On space-like surfaces in Minkowski 4-space with pointwise 1-type Gauss map of the second kind

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Abstract. In this work, we study space-like surfaces in the Minkowski space $\mathbb{E}^4_1$ with pointwise 1-type Gauss map. We prove that a maximal surface in $\mathbb{E}^4_1$ has pointwise 1-type Gauss map of the second kind if and only if it is an open part of a space-like plane. We also give a classification of surfaces in $\mathbb{E}^4_1$ with flat normal bundle, non-zero constant curvature and pointwise 1-type Gauss map of the second kind.

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1 Introduction

The notion of finite type submanifolds of Euclidean spaces was introduced by B.Y Chen in late 1970's, [6]. Since then many works have been done to characterize or classify submanifolds of Euclidean spaces or pseudo-Euclidean spaces in terms of finite type. Also, B. Y. Chen and P. Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds in [9]. A smooth map ϕ on a submanifold M of a Euclidean space or a pseudo-Euclidean space is said to be of *finite type* if ϕ can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of M, that is, $\phi = \phi_0 + \sum_{i=1}^{k} \phi_i$, where ϕ_0 is a constant map, $\phi_1, \ldots, \phi_k$ non-constant maps such that $\Delta\phi_i = \lambda_i\phi_i,\ \lambda_i \in \mathbb{R},\ i = 1, \ldots, k$.

If a submanifold M of a Euclidean space or a pseudo-Euclidean space has 1-type Gauss map ν, then ν satisfies $\Delta\nu = \lambda(\nu+C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. In [9], B. Y. Chen and P. Piccinni studied compact submanifolds of Euclidean spaces with finite type Gauss map. Several articles also appeared on submanifolds with finite type Gauss map (cf. [2, 3, 4, 5, 23, 24]).

However, the Laplacian of the Gauss map of several surfaces and hypersurfaces such as helicoids of the 1st, 2nd, and 3rd kind, conjugate Enneper's surface of the second kind and B-scrolls in a 3-dimensional Minkowski space $\mathbb{E}^3_1$, generalized catenoids,

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spherical n-cones, hyperbolical n-cones and Enneper's hypersurfaces in $\mathbb{E}_1^{n+1}$ take the form

$$\Delta\nu = f(\nu + C) \tag{1.1}$$

for some smooth function f on M and some constant vector C ([13, 20]). A submanifold of a pseudo-Euclidean space is said to have *pointwise 1-type Gauss map* if its Gauss map satisfies (1.1) for some smooth function f on M and some constant vector C. In particular, if C is zero, it is said to be of *the first kind.* Otherwise, it is said to be of *the second kind* (cf. [1, 7, 10, 12, 14, 19, 21]).

The complete classification of ruled surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the first kind was obtained in [20]. Recently, ruled surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind were studied in [11, 16]. Also, a complete classification of rational surfaces of revolution in $\mathbb{E}_1^3$ satisfying (1.1) was given in [19], and it was proved that a right circular cone and a hyperbolic cone in $\mathbb{E}_1^3$ are the only rational surfaces of revolution in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. The first author studied rotational hypersurfaces in Lorentz-Minkowski space with pointwise 1-type Gauss map, [13]. Moreover, in [22] a complete classification of cylindrical and non-cylindrical surfaces in $\mathbb{E}_1^m$ with pointwise 1-type Gauss map of the first kind was obtained.

In [1], the first author and G. G. Arsan gave some classification and characterization theorems on surfaces of the Euclidean 4-space satisfying (1.1). Recently, the authors extended this study to Minkowski space and obtained some results on space-like surfaces in the Minkowski space $\mathbb{E}_1^4$ with pointwise 1-type Gauss map of the first kind, [18].

In this paper, we present some results on space-like surfaces in $\mathbb{E}_1^4$ with pointwise 1-type Gauss map of the second kind. We focus on maximal surfaces and surfaces with constant mean curvature in $\mathbb{E}_1^4$. First, we show that a maximal surface in $\mathbb{E}_1^4$ has pointwise 1-type Gauss map of the second kind if and only if it is an open portion of a space-like plane. Then, we give a complete classification of maximal surfaces in $\mathbb{E}_1^4$ with 1-type Gauss map. Finally, we classify all space-like surfaces in $\mathbb{E}_1^4$ with flat normal bundle, constant mean curvature and pointwise 1-type Gauss map of the second kind.

2 Prelimineries

Let $\mathbb{E}_s^m$ denote the pseudo-Euclidean m-space with the canonical pseudo-Euclidean metric tensor of index s given by

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2,$$

where $(x_1, x_2, \dots, x_m)$ is a rectangular coordinate system in $\mathbb{E}_s^m$.

A vector $\zeta \neq 0 \in T_p(\mathbb{E}_s^m) \equiv \mathbb{E}_s^m$ is called space-like (resp., time-like or light-like) if $\langle\zeta,\zeta\rangle > 0$ (resp., $\langle\zeta,\zeta\rangle < 0$ or $\langle\zeta,\zeta\rangle = 0$), where $T_p(\mathbb{E}_s^m)$ denotes the tangent space of E_s^m at p. A submanifold M of $\mathbb{E}_s^m$ is said to be space-like if every non-zero tangent vector on M is space-like.

Let M be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space $\mathbb{E}^m_s$. We denote Levi-Civita connections of $\mathbb{E}^m_s$ and M by $\widetilde{\nabla}$ and ∇, respectively. In this section, we shall use letters $X,\ Y,\ Z,\ W$ (resp., $\xi,\ \eta$) to denote vectors fields tangent(resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X\xi, \tag{2.2}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

The Gauss and Ricci equations are given, respectively, by

$$\langle R(X,Y,)Z,W\rangle = \langle h(Y,Z),h(X,W)\rangle - \langle h(X,Z),h(Y,W)\rangle, \tag{2.3}$$

$$\langle R^D(X,Y)\xi,\eta\rangle = \langle [A_\xi,A_\eta]X,Y\rangle, \tag{2.4}$$

where R, R^D are the curvature tensors associated with connections ∇ and D respectively.

Now, we assume M is a space-like surface in $\mathbb{E}^4_1$. Let $\{e_1,e_2,e_3,e_4\}$ with $\varepsilon_A = \langle e_A,e_A\rangle = \mp 1$ be a given local, orthonormal frame field on M and $\{\omega_{AB}\}$ with $\omega_{AB}+\omega_{BA}=0$ be the connection 1-forms associated to this frame field. Then we have

$$\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{2} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{\beta=3}^{4} \varepsilon_\beta h^\beta_{ik} e_\beta$$

and

$$\widetilde{\nabla}_{e_k} e_\beta = -\sum_{j=1}^{2} \varepsilon_j h^\beta_{kj} e_j + \sum_{\nu=3}^{4} \varepsilon_\nu \omega_{\beta\nu}(e_k) e_\nu$$

for $i,k=1,2$ and $\beta=3,4$, where h^β_{ij}'s are the coefficients of the second fundamental form h. If $\{\omega_1,\omega_2\}$ denotes the dual basis corresponding to $\{e_1,e_2\}$, then the first structural equations of M become

$$dw_1 = w_{12}\wedge w_2, \quad dw_2 = w_{21}\wedge w_1. \tag{2.5}$$

The Codazzi equation of M is given by

$$\begin{aligned} & h^\beta_{ij,k} = h^\beta_{jk,i}, \quad i,j,k=1,2,\ \beta=3,4 \\ & h^\beta_{jk,i} = e_i(h^\beta_{jk}) + \sum_{\gamma=3}^{4} \varepsilon_\gamma h^\gamma_{jk}\omega_{\gamma\beta}(e_i) - \sum_{\ell=1}^{2}\left(\omega_{j\ell}(e_i)h^\beta_{\ell k} + \omega_{k\ell}(e_i)h^\beta_{j\ell}\right). \end{aligned} \tag{2.6}$$

On the other hand, a space-like surface M in $\mathbb{E}^4_1$ is said to have flat normal bundle if its normal curvature tensor R^D vanishes identically. Note that the Ricci equation (2.4) implies that if M has flat normal bundle, then the shape operators $A_{e_3} = A_3$ and $A_{e_4} = A_4$ can be simultaneously diagonalized.

For a surface M in E^4_1, the squared length $\|h\|^2$ of the second fundamental form h is defined by $\|h\|^2 = \sum\limits_{i,j,\beta} \varepsilon_i\varepsilon_j\varepsilon_\beta h^\beta_{ij} h^\beta_{ji}$. Gradient of a smooth function f on M

is defined by $\nabla f = \sum_{i=1}^{2} \varepsilon_i e_i(f) e_i$, and the Laplace operator acting on M is $\Delta = \sum_{i=1}^{2} \varepsilon_i (\nabla_{e_i} e_i - e_i e_i)$.

Let $G(m-n, m)$ be the Grassmannian manifold consisting of all oriented $(m-n)$-planes through the origin of $\mathbb{E}_t^m$ and $\bigwedge^{m-n} \mathbb{E}_t^m$ the vector space obtained by the exterior product of $m-n$ vectors in $\mathbb{E}_t^m$. Let $f_{i_1} \wedge \cdots \wedge f_{i_{m-n}}$ and $g_{i_1} \wedge \cdots \wedge g_{i_{m-n}}$ be two vectors in $\bigwedge^{m-n} \mathbb{E}_t^m$, where $\{f_1, f_2, \ldots, f_m\}$ and $\{g_1, g_2, \ldots, g_m\}$ are two orthonormal bases of $\mathbb{E}_t^m$. Define an indefinite inner product $\langle , \rangle$ on $\bigwedge^{m-n} \mathbb{E}_t^m$ by

$$\langle f_{i_1} \wedge \cdots \wedge f_{i_{m-n}}, g_{i_1} \wedge \cdots \wedge g_{i_{m-n}} \rangle = \det(\langle f_{i_\ell}, g_{j_k} \rangle). \tag{2.7}$$

Therefore, for some positive integer s, we may identify $\bigwedge^{m-n} \mathbb{E}_t^m$ with some pseudo-Euclidean space $\mathbb{E}_s^N$, where $N = \binom{m}{m-n}$. Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an oriented local orthonormal frame on an n-dimensional pseudo-Riemannian submanifold M in $\mathbb{E}_t^m$ with $\varepsilon_B = \langle e_B, e_B \rangle = \pm 1$ such that $e_1, \ldots, e_n$ are tangent to M and $e_{n+1}, \ldots, e_m$ are normal to M. The map $\nu : M \to G(m-n, m) \subset \mathbb{E}_s^N$ from an oriented pseudo-Riemannian submanifold M into $G(m-n, m)$ defined by

$$\nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_m)(p) \tag{2.8}$$

is called the *Gauss map* of M that is a smooth map which assigns to a point p in M the oriented $(m-n)$-plane through the origin of $\mathbb{E}_t^m$ and parallel to the normal space of M at p, [21].

We put $\varepsilon = \langle \nu, \nu \rangle = \varepsilon_{n+1} \varepsilon_{n+2} \cdots \varepsilon_m = \pm 1$ and

$$\widetilde{M}_s^{N-1}(\varepsilon) = \begin{cases} \mathbb{S}_s^{N-1}(1) & \text{in } \mathbb{E}_s^N \quad \text{if } \varepsilon = 1 \\ \mathbb{H}_{s-1}^{N-1}(-1) & \text{in } \mathbb{E}_s^N \quad \text{if } \varepsilon = -1. \end{cases}$$

Then the Gauss image $\nu(M)$ can be viewed as $\nu(M) \subset \widetilde{M}_s^{N-1}(\varepsilon)$.

In [18], the authors gave the following Lemma

Lemma 2.1. *[18] Let M be an n-dimensional oriented submanifold of a pseudo-Euclidean space $\mathbb{E}_t^{n+2}$. Then the Laplacian of Gauss map $\nu = e_{n+1} \wedge e_{n+2}$ is given by*

$$\begin{aligned} \Delta \nu \;=\; & \|h\|^2 \nu + 2 \sum_{1 \le j < k \le n} \varepsilon_j \varepsilon_k R^D(e_j, e_k; e_{n+1}, e_{n+2}) e_j \wedge e_k + \nabla(\mathrm{tr} A_{n+1}) \wedge e_{n+2} \\ & + e_{n+1} \wedge \nabla(\mathrm{tr} A_{n+2}) + n \sum_{j=1}^{n} \varepsilon_j \omega_{(n+1)(n+2)}(e_j) H \wedge e_j, \end{aligned} \tag{2.9}$$

where $\|h\|^2$ is the squared length of the second fundamental form, R^D the normal curvature tensor and $\nabla \mathrm{tr} A_r$ the gradient of $\mathrm{tr} A_r$.

We will also use the following theorems, proposition and remark:

Theorem 2.2. *[18] Let M be an oriented non-maximal space-like surface in $\mathbb{E}_1^4$. Then M has pointwise 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector.*

Theorem 2.3. *[18] An oriented maximal surface with harmonic Gauss map in the Minkowski space $\mathbb{E}^4_1$ is either an open part of a space-like plane or congruent to a surface given by*

$$x(u,v) = (\phi(u,v), u, v, \phi(u,v)). \tag{2.10}$$

for a smooth harmonic function $\phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, where Ω is an open set in $\mathbb{R}^2$.

Proposition 2.4. *[18] Let M be an oriented maximal surface in the Minkowski space $\mathbb{E}^4_1$. Then M has (global) 1-type Gauss map of the first kind if and only if the Gauss map ν of M is harmonic.*

Remark 2.1. [18] The Gauss map ν of a plane M in $\mathbb{E}^4_1$ is a constant vector in $\mathbb{E}^6_3$ and $\Delta\nu = 0$, i.e., it is harmonic. For $f = 0$ if we write $\Delta\nu = 0 \cdot \nu$, then M has pointwise 1-type Gauss map of the first kind. If we choose $C = -\nu$, then (1.1) holds for any non-zero smooth function f. In this case M has pointwise 1-type Gauss map of the second kind. Therefore, a plane in $\mathbb{E}^4_1$ is a trivial surface with pointwise 1-type Gauss map of both the first kind and the second kind.

3 Space-like surfaces with pointwise 1-type Gauss map of the second kind

In this section, we study space-like surfaces in the Minkowski space $\mathbb{E}^4_1$ with pointwise 1-type Gauss map of the second kind.

Let M be a space-like surface in $\mathbb{E}^4_1$. We choose a local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ defined on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M. Let C be a vector field in $\Lambda^2\mathbb{E}^4_1 \equiv \mathbb{E}^6_3$. Since the set $\{e_A \wedge e_B | 1 \le A < B \le 4\}$ is an orthonormal basis for $\mathbb{E}^6_3$, C can be expressed as

$$C = \sum_{1\le A<B\le 4} \varepsilon_A \varepsilon_B C_{AB}\, e_A \wedge e_B, \tag{3.1}$$

where $C_{AB} = \langle C, e_A \wedge e_B\rangle$. As e_1, e_2 are space-like, we have $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_4 = -\varepsilon_3$.

By a direct calculation using the Gauss and Weingarten formulas, we obtain that

$$\begin{aligned}
e_i(C) &= \sum_{1\le A<B\le 4} \varepsilon_A \varepsilon_B e_i(C_{AB} e_A \wedge e_B) \\
&= \left(e_i\left(C_{12}\right) - \varepsilon_3 h^3_{i2} C_{13} + \varepsilon_3 h^4_{i2} C_{14} + \varepsilon_3 h^3_{i1} C_{23} - \varepsilon_3 h^4_{i1} C_{24}\right) e_1 \wedge e_2 \\
&\quad + \left(e_i\left(C_{13}\right) + h^3_{i2} C_{12} + \varepsilon_3 \omega_{34}(e_i) C_{14} - \omega_{12}(e_i) C_{23} - \varepsilon_3 h^4_{i1} C_{34}\right) e_1 \wedge e_3 \\
&\quad + \left(e_i\left(C_{14}\right) + h^4_{i2} C_{12} + \varepsilon_3 \omega_{34}(e_i) C_{13} - \omega_{12}(e_i) C_{24} - \varepsilon_3 h^3_{i1} C_{34}\right) e_1 \wedge e_4 \\
&\quad + \left(e_i\left(C_{23}\right) - h^3_{i1} C_{12} + \omega_{12}(e_i) C_{13} + \varepsilon_3 \omega_{34}(e_i) C_{24} - \varepsilon_3 h^4_{i2} C_{34}\right) e_2 \wedge e_3 \\
&\quad + \left(e_i\left(C_{24}\right) - h^4_{i1} C_{12} + \omega_{12}(e_i) C_{14} + \varepsilon_3 \omega_{34}(e_i) C_{23} - \varepsilon_3 h^3_{i2} C_{34}\right) e_2 \wedge e_4 \\
&\quad + \left(e_i\left(C_{34}\right) - h^4_{i1} C_{13} + h^3_{i1} C_{14} - h^4_{i2} C_{23} + h^3_{i2} C_{24}\right) e_3 \wedge e_4.
\end{aligned}$$

Hence we state

Lemma 3.1. *A vector C in $\Lambda^2\mathbb{E}^4_1 \equiv \mathbb{E}^6_3$ written by* (3.1) *is constant if and only if the following equations are satisfied for* $i = 1, 2$

$$
\begin{array}{lrcl}
(3.2) & e_i\,(C_{12}) & = & \varepsilon_3 h^3_{i2} C_{13} - \varepsilon_3 h^4_{i2} C_{14} - \varepsilon_3 h^3_{i1} C_{23} + \varepsilon_3 h^4_{i1} C_{24}, \\
(3.3) & e_i\,(C_{13}) & = & -h^3_{i2} C_{12} - \varepsilon_3 \omega_{34}(e_i) C_{14} + \omega_{12}(e_i) C_{23} + \varepsilon_3 h^4_{i1} C_{34}, \\
(3.4) & e_i\,(C_{14}) & = & -h^4_{i2} C_{12} - \varepsilon_3 \omega_{34}(e_i) C_{13} + \omega_{12}(e_i) C_{24} + \varepsilon_3 h^3_{i1} C_{34}, \\
(3.5) & e_i\,(C_{23}) & = & h^3_{i1} C_{12} - \omega_{12}(e_i) C_{13} - \varepsilon_3 \omega_{34}(e_i) C_{24} + \varepsilon_3 h^4_{i2} C_{34}, \\
(3.6) & e_i\,(C_{24}) & = & h^4_{i1} C_{12} - \omega_{12}(e_i) C_{14} - \varepsilon_3 \omega_{34}(e_i) C_{23} + \varepsilon_3 h^3_{i2} C_{34}, \\
(3.7) & e_i\,(C_{34}) & = & h^4_{i1} C_{13} - h^3_{i1} C_{14} + h^4_{i2} C_{23} - h^3_{i2} C_{24}.
\end{array}
$$

Now, we focus on maximal surfaces in $\mathbb{E}^4_1$. In the Euclidean space $\mathbb{E}^4$, there exist non-planar minimal surfaces with pointwise 1-type Gauss map of the second kind (cf. [15, 17]). However, in the Minkowski space $\mathbb{E}^4_1$ we obtain the following theorem:

Theorem 3.2. *Let M be an oriented maximal surface in the Minkowski space $\mathbb{E}^4_1$. Then M has pointwise 1-type Gauss map of the second kind if and only if it is an open portion of a space-like plane.*

Proof. Let M be an oriented maximal surface in $\mathbb{E}^4_1$, i.e., $H \equiv 0$. Then there exists a frame field $\{e_1, e_2, e_3, e_4\}$ defined on M such that $\varepsilon_3 = -\varepsilon_4 = 1$ and the corresponding shape operators are of the form

$$
(3.8) \qquad A_3 = \begin{pmatrix} h^3_{11} & 0 \\ 0 & -h^3_{11} \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} h^4_{11} & h^4_{12} \\ h^4_{12} & -h^4_{11} \end{pmatrix}.
$$

Thus, (2.9) implies

$$
(3.9) \qquad \Delta\nu \;=\; \|h\|^2 \nu + 2R^D(e_1, e_2; e_3, e_4) e_1 \wedge e_2.
$$

Now, we assume M has pointwise 1-type Gauss map of the second kind. Then there exist a smooth function f and a non-zero constant vector $C \in \mathbb{E}^6_3$ such that (1.1) is satisfied. From (1.1), (3.1) and (3.9), we get $f(\nu + C) = \|h\|^2\nu + 2R^D(e_1, e_2; e_3, e_4) e_1 \wedge e_2$ which implies

$$
(3.10) \qquad C_{13} = C_{14} = C_{23} = C_{24} = 0.
$$

Since C is a constant vector, the functions $C_{AB},\ A, B = 1, 2, 3, 4.$ satisfy (3.2)-(3.7) because of Lemma 3.1. By using (3.8) and (3.10) in equations (3.3) and (3.6) for $i = 1, 2$, we obtain

$$
(3.11) \qquad C_{12} h^4_{11} = C_{34} h^4_{11} = 0,
$$

$$
(3.12) \qquad C_{12} h^3_{11} + C_{34} h^4_{12} = C_{12} h^4_{12} - C_{34} h^3_{11} = 0.
$$

Since C is non-zero, one of the functions C_{12} and C_{34} is non-zero. Therefore, (3.11) and (3.12) imply $h^3_{11} = h^4_{11} = h^4_{12} = 0$. Hence, we have $A_3 = A_4 = 0$ which yields M is an open portion of a space-like plane in $\mathbb{E}^4_1$.

The converse follows from Remark 2.1. □

Considering Proposition 2.3, Proposition 2.4 and Theorem 3.2, we state following classification theorem:

Theorem 3.3. *Let M be an oriented maximal surface in the Minkowski space $\mathbb{E}^4_1$. Then M has (global) 1-type Gauss map if and only if M is either an open part of an space-like plane or congruent to the surface given by* (2.10).

Next, we study space-like surfaces in $\mathbb{E}^4_1$ with constant mean curvature. First, we have some examples of space-like surfaces with pointwise 1-type Gauss map of the second kind.

Example 1. *Let M be a helical cylinder in $\mathbb{E}^4_1$ given by*

$$x_1(s,t) = (a_1 s, b_1 \cos s, b_1 \sin s, t), \tag{3.13}$$

where a_1 and b_1 are some non-zero constants with $b_1^2 - a_1^2 > 0$. Then M is a space-like surface with constant mean curvature and flat normal bundle. Moreover, its Gauss map ν satisfies (1.1) *for the smooth function $f = \frac{1}{b_1^2 - a_1^2}$ and the constant vector $C = \frac{a_1^2}{b_1^2 - a_1^2}\nu + \frac{a_1 b_1}{b_1^2 - a_1^2} e_1 \wedge e_3$. Therefore, M has pointwise 1-type Gauss map of the second kind.*

Example 2. *The same arguments hold for the helical cylinders given by*

$$x_2(s,t) = (b_2 \cosh s, b_2 \sinh s, a_2 s, t) \tag{3.14}$$

and

$$x_3(s,t) = (b_3 \sinh s, b_3 \cosh s, a_3 s, t), \tag{3.15}$$

for some non-zero constants $a_2,\ a_3,\ b_2,\ b_3$ with $a_3^2 - b_3^2 > 0$.

We need the following lemma for later use:

Lemma 3.4. *Let M be an oriented space-like surface in the Minkowski space $\mathbb{E}^4_1$. If there exists an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ defined on M such that the corresponding connections forms satisfy*

$$\omega_{13} = -\alpha\omega_1, \quad \omega_{34} = \beta\omega_1, \quad \omega_{12} = \omega_{14} = \omega_{23} = \omega_{24} = 0 \tag{3.16}$$

for some constants $\alpha \neq 0$ and $\beta \neq 0$ with $\varepsilon_3\alpha^2 - \beta^2 \neq 0$, then M is congruent to one of the helical cylinders given by (3.13), (3.14) *and* (3.15).

Proof. Let the connection forms of M relative to an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ be given by (3.16). Then, we have $\omega_{34}(e_1) = \beta$, $\omega_{34}(e_2) = 0$, $A_3 = \mathrm{diag}(\alpha, 0)$ and $A_4 = 0$. The first structural equation (2.5) implies $d\omega_1 = d\omega_2 = 0$ as $\omega_{12} = 0$. Thus, the dual forms ω_1 and ω_2 are exact, i.e., there exists a local coordinate system $\{u, v\}$ such that $\omega_1 = du$ and $\omega_2 = dv$ which imply $e_1 = \partial_u$ and $e_2 = \partial_v$

Let $x = x(u,v)$ be the position vector of M in $\mathbb{E}^4_1$ defined on an open set Ω of $\mathbb{R}^2$. Since $\omega_{12} = 0$, we have $\widetilde{\nabla}_{e_1} e_1 = x_{uu} = h(e_1, e_1)$, $\widetilde{\nabla}_{e_1} e_2 = \widetilde{\nabla}_{e_2} e_1 = x_{uv} = h(e_1, e_2)$ and $\widetilde{\nabla}_{e_2} e_2 = x_{vv} = h(e_2, e_2)$. From these equations, (2.1) and (2.2) we obtain

$$x_{uu} = \varepsilon_3 \alpha e_3, \quad x_{uv} = 0, \quad x_{vv} = 0, \tag{3.17}$$

$$\widetilde{\nabla}_{e_1} e_3 = (e_3)_u = -\alpha x_u - \varepsilon_3 \beta e_4, \quad \widetilde{\nabla}_{e_2} e_4 = (e_3)_v = 0, \tag{3.18}$$

$$\widetilde{\nabla}_{e_1} e_4 = (e_4)_u = -\varepsilon_3 \beta e_3, \quad \widetilde{\nabla}_{e_2} e_4 = (e_4)_v = 0. \tag{3.19}$$

The second and third equations in (3.17) imply

$$x = vL_1 + B(u), \tag{3.20}$$

where $L_1 \in \mathbb{E}^4_1$ is a constant vector and B is a vector-valued function into $\mathbb{E}^4_1$. Note that (3.18) and (3.19) show that the vector fields e_3 and e_4 depend only on u. In addition, from the first equation in (3.17) we obtain that

$$\langle x_{uu}, x_{uu}\rangle = \varepsilon_3\alpha^2 \tag{3.21}$$

and

$$B'' = \varepsilon_3\alpha e_3 \tag{3.22}$$

where $'$ denotes derivative with respect to u.

By differentiating the first equation in (3.18) and using (3.19), (3.20) we obtain

$$e_3'' + (\varepsilon_3\alpha^2 - \beta^2)e_3 = 0. \tag{3.23}$$

Considering the sign of the constant $\varepsilon_3\alpha^2 - \beta^2$, the general solution of (3.23) can be written in terms of hyperbolic or trigonometric functions. Therefore, we have two cases:

Case 1. $\varepsilon_3\alpha^2 - \beta^2 = -a^2 < 0$. By solving (3.23) we obtain $e_3 = \cosh(au)\,\tilde{L}_2 + \sinh(au)\,\tilde{L}_3$ for some constant vectors $\tilde{L}_2, \tilde{L}_3 \in \mathbb{E}^4_1$. Thus, (3.22) becomes

$$B''(u) = \varepsilon_3\alpha\left(\cosh(au)\,\tilde{L}_2 + \sinh(au)\,\tilde{L}_3\right)$$

from which we have

$$B(u) = \cosh(au)\,L_2 + \sinh(au)\,L_3 + uL_4 + L_5 \tag{3.24}$$

for some constant vectors $L_2,\ L_3,\ L_4,\ L_5 \in \mathbb{E}^4_1$. Without loss of generality, we may take $L_5 = 0$. Thus, from (3.20) and (3.24) we get

$$x = L_1 v + \cosh(au)\,L_2 + \sinh(au)\,L_3 + uL_4. \tag{3.25}$$

From $\langle x_u, x_u\rangle = 1$, $\langle x_u, x_v\rangle = 0$, $\langle x_v, x_v\rangle = 1$ and (3.21) we obtain that $L_1,\ L_2,\ L_3$ and L_4 are mutually perpendicular and that

$$\langle L_1, L_1\rangle = 1, \quad \langle L_2, L_2\rangle = -\langle L_3, L_3\rangle = \tfrac{\varepsilon_3\alpha^2}{(\varepsilon_3\alpha^2-\beta^2)^2}, \quad \langle L_4, L_4\rangle = \tfrac{\beta^2}{\beta^2-\varepsilon_3\alpha^2}.$$

Considering $\varepsilon_3 = 1$ or $\varepsilon_3 = -1$, we see that there are exactly two different choice of $L_2,\ L_3$ and L_4, up to linear isometries of $\mathbb{E}^4_1$. Thus, we have two subcases:

Case 1a. $\varepsilon_3 = 1$. After a suitable isometry of $\mathbb{E}^4_1$, we may assume that $L_1 = (0,0,0,1),\ L_2 = \frac{\alpha}{\alpha^2-\beta^2}(0,1,0,0),\ L_3 = \frac{\alpha}{\alpha^2-\beta^2}(1,0,0,0),\ L_4 = \frac{\beta}{\sqrt{\beta^2-\alpha^2}}(0,0,1,0)$. Hence, by choosing suitable coordinates, putting $\sqrt{\beta^2-\alpha^2}u = s$, $\dfrac{\beta}{\beta^2-\alpha^2} = a_3$ and $\dfrac{\alpha}{\alpha^2-\beta^2} = b_3$ and replacing v by t, we obtain (3.15).

Case 1b. $\varepsilon_3 = -1$. Up to isometries of $\mathbb{E}^4_1$, we may choose $L_1 = (0,0,0,1)$, $L_2 = \frac{\alpha}{\alpha^2+\beta^2}(1,0,0,0)$, $L_3 = \frac{\alpha}{\alpha^2+\beta^2}(0,1,0,0)$, $L_4 = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}(0,0,1,0)$. After a suitable choice of Minkowskian coordinate system $\{s,t\}$ and constants a_2, b_2, we can see that M is congruent to the surface given by (3.14).

Case 2. $\varepsilon_3\alpha^2 - \beta^2 = a^2 > 0$. In this case, the general solution of (3.23) is

$$e_3 = \cos(au)\,\tilde{L}_2 + \sin(au)\,\tilde{L}_3$$

for some constant vectors $\tilde{L}_2, \tilde{L}_3 \in \mathbb{E}^4_1$ and we have only $\varepsilon_3 = 1$. By a similar way to Case 1a, we can see that M is congruent to the surface given by (3.13). □

Theorem 3.5. *Let M be an oriented space-like surface in the Minkowski space $\mathbb{E}^4_1$ with flat normal bundle and non-zero constant mean curvature. Then, M has pointwise 1-type Gauss map of the second kind if and only if it is congruent to one of the helical cylinders given by* (3.13), (3.14) *and* (3.15).

Proof. Since M has non-zero constant mean curvature, there exists a local orthonormal frame field $\{e_3, e_4\}$ of normal bundle of M such that the mean curvature vector H of M is proportional to e_3. Moreover, since M has flat normal bundle, shape operators can be simultaneously diagonalized by choosing a proper basis $\{e_1, e_2\}$ of tangent bundle of M. Therefore, the shape operators are of the form

$$A_3 = \operatorname{diag}(h^3_{11}, h^3_{22}), \quad A_4 = \operatorname{diag}(h^4_{11}, -h^4_{11}). \tag{3.26}$$

Let $\alpha = h^3_{11} + h^3_{22} \neq 0$ which is a constant. From (2.9) and (3.26) we obtain that

$$\Delta\nu = \|h\|^2\nu - \varepsilon_3\alpha\omega_{34}(e_1)e_1 \wedge e_3 - \varepsilon_3\alpha\omega_{34}(e_2)e_2 \wedge e_3. \tag{3.27}$$

We assume M has pointwise 1-type Gauss map of the second kind. Now, we are going to determine the connection forms of M. According to the assumption, (1.1) is satisfied for some function $f \neq 0$ and non-zero constant vector $C \in \mathbb{E}^6_3$. From (1.1), (3.1) and (3.27) we have

$$f(1 - C_{34}) = \|h\|^2, \tag{3.28}$$
$$fC_{13} = -\alpha\omega_{34}(e_1), \tag{3.29}$$
$$fC_{23} = -\alpha\omega_{34}(e_2), \tag{3.30}$$
$$C_{12} = C_{14} = C_{24} = 0. \tag{3.31}$$

Since C is a non-zero constant vector, its components satisfy (3.2)-(3.7) for $i = 1,2$ because of Lemma 3.1. From (3.4) and (3.6) for $i = 1,2$, we obtain that

$$-\omega_{34}(e_1)C_{13} + h^3_{11}C_{34} = 0. \tag{3.32}$$
$$\omega_{34}(e_2)C_{13} = 0, \tag{3.33}$$
$$\omega_{34}(e_1)C_{23} = 0, \tag{3.34}$$
$$-\omega_{34}(e_2)C_{23} + h^3_{22}C_{34} = 0. \tag{3.35}$$

Note that if $\omega_{34}(e_1) = \omega_{34}(e_2) = 0$, then M has parallel mean curvature vector which is a contradiction because of Theorem 2.2. Therefore, without loss of generality, we may assume $\omega_{34}(e_1) \neq 0$. So, (3.29) implies that $C_{13} \neq 0$. From (3.33) we get $\omega_{34}(e_2) = 0$. Thus, (3.30) implies $C_{23} = 0$. Hence, C becomes

$$C = \varepsilon_3 C_{13} e_1 \wedge e_3 - C_{34} e_3 \wedge e_4. \tag{3.36}$$

On the other hand, (3.35) gives $C_{34}h^3_{22} = 0$ as $C_{23} = 0$. Note that if $C_{34} = 0$, then (3.32) implies $\omega_{34}(e_1)C_{13} = 0$ which is a contradiction. So, we have $h^3_{22} = 0$. Therefore, from (3.26) we have

$$A_3 = \mathrm{diag}(\alpha, 0), \quad A_4 = \mathrm{diag}(h^4_{11}, -h^4_{11}). \tag{3.37}$$

Thus, the Codazzi equations $h^3_{11,2} = h^3_{12,1}$, $h^3_{22,1} = h^3_{12,2}$ and $h^4_{22,1} = h^4_{12,2}$ become, respectively,

$$\alpha\omega_{12}(e_1) = 0, \tag{3.38}$$

$$\varepsilon_4 h^4_{11}\omega_{34}(e_1) = \alpha\omega_{12}(e_2), \tag{3.39}$$

$$e_1(-h^4_{11}) = 2h^4_{11}\omega_{12}(e_2). \tag{3.40}$$

In addition, the Gauss equation $\langle R(e_1, e_2)e_1, e_2\rangle = \varepsilon_3(\det A_3 - \det A_4)$ implies

$$e_1\big(\omega_{12}(e_2)\big) = \varepsilon_3 \left(h^4_{11}\right)^2 - \big(\omega_{12}(e_2)\big)^2. \tag{3.41}$$

From (3.38) we obtain $\omega_{12}(e_1) = 0$ as $\alpha \neq 0$.

Now, we will show that $h^4_{11} = 0$. Suppose that $h^4_{11} \neq 0$. Multiplying (3.29) by $\omega_{34}(e_1)$ and using (3.32), we obtain that

$$fC_{34} = -\left(\omega_{34}(e_1)\right)^2 \tag{3.42}$$

as $h^3_{11} = \alpha \neq 0$. Thus, (3.28) implies

$$f = \varepsilon_3(\alpha^2 - 2(h^4_{11})^2) - (\omega_{34}(e_1))^2. \tag{3.43}$$

From (3.29), (3.36) and (3.42) we obtain that

$$C = \frac{-\omega_{34}(e_1)}{f}\left(\varepsilon_3\alpha e_1 \wedge e_3 - \omega_{34}(e_1) e_3 \wedge e_4\right). \tag{3.44}$$

Next, we define a vector field $\hat{C} = \varepsilon_3\alpha e_1 \wedge e_3 - \omega_{34}(e_1)e_3 \wedge e_4$ and a function $\hat{f} = -\omega_{34}(e_1)/f$. Then (3.44) infers $C = \hat{f}\hat{C}$. Since C is a constant vector, we get

$$e_1(C) = e_1(\hat{f})\hat{C} + \hat{f}e_1(\hat{C}) = 0. \tag{3.45}$$

Note that if $\hat{C}$ and $e_1(\hat{C})$ linearly independent, (3.45) implies $\hat{f} = 0$ which is a contradiction. In addition, by a direct calculation using Gauss and Weingarten formulas (2.1) and (2.2), we obtain that

$$e_1(\hat{C}) = -h^4_{11}\omega_{34}(e_1)e_1 \wedge e_3 + \Big(\alpha h^4_{11} - e_1\big(\omega_{34}(e_1)\big)\Big)e_3 \wedge e_4 \tag{3.46}$$

which implies $e_1(\hat{C}) \neq 0$ as $h^4_{11} \neq 0$ and $\omega_{34}(e_1) \neq 0$. Thus, $\hat{C}$ and $e_1(\hat{C})$ are linearly dependent. By differentiating (3.39), we obtain

$$\varepsilon_4 e_1(h^4_{11})\omega_{34}(e_1) + \varepsilon_4 h^4_{11} e_1\big(\omega_{34}(e_1)\big) = \alpha e_1(\omega_{12}(e_2))$$

from which and (3.39)-(3.41) we get

$$h^4_{11}\Big(e_1\big(\omega_{34}(e_1)\big) + \alpha h^4_{11} - \omega_{12}(e_2)\omega_{34}(e_1)\Big) = 0. \tag{3.47}$$

As $h^4_{11} \neq 0$, (3.47) implies

$$e_1\big(\omega_{34}(e_1)\big) = -\alpha h^4_{11} + \omega_{12}(e_2)\omega_{34}(e_1). \tag{3.48}$$

From (3.39), (3.46) and (3.48) we obtain

$$e_1(\hat{C}) \quad = \quad \omega_{12}(e_2)\hat{C} + 2\alpha h^4_{11} e_3 \wedge e_4. \tag{3.49}$$

Since $e_1(\hat{C})$ and $\hat{C}$ linearly dependent, (3.49) implies $h^4_{11} = 0$ which is a contradiction. Therefore, we proved $h^4_{11} = 0$. As $h^4_{11} = 0$, (3.39) implies $\omega_{12}(e_2) = 0$. On the other hand, from (3.43) and (3.44) we have $\varepsilon_3\alpha^2\langle C, C\rangle = (1 + \langle C, C\rangle)\omega_{34}(e_1)$ which implies $\omega_{34}(e_1) = \beta$, where

$$\beta = \frac{\varepsilon_3\alpha^2\langle C, C\rangle}{1 + \langle C, C\rangle} \neq 0.$$

Moreover, (3.43) implies $f = \varepsilon_3\alpha^2 - \beta^2 \neq 0$.

Consequently, we have the connection forms of M in $\mathbb{E}^4_1$ as

$$\omega_{13} = -\alpha\omega_1, \quad \omega_{34} = \beta\omega_1, \quad \omega_{12} = \omega_{14} = \omega_{23} = \omega_{24} = 0.$$

Considering Lemma 3.4, the connection forms of M and helical cylinders given by (3.13), (3.14) and (3.15) coincides. Therefore, considering the fundemental theorem of submanifolds, M is congruent to one of the helical cylinders given by (3.13), (3.14) and (3.15). □

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