonic psudoumbilical hypersurface of* $\mathbb{E}^{n+1}$ *is minimal.*

This result includes the case $n = 4$. Thus, we recovered Dimitric's theorem 4.1. Finally, we consider polyharmonic psudoumbilical hypersurfaces with constant mean curvature.

Proposition 4.4. *Any polyharmonic psudoumbilical hypersurface with constant mean curvature is minimal, namely, if* $\triangle^k \mathbf{H} = 0$*, then* $\mathbf{H} = 0$*, for all* k $(= 1, 2, \cdots)$.

Proof. Direct computation shows that $|A_{e_{n+1}}|^2 = nH_{n+1}^2$. Using this and Lemma 3.1, we have

$$\triangle \mathbf{H} = -H_{n+1}|A_{e_{N+1}}|^2 e_{n+1} = -nH_{n+1}^3 e_{n+1}.$$

Furthermore, using Lemma 3.1, we obtain

$$\triangle^k \mathbf{H} = (-1)^k n^k H_{n+1}^{2k+1} e_{n+1}.$$

Therefore, M is minimal. □

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