Constant curvature conditions for Kropina spaces

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Abstract. The characterization of Finsler spaces of constant curvature is an old and cumbersome one. In the present paper we obtain the conditions for a Kropina space to be of constant curvature improving in this way the characterization given by Matsumoto ([6]) as well as our past results ([13]).

M.S.C. 2010: 58B20, 53B21.
Key words: Riemannian spaces; Killing vector fields; Finsler metrics; Kropina metrics; curvature.

1 Introduction

Randers spaces $(M, F = \alpha + \beta)$ of constant flag curvature have been studied by [2], [7] and [12]. Remarkably, these spaces can be characterized by means of Zermelo navigation on Riemannian manifolds using a new Riemannian metric h and a vector field W satisfying $h(W, W) < 1$ ([3]). In the present paper, we investigate a similar characterization for Kropina spaces.

C. Shibata started the study of Kropina spaces as Finsler spaces ([11]) being followed by Makoto Matsumoto who obtained the necessary and sufficient conditions for a Kropina space to be of constant curvature and gave a characterization theorem of these spaces in terms of five conditions ([8], [9]).

In [13], we have characterized Kropina spaces by means of some Riemannian metric h and a unit vector field W on the same manifold M, and have represented Matsumoto's conditions using h and W. However, a few years after we noticed that our results in [13] can be improved, therefore we reformulate the problem in a different way.

We point out that by Legendre duality a Kropina spaces $(M, F = \alpha^2/\beta)$ on TM corresponds to a Randers space $(M, \overline{F} = \overline{\alpha} + \overline{\beta})$ on T^*M only in the case $b^2 = 1$, where b^2 is the Riemannian length of β. Moreover, for regular Lagrangians, a Finsler space is of constant flag curvature K if and only if its dual space is also of constant flag curvature $\overline{K}$ ([4], [5]).

However, the results about Randers metrics of constant flag curvature in [2] are about strongly convex Randers metrics while the Randers metric corresponding to a Kropina one through the Legendre duality is not strongly convex. Moreover, in the

Balkan Journal of Geometry and Its Applications, Vol.17, No.1, 2012, pp. 115-124.

case of Kropina metrics, the Lagrangian $L = F^2$ is not a regular one and therefore the Legendre transformation is not a local diffeomorphism in all TM, so the results in ([4], [5]) must be used with precaution. Using the Legendre duality between Randers and Kropina spaces, certainly some formulas from the theory of Randers spaces can be transformed and used in the study of the present topic. We prefer however to take another way.

In the present paper, we express the conditions for a Kropina space to be of constant curvature using a Riemannian metric h and a vector field W and obtain the minimal necessary and sufficient conditions for a Kropina space to be of constant curvature.

In section 2, we shall describe a Kropina space in terms of some Riemannian metric h and a unit vector field W, and in section 3, we shall express the coefficients of the geodesic spray in a Kropina space using the Riemannian metric h and the unit vector field W.

Indeed, the necessary and sufficient condition for a Kropina space to be of constant curvature is not new ([8]). We express this condition by h and W, and obtain the necessary and sufficient conditions for a Kropina space to be of constant curvature by straightforward calculations. Our main results are Theorem 4.9 and Theorem 4.10. The former is the improved version of Theorem 2 in [13] and the latter is an important result which is used in considering the geodesics in Kropina spaces. Therefore, this paper is the improved version of [8] and [13]. Since the calculations are quite long and complicated, we give here only the outline of the proofs. The detailed computations can be found in [14].

2 The description of a Kropina metric

Let (M, α) be an $n(\geq 2)$-dimensional differential manifold endowed with a Riemannian metric α. A Kropina space $(M, \alpha^2/\beta)$ is a Finsler space whose fundamental function is given by $F = \alpha^2/\beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Even though Kropina spaces can be studied in more general case ([8], [10]), in this paper, we suppose that the matrix (a_{ij}) is positive definite.

Let us remark that for a Kropina space $(M, \alpha^2/\beta)$ the Kropina metric $F = \alpha^2/\beta$ can be rewritten as follows:

$$e^{\kappa(x)} a_{ij} \frac{y^i}{F}\frac{y^j}{F} - e^{\kappa(x)} a_{ij} \frac{y^i}{F} b^j + \frac{1}{4} e^{\kappa(x)} a_{ij} b^i b^j = \frac{1}{4} e^{\kappa(x)} b^2, \tag{2.1}$$

where $\kappa(x)$ is a function of (x^i) alone, $b^2 = a_{ij}(x)b^i b^j$, $b^i = a^{ij} b_j$ and the matrix $(a^{ij}(x))$ is the inverse one of $(a_{ij}(x))$.

Define a new Riemannian metric $h = \sqrt{h_{ij}(x)y^i y^j}$ and a vector field $W = W^i(\partial/\partial x^i)$ on M by

$$h_{ij} = e^{\kappa(x)} a_{ij} \quad \textit{and} \quad 2W_i = e^{\kappa(x)} b_i, \tag{2.2}$$

where $W_i = h_{ij} W^j$, then the equation (2.1) reduces to $|\frac{y}{F} - W| = |W|$. In the previous equation, the notation $|\cdot|$ denotes the length of a vector with respect to the Riemannian metric h.

We notice that the equation $|W| = 1$ holds if and only if the function $\kappa(x)$ satisfies the condition

$$e^{\kappa(x)} b^2 = 4. \tag{2.3}$$

Suppose that the function $\kappa(x)$ satisfies (2.3), then we have $|W| = 1$ and

$$\left| \frac{y}{F} - W \right| = 1. \tag{2.4}$$

Therefore, in each tangent space T_xM, the indicatrix of the Kropina metric necessarily goes through the origin.

Conversely, consider a Riemannian space (M, h), where $h = \sqrt{h_{ij}(x)y^iy^j}$, and a unit vector field $W = W^i(\partial/\partial x^i)$ on it. If we consider the metric F characterized by (2.4), then by solving (2.4) for F, we get $F = |y|^2/\{\sqrt{2}h(y, W)\}^2$.

Comparing the above equality with a Kropina metric $F = \alpha^2/\beta$, we obtain (2.2) and from the assumption $|W| = 1$ we get (2.3).

Summarizing the above discussion, we obtain

Theorem 2.1. *Let (M, α) be an $n(\geq 2)$-dimensional Riemannian space with the metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$. For a Kropina space $(M, F = \alpha^2/\beta)$, where $\beta = b_i(x)y^i$, we define a new Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ and a unit vector field $W = W^i(\partial/\partial x^i)$ by (2.2) and (2.3). Then, the Kropina metric F satisfies the equation (2.4).*

Conversely, suppose that $h = \sqrt{h_{ij}(x)y^iy^j}$ is a Riemannian metric and $W = W^i(\partial/\partial x^i)$ is a unit vector field on (M, h). Consider the metric F defined by (2.4). Then, defining $a_{ij}(x) := e^{-\kappa(x)}h_{ij}(x)$ and $b_i(x) := 2e^{-\kappa(x)}W_i$ by (2.2) using some function $\kappa(x)$ of (x^i) alone, we get $F = \alpha^2/\beta$ and it follows the function $\kappa(x)$ satisfies (2.3).

3 The coefficients of the geodesic spray

From the theory of Riemannian spaces, we have the following:

Theorem 3.1. *Let (M, g) and $(M, g^* = e^\rho g)$, where $g = \sqrt{g_{ij}(x)y^iy^j}$ and $g^* = \sqrt{g^*_{ij}(x)y^iy^j}$ respectively, be two n-dimensional Riemannian spaces which are conformal to each other. Furthermore, let $\gamma_j{}^i{}_k$ and $\gamma^{*}_j{}^i{}_k$ be the coefficients of Levi-Civita connection of (M, g) and (M, g^*), respectively. Then, we have*

$$g^*_{ij} = e^{2\rho}g_{ij}, \qquad g^{*ij} = e^{-2\rho}g^{ij} \qquad \text{and} \qquad \gamma^{*}_j{}^i{}_k = \gamma_j{}^i{}_k + \rho_j\delta^i{}_k + \rho_k\delta^i{}_j - \rho^i g_{jk},$$

where $\rho_i = \partial\rho/\partial x^i$ and $\rho^i = g^{ij}\rho_j$.

From (2.2), we have $h_{ij} = e^\kappa a_{ij}$. Applying Theorem 3.1, we get

$$^h\gamma_j{}^i{}_k = {}^\alpha\gamma_j{}^i{}_k + \frac{1}{2}\kappa_j\delta^i{}_k + \frac{1}{2}\kappa_k\delta^i{}_j - \frac{1}{2}\kappa^i a_{jk}, \tag{3.1}$$

where $^h\gamma_j{}^i{}_k$ and $^\alpha\gamma_j{}^i{}_k$ are the coefficients of Levi-Civita connection of (M, h) and (M, α) respectively, $\kappa_i = \partial\kappa/\partial x^i$ and $\kappa^i = a^{ij}\kappa_j$. Transvecting (3.1) by y^jy^k, we get

$$^h\gamma_0{}^i{}_0 = {}^\alpha\gamma_0{}^i{}_0 + \kappa_0 y^i - \frac{1}{2}h_{00}\overline{\kappa}^i, \tag{3.2}$$

where $\overline{\kappa}^i = h^{ij}\kappa_j$ and the index $_0$ means the transvection by y^i .

We denote the covariant derivative in the Riemannian space (M, α) by $(_{;i})$ and introduce the following notations: $s_{ij} := \frac{b_{i;j}-b_{j;i}}{2}$, $r_{ij} := \frac{b_{i;j}+b_{j;i}}{2}$, $s_j := b^i s_{ij}$, $r_j := b^i r_{ij}$.

In [1], the authors have shown that the coefficients G^i of the geodesic spray in a Finsler space $(M, F = \alpha\phi(s))$, where $s = \beta/\alpha$ and ϕ is a differential function of s alone, are given by

$$2G^i = {}^{\alpha}\gamma_0{}^i{}_0 + 2\omega\alpha s^i{}_0 + 2\Theta(r_{00} - 2\alpha\omega s_0)\left(\frac{y^i}{\alpha} + \frac{\omega'}{\omega - s\omega'}b^i\right), \tag{3.3}$$

where $\omega := \frac{\phi'}{\phi - s\phi'}$, and $\Theta := \frac{\omega - s\omega'}{2\{1+s\omega+(b^2-s^2)\omega'\}}$.

For a Kropina space, we have $\phi(s) = 1/s$, hence by straightforward computation we obtain

$$2G^i = {}^{h}\gamma_0{}^i{}_0 - \kappa_0 y^i + \frac{1}{2}h_{00}\overline{\kappa}^i - F s^i{}_0 - \frac{1}{b^2}(r_{00} + F s_0)(\frac{2}{F}y^i - b^i).$$

From Theorem 2.1, for a Kropina space $(M, \alpha^2/\beta)$, a new Riemannian metric $h = \sqrt{h_{ij}(x)y^i y^j}$ and a vector field $W = W^i(\partial/\partial x^i)$ are defined by (2.2) and (2.3). So, the vector field W satisfies the condition $|W| = 1$ and we have $F = h_{00}/2W_0$.

Therefore, we get

$$2G^i = {}^{h}\gamma_0{}^i{}_0 + 2\Phi^i, \tag{3.4}$$

where

$$2\Phi^i := -\kappa_0 y^i + \frac{1}{2}h_{00}\overline{\kappa}^i - \frac{h_{00}}{2W_0}s^i{}_0 - \frac{1}{b^2}(r_{00} + \frac{h_{00}s_0}{2W_0})(\frac{4W_0}{h_{00}}y^i - b^i). \tag{3.5}$$

Using (3.1), we have $b_{i;j} = 2e^{-\kappa}W_{i||j} + e^{-\kappa}(\kappa_i W_j - \kappa_j W_i - W_r\overline{\kappa}^r h_{ij})$, where the notation $(_{||i})$ stands for the h-covariant derivative in the Riemannian space (M, h).

Remark 3.1. We can introduce a Finsler connection $\Gamma^* = ({}^h\gamma_j{}^i{}_k(x), N_j{}^i := {}^h\gamma_j{}^i{}_k(x)y^k, C_j{}^i{}_k)$ associated with the linear connection ${}^h\gamma_j{}^i{}_k(x)$ of the Riemannian space (M, h). The *h-covariant derivative* are defined as follows ([6]):

For a vector field $W^i(x)$ on M,

(1) $W^i(x)_{||j} = \frac{\partial W^i}{\partial x^j} - \frac{\partial W^i}{\partial y^s}N_j{}^s +^h \gamma_j{}^i{}_s W^s = \frac{\partial W^i}{\partial x^j} +^h \gamma_j{}^i{}_s W^s$.

For a reference vector y^i,

(2) $y^i{}_{||j} = \frac{\partial y^i}{\partial x^j} - \frac{\partial y^i}{\partial y^s}N_j{}^s +^h \gamma_j{}^i{}_s y^s = -N_j{}^i + N_j{}^i = 0$.

We put

$$\mathtt{R}_{ij} := \frac{W_{i||j} + W_{j||i}}{2}, \quad \mathtt{S}_{ij} := \frac{W_{i||j} - W_{j||i}}{2}, \qquad \mathtt{R}^i{}_j := h^{ir}\mathtt{R}_{rj}, \quad \mathtt{S}^i{}_j := h^{ir}\mathtt{S}_{rj},$$
$$\mathtt{R}_i := W^r\mathtt{R}_{ri}, \quad \mathtt{S}_i := W^r\mathtt{S}_{ri}, \qquad \mathtt{R}^i := h^{ir}\mathtt{R}_r, \quad \mathtt{S}^i := h^{ir}\mathtt{S}_r.$$

It follows $r_{ij} = 2e^{-\kappa}\left(\mathtt{R}_{ij} - \frac{1}{2}W_r\overline{\kappa}^r h_{ij}\right)$, $s_{ij} = 2e^{-\kappa}\left(\mathtt{S}_{ij} + \frac{\kappa_i W_j - \kappa_j W_i}{2}\right)$.

$$r_{ij} = 2e^{-\kappa}\left(\mathtt{R}_{ij} - \frac{1}{2}W_r\overline{\kappa}^r h_{ij}\right), \quad s_{ij} = 2e^{-\kappa}\left(\mathtt{S}_{ij} + \frac{\kappa_i W_j - \kappa_j W_i}{2}\right).$$

Furthermore, we get

$$s^i{}_j = 2\mathtt{S}^i{}_j + \overline{\kappa}^i W_j - \kappa_j W^i, \qquad s^i{}_0 = 2\mathtt{S}^i{}_0 + W_0\overline{\kappa}^i - \kappa_0 W^i,$$

$$s_i = 2e^{-\kappa}\left(2\mathtt{S}_i + W_r\overline{\kappa}^r W_i - \kappa_i\right), \qquad s_0 = 2e^{-\kappa}\left(2\mathtt{S}_0 + W_r\overline{\kappa}^r W_0 - \kappa_0\right),$$

$$r_{00} = 2e^{-\kappa}\left(\mathtt{R}_{00} - \frac{1}{2}W_r\overline{\kappa}^r h_{00}\right), \qquad b^i = a^{ir}b_r = e^{\kappa}h^{ir}\frac{2W_r}{e^{\kappa}} = 2W^i.$$

Substituting the above equalities in (3.5), we have

$$2\Phi^i = \frac{h_{00}}{W_0}(\mathtt{S}_0 W^i - \mathtt{S}^i{}_0) + (\mathtt{R}_{00}W^i - 2\mathtt{S}_0 y^i) - \frac{2W_0}{h_{00}}\mathtt{R}_{00}y^i. \tag{3.6}$$

Multiplying now the above equalities by $2h_{00}W_0$, we get

$$4h_{00}W_0\Phi^i = (h_{00})^2 A^i_{(1)} + h_{00}W_0 A^i_{(2)} + (W_0)^2 A^i_{(3)}, \tag{3.7}$$

where $A^i_{(1)} := 2(\mathtt{S}_0 W^i - \mathtt{S}^i{}_0)$, $A^i_{(2)} := 2(\mathtt{R}_{00}W^i - 2\mathtt{S}_0 y^i)$, $A^i_{(3)} := -4\mathtt{R}_{00}y^i$.

4 The necessary and sufficient conditions for constant curvature

In this section, we consider a Kropina space $(M, F = \alpha^2/\beta)$ of constant curvature K, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Furthermore, we suppose that the matrix (a_{ij}) is always positive definite and that the dimension n is greater than or equal two. Hence, it follows that α^2 is not divisible by β. This is an important relation and is equivalent to that h_{00} is not divisible by W_0. Using these, we shall obtain the necessary and sufficient conditions for a Kropina space to be of constant curvature.

4.1 The curvature tensor of a Kropina space

Let $R_j{}^i{}_{kl}$ be the h-curvature tensor of Cartan connection in Finsler space. The Berwald spray curvature tensor is

$${}^{(b)}R_j{}^i{}_{kl} = A_{(kl)}\left\{\frac{\partial G_j{}^i{}_k}{\partial x^l} + G_j{}^r{}_k G_r{}^i{}_l\right\}, \tag{4.1}$$

where the symbol $A_{(kl)}$ denotes the interchange of indices k and l and subtraction. It is well-known that the equality $R_0{}^i{}_{kl} = {}^{(b)}R_0{}^i{}_{kl}$ holds good ([6]).

From $2G^i = {}^h\gamma_0{}^i{}_0 + 2\Phi^i$, it follows $G^i{}_j = {}^h\gamma_0{}^i{}_j + \Phi^i{}_j$ and $G_j{}^i{}_k = {}^h\gamma_j{}^i{}_k + \Phi_j{}^i{}_k$, where $\Phi^i{}_j := \frac{\partial\Phi^i}{\partial y^j}$ and $\Phi^i_{jk} := \frac{\partial\Phi^i_j}{\partial y^k}$. Substituting the above equalities in (4.1), we get ${}^{(b)}R_j{}^i{}_{kl} = {}^hR_j{}^i{}_{kl} + A_{(kl)}\{\Phi_j{}^i{}_{k||l} + \Phi_j{}^r{}_k\Phi_r{}^i{}_l\}$.

The following result is well-known ([6]):

Proposition 4.1. *The necessary and sufficient condition for a Finsler space* (M, F) *to be of scalar curvature* K *is that the equality*

$$R_0{}^i{}_{0l} = KF^2(\delta^i{}_l - l^i l_l), \tag{4.2}$$

where $l^i = y^i/F$ *and* $l_l = \partial F/\partial y^l$*, holds.*

If the equality (4.2) holds and K is constant, then the Finsler space is called of *constant curvature* K.

For a Kropina space of constant curvature K, since $F = h_{00}/(2W_0)$, we have

$$\delta^i{}_l - l^i l_l = \delta^i{}_l - \frac{2W_0 h_{0l} - h_{00}W_l}{h_{00}W_0} y^i.$$

Using the curvature we obtained above, we have $R_0{}^i{}_{0l} = {}^hR_0{}^i{}_{0l} + 2\Phi^i{}_{||l} - \Phi^i{}_{l||0} + 2\Phi^r\Phi_r{}^i{}_l - \Phi^r{}_l\Phi^i{}_r$.

Substituting the above equalities in (4.2), we get

$$KF^2\left(\delta^i{}_l - \frac{2W_0h_{0l} - h_{00}W_l}{h_{00}W_0}y^i\right) = {}^hR_0{}^i{}_{0l} + 2\Phi^i{}_{||l} - \Phi^i{}_{l||0} + 2\Phi^r\Phi_r{}^i{}_l - \Phi^r{}_l\Phi^i{}_r. \tag{4.3}$$

Multiplying (4.3) by $16(h_{00})^4(W_0)^4$ and using $F^2 = (h_{00})^2/\{4(W_0)^2\}$, we have the equality

$$\begin{aligned} 4K(h_{00})^6(W_0)^2h^i{}_l &= 16(h_{00})^4(W_0)^4 \cdot {}^hR_0{}^i{}_{0l} + 8(h_{00})^3(W_0)^2 \cdot 4h_{00}(W_0)^2\Phi^i{}_{||l} \\ &- 4(h_{00})^2W_0 \cdot 4(h_{00})^2(W_0)^3\Phi^i{}_{l||0} + 32(h_{00})^4(W_0)^4\Phi^r\Phi_r{}^i{}_l - 16(h_{00})^4(W_0)^4\Phi^r{}_l\Phi^i{}_r, \end{aligned}$$

where $h^i{}_l = \delta^i{}_l - l^i l_l$. Computing the quantities $\Phi^i{}_{||l}$, $\Phi^i{}_l$, $\Phi^i{}_{l||0}$, $\Phi^r\Phi_r{}^i{}_l$, $\Phi^r{}_l\Phi^i{}_r$ (see [14] for detailed computations) in the above equality, by straightforward computation we finally obtain

$$(h_{00})^4P^i_{(5)_l} + (h_{00})^2Q^i_{(9)_l} + (W_0)^4R^i_{(9)_l} = 0, \tag{4.4}$$

where $P^i_{(5)_l}$, $Q^i_{(9)_l}$ and $R^i_{(9)_l}$ are homogeneous polynomials of degrees 5, 9, and 9 in y^i, respectively (see [14] for concrete expressions). They are called the *curvature part*, the *vanishing part* and the *Killing part*, respectively.

We conclude:

Proposition 4.2. *The necessary and sufficient condition for a Kropina space* (M, F) *with* $F = \alpha^2/\beta = h_{00}/(2W_0)$ *to be of constant curvature* K *is that* (4.4) *holds good.*

4.2 The Killing part

We consider the Killing part $R^i_{(9)_l}$ and obtain the conclusion that the vector field W is Killing. By computation we have

$$R^i_{(9)_l} = -32h_{00}\mathtt{R}_{00}\{W_0(2\mathtt{R}_{00}h_{00}\delta^i{}_l + 2h_{00}\mathtt{R}_{0l}y^i + 7\mathtt{R}_{00}h_{0l}y^i) - 8\mathtt{S}_0h_{00}h_{0l}y^i + \mathtt{R}_{00}h_{00}W_ly^i\}.$$

Substituting the above equality in (4.4) and dividing it by W_0h_{00}, we get

$$\begin{aligned} (h_{00})^3P^i_{(5)_l} + h_{00}Q^i_{(9)\_}32(W_0)^4\mathtt{R}_{00}\{W_0(2\mathtt{R}_{00}h_{00}\delta^i{}_l + 2h_{00}\mathtt{R}_{0l}y^i + 7\mathtt{R}_{00}h_{0l}y^i)l \\ - 8\mathtt{S}_0h_{00}h_{0l}y^i + \mathtt{R}_{00}h_{00}W_ly^i\} = 0. \end{aligned} \tag{4.5}$$

Lemma 4.3. *In the equation* (4.5)*, it follows that* $\mathtt{R}_{00}$ *is divisible by* h_{00}.

Proof. Suppose that $\mathtt{R}_{00}$ is not divisible by h_{00} and since (h_{ij}) is positive definite, it follows that $(\mathtt{R}_{00})^2$ is not divisible by h_{00}.

Taking into account that $P^i_{(5)_l}$ and $Q^i_{(9)_l}$ are homogeneous polynomials of y^i and that $(W_0)^2$ is not divisible by h_{00}, it follows that the equation $h_{0l}y^i = h_{00}\eta^i{}_l(x)$, where $\eta_l{}^i(x)$ is a function of (x^i) alone. Transvecting the above equation by W^l, we get $W_0 y^i = h_{00}\eta_l{}^i(x)W^l$. Since h_{00} is not divisible by W_0, the above equation is impossible. □

Therefore, it follows that $\mathtt{R}_{00}$ is divisible by h_{00} and that $\mathtt{R}_{00} = c(x)h_{00}$, where $c(x)$ is a function of (x^i) alone. Derivating the above equation by y^i and y^j, we get $W_{i||j} + W_{j||i} = 2c(x)h_{ij}$. Transvecting the previous relation by W^iW^j, we get $W_{i||j}W^iW^j = c(x)h_{ij}W^iW^j$ and using $h_{ij}W^iW^j = |W|^2 = 1$ and $W_{i||r}W^i = 0$, we obtain $c(x) = 0$. Therefore, it follows that the equality $\mathtt{R}_{ij} = 0$ holds good. Hence, we have that W is a Killing vector field. Therefore, we can state

Lemma 4.4. *If a Kropina space* $(M, \alpha^2/\beta)$ *is of constant curvature* K*, then*

1. $W(x)$ *is a Killing vector field,*
2. *the Killing part* $R^i_{(9)l} = 0$.

The equation (4.5) reduces now to $(h_{00})^2 P^i_{(5)_l} + Q^i_{(9)_l} = 0$ and we have following equalities:

$$W_{i||j} = \mathtt{S}_{ij}, \quad \mathtt{S}_j = W_{i||j}W^i = 0, \quad W_{0||j} = \mathtt{S}_{0j}, \quad W_{i||0} = \mathtt{S}_{i0}, \quad W_{0||0} = 0. \tag{4.6}$$

4.3 The vanishing part

We obtain further that the equality $Q^i_{(9)l} = 0$ holds from the relation $\mathtt{R}_{00} = 0$ obtained in the previous subsection. Indeed, one can easily see that all coefficients $(h_{00})^i$, $(i = 1, 2, 3)$ and $(W_0)^j$, $j = 3, 4, 5$ of $Q^i_{(9)l}$ vanish respectively and hence Lemma 4.4 implies

Lemma 4.5. *If a Kropina space* $(M, \alpha^2/\beta)$ *is of constant curvature* K*, then we have* $Q^i_{(9)l} = 0$ *and* $P^i_{(5)l} = 0$.

4.4 The curvature part

In this subsection, we shall see that Lemma 4.5 implies that (M, h) is a Riemannian space of constant curvature K. Indeed, by Lemma 4.5 and some further computations, we have

(4.7)

$$\begin{aligned}
&-\frac{1}{4}P^i_{(5)_l} = (h_{00})^2 W_0(W^i{}_{||r}W^r{}_{||l} + K\delta^i{}_l) + (h_{00})^2(W^i{}_{||r}W^r{}_{||0}W_l + KW_l y^i)\\
&+ 2h_{00}(W_0)^2(2W^i{}_{||0||l} - W^i{}_{||l||0}) + 2h_{00}W_0(W^i{}_{||0||0}W_l - W^i{}_{||r}W^r{}_{||0}h_{0l} - Kh_{0l}y^i)\\
&- 4(W_0)^3 \cdot {}^hR_0{}^i{}_{0l} - 4(W_0)^2 W^i{}_{||0||0}h_{0l} = 0,
\end{aligned}$$

First, we consider the term $(h_{00})^2(W^i{}_{||r}W^r{}_{||0}W_l+KW_ly^i)$ which does not contain W_0. Taking into account that $(h_{00})^2$ is not divisible by W_0, we get the equality

$$W^i{}_{||r}W^r{}_{||0}W_l + KW_ly^i = W_0c_l{}^i(x), \tag{4.8}$$

where $c_l{}^i(x)$ are functions of (x^i) alone.

Some computations shall lead to the relation (for details see [14]) ${}^hR_k{}^i{}_{jl} = K(h_{jk}\delta^i{}_l - h_{kl}\delta^i{}_j)$, that is, the Riemannian space (M,h) is of constant curvature K.

Therefore, we obtain

Theorem 4.6. *Let M be an $n(\geq 2)$-dimensional Riemannian manifold. Put $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. Let $(M,\alpha^2/\beta)$ be a Kropina space and define a new Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ and a unit vector field W by (2.2) and (2.3).*

If the Kropina space $(M,\alpha^2/\beta)$ is of constant curvature K, then the vector field W is a Killing one and the Riemannian space (M,h) is of constant curvature K .

4.5 The converse of Theorem 4.6

Let $(M,\alpha^2/\beta)$ be a Kropina space and let us define a new Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ and a unit vector field W by (2.2) and (2.3). Suppose that the vector field W is a Killing one and that the Riemannian space (M,h) is of constant curvature K. To prove that the Kropina space $(M,\alpha^2/\beta)$ is of constant curvature K, we have only to show that the equality (4.4) holds. Since the vector field W is a Killing one, we have $\mathtt{R}_{00} = 0$. Taking into account 2 of Lemma 4.4 and the first equation of Lemma 4.5, the Killing part R and the vanishing part Q vanishes respectively, so we have only to show that the curvature part $P^i_{(5)l}$ defined in (4.7) vanishes and we are going to prove it in the following.

First, we give the following result of Riemannian geometry (see [14] for a proof):

Lemma 4.7. *For a unit Killing vector field $W = W^i(\partial/\partial x^i)$, the equality*

$$W_{i||j||k} = W_r\ {}^hR_k{}^r{}_{ij} \tag{4.9}$$

holds good.

From the assumption that the Riemannian space (M,h) is of constant curvature K, we have

$${}^hR_k{}^r{}_{ji} = K(h_{kj}\delta^r{}_i - h_{ki}\delta^r{}_j). \tag{4.10}$$

Using the above equality we get

$$W^i{}_{||j||k} = K(\delta^i{}_kW_j - h_{kj}W^i) \tag{4.11}$$

and from here and $y^i{}_{||j} = 0$ (See Remark 3.1), it follows

$$\begin{aligned} &W^i{}_{||0||l} = K(\delta^i{}_lW_0 - h_{l0}W^i), \quad W^i{}_{||0||0} = K(y^iW_0 - h_{00}W^i), \\ &W^i{}_{||l||0} = K(y^iW_l - h_{0l}W^i). \end{aligned} \tag{4.12}$$

From (4.10), we have

$${}^hR_0{}^i{}_{0l} = K(h_{00}\delta^i{}_l - h_{0l}y^i) \tag{4.13}$$

and applying the h-covariant derivative $_{||i}$ to the equality $|W|^2 = W_r W_s h^{rs} = 1$, we get $W_{r||i} W^r = -W_{i||r} W^r = 0$. Furthermore, applying the h-covariant derivative $_{||l}$ to the above equality, we obtain $W_{i||r} W^r{}_{||l} + W_{i||r||l} W^r = 0$. From the above equality and (4.11), we have

$$W_{i||r} W^r{}_{||l} = -W_{i||r||l} W^r = K(h_{lr} W_i - h_{li} W_r) W^r = K(W_l W_i - h_{li}). \tag{4.14}$$

Substituting the equalities (4.12)-(4.14) in the first equality in (4.9), we can easily recognize the curvature part $P^i_{(5)\,l} = 0$. Therefore, Proposition 4.1 holds good. Hence, we get

Theorem 4.8. *Let $(M, \alpha^2/\beta)$ be an $n(\geq 2)$-dimensional Kropina space, where $\alpha^2 = a_{ij}(x) y^i y^j$, $\beta = b_i(x) y^i$ and the matrix (a_{ij}) is positive definite. For this Kropina space, we define a new Riemannian metric $h = \sqrt{h_{ij}(x) y^i y^j}$ and a unit vector field $W = W^i(\partial/\partial x^i)$ on (M, h) by (2.2) and (2.3).*

If the vector field $W = W^i(\partial/\partial x^i)$ is a Killing one and the Riemannian space (M, h) is of constant curvature K, the Kropina space $(M, \alpha^2/\beta)$ is of constant curvature K.

From Theorems 4.6 and 4.8, we have

Theorem 4.9. *Let $(M, \alpha^2/\beta)$ be an $n(\geq 2)$-dimensional Kropina space, where $\alpha^2 = a_{ij}(x) y^i y^j$, $\beta = b_i(x) y^i$ and the matrix (a_{ij}) is positive definite. For this Kropina space, we define a new Riemannian metric $h = \sqrt{h_{ij}(x) y^i y^j}$ and a unit vector field $W = W^i(\partial/\partial x^i)$ on (M, h) by (2.2) and (2.3).*

Then, the Kropina space $(M, \alpha^2/\beta)$ is of constant curvature K if and only if the following conditions hold:

1. *$W_{i||j} + W_{j||i} = 0$, that is, $W = W^i(\partial/\partial x^i)$ is a Killing vector field.*
2. *The Riemannian space (M, h) is of constant curvature K.*

Remark 4.1. Randers metrics of constant flag curvature are characterized by three conditions: the Basic Equation, the CC Equation and the Curvature Equation [2]. By some supplementary computations we can find the correspondence between these three conditions and our formulas, but it takes too much space to write them down here.

Let $(M, F = \alpha^2/\beta)$ be an $n(\geq 2)$-dimensional Kropina space. From Theorem 2.1, for this Kropina metric $F = \alpha^2/\beta$, we can define a Riemannian metric $h = \sqrt{h_{ij}(x) y^i y^j}$ and a unit vector field $W = W^i(\partial/\partial x^i)$ on (M, h) by (2.2) and (2.3). We suppose that the vector field W is a Killing one. Then, we have $\mathtt{R}_{00} = 0$. From this assumption, we get the second equation of (4.6), that is, $\mathtt{S}_0 = 0$. Substituting $\mathtt{R}_{00} = 0$, $\mathtt{S}_0 = 0$ and $F = h_{00}/(2W_0)$ in (3.6), we obtain the equation $\Phi^i = -F\mathtt{S}^i{}_0$. Substituting this in (3.4), we get

Theorem 4.10. *Let $(M, \alpha^2/\beta)$ be an $n(\geq 2)$-dimensional Kropina space, where $\alpha^2 = a_{ij}(x) y^i y^j$, $\beta = b_i(x) y^i$ and the matrix (a_{ij}) is positive definite. For this Kropina space, we define a new Riemannian metric $h = \sqrt{h_{ij}(x) y^i y^j}$ and a unit vector field $W = W^i(\partial/\partial x^i)$ on (M, h) by (2.2) and (2.3).*

Suppose that the vector field W is a Killing one, then the coefficients G^i of the geodesic spray of the Kropina space $(M, \alpha^2/\beta)$ is written as $2G^i = {}^h\gamma_0{}^i{}_0 - 2F\mathtt{S}^i{}_0$, where ${}^h\gamma_j{}^i{}_k$ are Christoffel symbols of the Riemannian space (M, h).

Acknowledgments. The authors would like to express their thanks to S. V. Sabau and H. Shimada for valuable advice and continuous encouragement.

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