Reidemeister torsion of product manifolds and its applications to quantum entanglement

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Abstract. Using symplectic chain complex, a formula for the Reidemeister torsion of product of oriented closed connected even dimensional manifolds is presented. In applications, the formula is applied to Riemann surfaces, Grassmannians, projective spaces and these results will be applied to manifolds of pure bipartite states with Schmidt ranks.

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1 Introduction

Reidemeister torsion was first introduced by Reidemeister in 1935 [12]. This is a topological invariant which is not homotopy invariant. With the help of Reidemeister torsion, he classified (up to PL equivalence) 3-dimensional lens spaces i.e. $\mathbb{S}^3/\Gamma$, where Γ is a finite cyclic group of fixed point free orthogonal transformations [12]. Franz extended Reidemeister torsion in 1935 and classified the higher dimensional lens spaces $\mathbb{S}^{2n+1}/\Gamma$, where Γ is a cyclic group acting freely and isometrically on the sphere $\mathbb{S}^{2n+1}$ [6].

In [5], de Rham extended the results of Reidemeister and Franz to spaces of constant curvature $+1$. Kirby and Siebenmann in 1969 proved the topological invariance of Reidemeister torsion for manifolds [7]. Chapman proved the topological invariance of Reidemeister torsion for arbitrary simplicial complexes [3, 4]. Hence, Reidemeister and Franz's classification of lens spaces was actually topological i.e. up to homeomorphism.

By using Reidemeister torsion, Milnor disproved *Hauptvermutung* in 1961. To be more precise, he constructed two combinatorially distinct but homeomorphic finite simplicial complexes. He, in 1962, identified Reidemeister torsion with Alexander polynomial which plays an important role in knot theory and links [8, 10].

In [17], Witten introduced symplectic chain complex. Let S be a compact 2-dimensional manifold, G be a compact gauge group, E be a $G-$bundle over S, with a connection

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A and curvature F. F is a two form with values in the adjoint bundle $ad(E)$. Let $\mathcal{M}$ be the moduli space of flat connections on E, upto gauge transformations. For orientable S, $\mathcal{M}$ has a natural symplectic form ω [1], and hence there exits a natural volume form $\theta = \omega^n/n!$ on $\mathcal{M}$, where $2n = dim\mathcal{M}$.

In [17], by using the symplectic chain complex and the Reidemeister torsion, Witten defined a volume element on $\mathcal{M}$, where S is orientable or not. In the orientable case, this volume form and θ coincide. Moreover, using this volume element, he computes the volume of $\mathcal{M}$ in [17].

By using sympletic chain complex and Thurston's geodesic lamination theory, we [13] presented a volume element on the moduli space of representation $\varrho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ of the fundamental group of compact oriented Riemann surface of genus ≥ 2 into $\mathrm{PSL}_2(\mathbb{R})$. Also, we explained in [14] the relation between Reidemeister torsion and Fubuni-Study form ω_{FS} of the complex projective $n-$space $\mathbb{CP}^n$ by using symplectic chain complex. Furthermore, this technique enabled us to prove the connection of Reidemeister torsion of compact oriented Riemann surface S of genus ≥ 1 and its period matrix [15].

In [16], we consider even dimensional oriented closed connected manifolds. With the help of symplectic chain complex, we proved a formula for computing the Reidemeister torsion of them. Moreover, we presented applications to Riemann surfaces and Grasmannians in [16]. In the present article, we prove a formula of Reidemeister torsion of product of oriented closed connected manifolds. We also present applications of this formula to Riemann surfaces and Grasmannians.

2 The Reidemeister torsion

In this section, the required definitions and the basic facts about the Reidemeister torsion are given. For more information and the detailed proof, we refer the reader to [11, 13, 17], and the references therein.

Throughout the paper, $\mathbb{F}$ is the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers. Let $(C_*, \partial_*) = (C_n \stackrel{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{F}$. Let $H_p(C_*) = Z_p(C_*)/B_p(C_*)$ be the $p-$th homology of C_*, where $B_p(C_*) = \mathrm{Im}\{\partial_{p+1} : C_{p+1} \to C_p\}$ and $Z_p(C_*) = \ker\{\partial_p : C_p \to C_{p-1}\}$.

There are the following short-exact sequences: $0 \to Z_p(C_*) \to C_p \to B_{p-1}(C_*) \to 0$ and $0 \to B_p(C_*) \to Z_p(C_*) \to H_p(C_*) \to 0$. Let $\mathbf{b}_p$, $\mathbf{h}_p$ be bases of $B_p(C_*)$, $H_p(C_*)$, respectively. Also, let $\ell_p : H_p(C_*) \to Z_p(C_*)$, $s_p : B_{p-1}(C_*) \to C_p$ be sections of $Z_p(C_*) \to H_p(C_*)$, $C_p \to B_{p-1}(C_*)$, respectively. Then, one gets a new basis of C_p, more precisely, $\mathbf{b}_p \oplus \ell_p(\mathbf{h}_p) \oplus s_p(\mathbf{b}_{p-1})$.

The *Reidemeister torsion* of C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is the alternating product

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n) = \prod_{p=0}^{n} [\mathbf{b}_p \oplus \ell_p(\mathbf{h}_p) \oplus s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]^{(-1)^{(p+1)}},$$

where $[\mathbf{e}_p, \mathbf{f}_p]$ is the determinant of the change-base-matrix from basis $\mathbf{f}_p$ to $\mathbf{e}_p$ of C_p.

It is proved by Milnor that the Reidemeister torsion is independent of bases $\mathbf{b}_p$, sections s_p, ℓ_p [9]. If $\mathbf{c}'_p, \mathbf{h}'_p$ are other bases respectively for C_p, $H_p(C_*)$, then an easy computation gives the following change-base-formula:

$$\mathbb{T}(C_*, \{\mathbf{c}'_p\}_0^n, \{\mathbf{h}'_p\}_0^n) = \prod_{p=0}^{n} \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]} \right)^{(-1)^p} \mathbb{T}(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n). \tag{2.1}$$

Formula (2.1) follows easily from the independence of the Reidemeister torsion from $\mathbf{b}_p$ and sections s_p, ℓ_p.

From Snake Lemma it follows that for the short-exact sequence (2.2) of chain complexes

$$0 \to A_* \xrightarrow{\mathrm{i}} B_* \xrightarrow{\pi} D_* \to 0, \tag{2.2}$$

there is also the long-exact sequence of vector spaces C_* of length $3n + 2$. More precisely,

$$C_* : \cdots \to H_p(A_*) \xrightarrow{\mathrm{i}_p} H_p(B_*) \xrightarrow{\pi_p} H_p(D_*) \xrightarrow{\delta_p} H_{p-1}(A_*) \to \cdots, \tag{2.3}$$

where $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(A_*)$, and $C_{3p+2} = H_p(B_*)$.

The bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ clearly serve as bases for C_{3p}, C_{3p+1}, and C_{3p+2}, respectively. The following result of Milnor states that the alternating product of the torsions of the chain complexes in (2.2) is equal to the torsion of (2.3) in [9]. Using this statement we have the following sum-lemma:

Lemma 2.1. *Let A_*, D_* be two chain complexes. Let $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^D$ be bases of A_p, D_p, $H_p(A_*)$, and $H_p(D_*)$, respectively. Then,*

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \oplus \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \oplus \mathbf{h}_p^D\}_0^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n).$$

For detailed proof and further information, we may refer the readers to [16].

It is independently explained in [2, 13] that a general chain complex can (unnaturally) be splitted as a direct sum of an acyclic and $\partial-$zero chain complexes. Furthermore, it is showed independently in [2, Proposition 1.5] and [13, Theorem 2.0.4] that the Reidemeister torsion $\mathbb{T}(C_*)$ of a general complex C_* can be interpreted as an element of $\otimes_{p=0}^n (\det(H_p(C_*)))^{(-1)^{p+1}}$, where $\det(H_p(M)) = \bigwedge^{\dim_{\mathbb{R}} H_p(C_*)} H_p(C_*)$ is the top exterior power of $H_p(C_*)$, and where $\det(H_p(C_*))^{-1}$ is the dual of $\det(H_p(C_*))$. See [2, 13] for details.

A *symplectic chain complex* of length q is $(C_*, \partial_*, \{\omega_{*,q-*}\})$, where $C_* : 0 \to C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ is a chain complex with $q \equiv 2 (\bmod\ 4)$, and for $p = 0, \ldots, q/2$, $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{R}$ is a $\partial-$compatible anti-symmetric non-degenerate bilinear form. More precisely, $\omega_{p,q-p}(\partial_{p+1}a, b) = (-1)^{p+1}\omega_{p+1,q-(p+1)}(a, \partial_{q-p}b)$ and $\omega_{p,q-p}(a, b) = (-1)^{p(q-p)}\omega_{q-p,p}(b, a)$.

From $q \equiv 2 (\bmod\ 4)$ it follows easily that $\omega_{p,q-p}(a, b) = (-1)^p\omega_{q-p,p}(b, a)$. By the $\partial-$compatibility of the non-degenerate anti-symmetric bilinear maps $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{R}$, we can easily extend these to homologies [13].

Let C_* be a symplectic chain complex. Let $\mathbf{c}_p$ and $\mathbf{c}_{q-p}$ be bases of C_p and C_{q-p}, respectively. These bases are said to be $\omega-$*compatible* if the matrix of $\omega_{p,q-p}$ in bases $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ equals to the $k \times k$ identity matrix $\mathrm{I}_{k\times k}$ when $p \neq q/2$ and $\begin{bmatrix} 0_{l\times l} & \mathrm{I}_{l\times l} \\ -\mathrm{I}_{l\times l} & 0_{l\times l} \end{bmatrix}$ when $p = q/2$, where $k = \dim C_p = \dim C_{q-p}$ and $2l = \dim C_{q/2}$.

By considering $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{R}$, one can also define the $[\omega]$-compatibility of bases $\mathbf{h}_p$ of $H_p(C_*)$ and $\mathbf{h}_{q-p}$ of $H_{q-p}(C_*)$.

By using the existence of $\omega-$compatible bases, we were able to prove in [13] that a symplectic chain complex C_* can be splitted $\omega-$orthogonally as a direct sum of an exact and $\partial-$zero symplectic complexes. We already calculated the Reidemeister torsion of C_* with respect to $\{\mathbf{c}_p\}_0^q$, $\{\mathbf{h}_p\}_0^q$, in [13]. Then we have

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q) = \prod_{p=0}^{(q/2)-1} (\det[\omega_{p,q-p}])^{(-1)^p} \sqrt{\det[\omega_{q/2,q/2}]}^{(-1)^{q/2}}.$$

Here, $\det[\omega_{p,q-p}]$ is the determinant of the matrix of the non-degenerate pairing $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{R}$ in bases $\mathbf{h}_p$, $\mathbf{h}_{q-p}$.

For further applications of this result, we refer the reader to [14, 15, 16].

Let us define the Reidemeister torsion of a manifold. Let M be an $m-$manifold with a cell decomposition K. Let $\mathbf{c}_p = \{c_1^p, \ldots, c_{n_p}^p\}$ be the *geometric basis* for the $p-$cells $C_p(K;\mathbb{Z})$, $p = 0, \ldots, m$. Then, there is the following chain complex associated to M

$$0 \to C_m(K) \stackrel{\partial_m}{\to} C_{m-1}(K) \to \cdots \to C_1(K) \stackrel{\partial_1}{\to} C_0(K) \to 0,$$

where $\mathbb{Z}$ is the set of integers and ∂_p is the usual boundary operator.

Let M be an $m-$manifold with a cell decomposition K. For $p = 0, \ldots, m$, let $\mathbf{c}_p$ and $\mathbf{b}_p$ be bases of $C_p(K;\mathbb{Z})$ and $H_p(M;\mathbb{Z})$, respectively. $\mathbb{T}(C_*(K), \{\mathbf{c}_p\}_0^m, \{\mathbf{h}_p\}_0^m)$ is called the *Reidemeister torsion* of M. From ([14]) we know that the Reidemeister torsion of M is independent of cell decomposition. Thus, the Reidemeister torsion $\mathbb{T}(C_*(K), \{\mathbf{c}_p\}_0^m, \{\mathbf{h}_p\}_0^m)$ of M is well-defined.

From [2, Proposition 1.5] and [13, Theorem 2.0.4] we can conclude that the Reidemeister torsion of M is an element of the dual of $1-$dimensional vector space $\otimes_{p=0}^n(\det(H_p(M))^{(-1)^p}$.

3 Main Result

Let us introduce the following notation used throughout the paper. Let M be a closed connected oriented manifold of dimension m. For $p = 0, \ldots, m$, let $\mathbf{h}_p^M$ and $\mathbf{h}_{m-p}^M$ be bases of $H_p(M)$ and $H_{m-p}(M)$, respectively. Let $H_{p,m-p}(M)$ be the matrix of intersection pairing $(\cdot,\cdot)_{p,m-p} : H_p(M) \times H_{m-p}(M) \to \mathbb{R}$ in the bases $\mathbf{h}_p^M$ and $\mathbf{h}_{d-p}^M$. If $H_p(M) = H_{m-p}(M) = 0$, then we define $H_{p,m-p}(M) = 1$. Hence, we let $\mathbb{T}(M, \{\mathbf{h}_p\}_0^m)$ denote the Reidemeister torsion of M in the bases $\mathbf{h}_p$ of $H_p(M)$, $p = 0, \ldots, m$. So we are ready to prove the main result of this paper.

Theorem 3.1. *Let M, N be oriented closed connected $2m$, $2n-$manifold $(m, n \geq 1)$ respectively. Let $\mathbf{h}_p^M (p = 0, \ldots, 2m)$, $\mathbf{h}_q^N (q = 0, \ldots, 2n)$ be bases of $H_p(M)$, $H_q(N)$, respectively. Then,*

$$\left|\mathbb{T}(M \times N, \{\oplus_{p+q=2m+2n}\mathbf{h}_p^M \otimes \mathbf{h}_q^N\})\right| = \left|\mathbb{T}(M, \{\mathbf{h}_p^M\}_{p=0}^{2m})\right|^{\chi(N)} \left|\mathbb{T}(N, \{\mathbf{h}_p^N\}_{q=0}^{2n})\right|^{\chi(M)},$$

where χ is the Euler characteristic.

Proof. Let us assume $m \leq n$. We consider the cases: $n \leq 3m+2$ and $n > 3m+2$, separately. The proof of each case is similar, thus we shall completely give the proof of one case only.

Let us consider the case: $n > 3m+2$. From the Künneth formula it follows that for $p = 0, \ldots, 2m$

$$\begin{aligned} |\det H_{p,2m+2n-p}(M \times N)| &= \prod_{i=0}^{p} |\det H_{i,2m-i}(M)|^{\dim H_{p-i}(N)} \\ &\times \prod_{i=0}^{p} |\det H_{i,2n-i}(N)|^{\dim H_{p-i}(M)}, \end{aligned} \tag{3.1}$$

and for $p = 2m+1, \ldots, m+n-1$,

$$\begin{aligned} |\det H_{p,2m+2n-p}(M \times N)| &= \prod_{i=0}^{2m} |\det H_{i,2m-i}(M)|^{\dim H_{p-i}(N)} \\ &\times \prod_{i=p-2m}^{p} |\det H_{i,2n-i}(N)|^{\dim H_{p-i}(M)}. \end{aligned} \tag{3.2}$$

Finally,

$$\begin{aligned} |\det H_{m+n,m+n}(M \times N)| &= \prod_{i=0}^{2m} |\det H_{i,2m-i}(M)|^{\dim H_{m+n-i}(N)} \\ &\times \prod_{i=n-m}^{n+m} |\det H_{i,2n-i}(N)|^{\dim H_{n+m-i}(M)}. \end{aligned} \tag{3.3}$$

It follows from (3.1) that

$$\begin{aligned} \prod_{p=0}^{2m} |\det H_{p,2m+2n-p}(M \times N)|^{(-1)^p} &= \prod_{p=0}^{2m}\prod_{i=0}^{p} |\det H_{i,2m-i}(M)|^{(-1)^p \dim H_{p-i}(N)} \\ &\times \prod_{p=0}^{2m}\prod_{i=0}^{p} |\det H_{i,2n-i}(N)|^{(-1)^p \dim H_{p-i}(M)}. \end{aligned}$$

By changing the order of the products, we obtain

$$\begin{aligned} &\prod_{p=0}^{2m}\prod_{i=0}^{p} |\det H_{i,2m-i}(M)|^{(-1)^p \dim H_{p-i}(N)} \\ &= \prod_{i=0}^{2m} |\det H_{i,2m-i}(M)|^{(-1)^i \sum_{j=0}^{2m-i} (-1)^j \dim H_j(N)}, \end{aligned} \tag{3.4}$$

$$\prod_{p=0}^{2m}\prod_{i=0}^{p}|\det H_{i,2n-i}(N)|^{(-1)^p \dim H_{p-i}(M)}$$

$$= \prod_{i=0}^{2m}|\det H_{i,2n-i}(N)|^{(-1)^i \sum_{j=0}^{2m-i}(-1)^j \dim H_j(M)}. \tag{3.5}$$

From (3.2) it follows that

$$\prod_{p=2m+1}^{m+n-1}|\det H_{p,2m+2n-p}(M\times N)|^{(-1)^p}$$

$$= \prod_{p=2m+1}^{m+n-1}\prod_{i=0}^{2m}|\det H_{i,2m-i}(M)|^{(-1)^p \dim H_{p-i}(N)}$$

$$\times \prod_{p=2m+1}^{m+n-1}\prod_{i=p-2m}^{p}|\det H_{i,2n-i}(N)|^{(-1)^p \dim H_{p-i}(M)}.$$

Change of the order of products results that

$$\prod_{p=2m+1}^{m+n-1}\prod_{i=0}^{2m}|\det H_{i,2m-i}(M)|^{(-1)^p \dim H_{p-i}(N)}$$

$$= \prod_{i=0}^{2m}|\det H_{i,2m-i}(M)|^{(-1)^i \sum_{j=2m-i+1}^{m+n-i-1}(-1)^j \dim H_j(N)}. \tag{3.6}$$

By changing the order of the products and using $n > 3m+2$, we get

$$\prod_{p=2m+1}^{m+n-1}\prod_{i=p-2m}^{p}|\det H_{i,2n-i}(N)|^{(-1)^p \dim H_{p-i}(M)}$$

$$= \prod_{i=1}^{2m}|\det H_{i,2n-i}(N)|^{(-1)^i \sum_{j=2m-i+1}^{2m}(-1)^j \dim H_j(M)}$$

$$\times \prod_{i=2m+1}^{n-m-1}|\det H_{i,2n-i}(N)|^{(-1)^i \sum_{j=0}^{2m}(-1)^j \dim H_j(M)}$$

$$\times \prod_{i=n-m}^{m+n-1}|\det H_{i,2n-i}(N)|^{(-1)^i \sum_{j=0}^{m+n-i-1}(-1)^j \dim H_j(M)}. \tag{3.7}$$

Finally, it follows from (3.3) that

$$\sqrt{|\det H_{m+n,m+n}(M\times N)|}^{\,(-1)^{m+n}}$$

$$= \left(\prod_{i=0}^{2m}|\det H_{i,2m-i}(M)|^{(-1)^{m+n} \dim H_{m+n-i}(N)}\right)^{1/2}$$

$$\times \left(\prod_{i=n-m}^{m+n}|\det H_{i,2n-i}(N)|^{(-1)^{m+n} \dim H_{m+n-i}(M)}\right)^{1/2}.$$

An easy computation gives us that

$$\left(\prod_{i=0}^{2m} |\det H_{i,2m-i}(M)|^{(-1)^{m+n}\dim H_{m+n-i}(N)}\right)^{1/2}$$
$$= \prod_{i=0}^{m-1} |\det H_{i,2m-i}(M)|^{(-1)^{m+n}\dim H_{m+n-i}(N)}$$
$$\times \sqrt{|\det H_{m,m}(M)|}^{(-1)^{m+n}\dim H_n(N)} \tag{3.8}$$

and

$$\left(\prod_{i=n-m}^{m+n} |\det H_{i,2n-i}(N)|^{(-1)^{m+n}\dim H_{m+n-i}(M)}\right)^{1/2}$$
$$= \prod_{i=n-m}^{n-1} |\det H_{i,2n-i}(N)|^{(-1)^{m+n}\dim H_{i-n+m}(N)}$$
$$\times \sqrt{|\det H_{n,n}(N)|}^{(-1)^{m+n}\dim H_m(M)} \tag{3.9}$$

Finally, the product of (3.4), (3.6), and (3.8) yield that

$$\left|\mathbb{T}(M, \{\mathbf{h}_p^M\}_{p=0}^{2m})\right|^{\chi(N)}. \tag{3.10}$$

Note also that the product of (3.5), (3.7), and (3.9) is

$$\left|\mathbb{T}(N, \{\mathbf{h}_p^N\}_{p=0}^{2n})\right|^{\chi(M)}. \tag{3.11}$$

This is the end of proof of Theorem 3.1. □

Clearly, by Theorem 3.1, we have

Theorem 3.2. *For $i = 1, \ldots, n$, let M_i be oriented closed connected $2m_i-$manifold ($m_i \geq 1$) and let $M = \times_{i=1}^n M_i$ be the product manifold. For $i = 1, \ldots, n$, and $p = 0, \ldots, 2m_i$, let $\mathbf{h}_{p,i}$ be a basis of $H_p(M_i)$. Then,*

$$\left|\mathbb{T}(M, \left\{\oplus_{|\alpha|=p}\mathbf{h}_{\alpha_1,1} \otimes \cdots \otimes \mathbf{h}_{\alpha_n,n}\right\}_{p=0}^{2m})\right| = \prod_{i=1}^{n} \left|\mathbb{T}(M_i, \{\mathbf{h}_{p,i}\}_{p=0}^{2m_i})\right|^{\chi(M)/\chi(M_i)}$$

where $m = \sum_{i=1}^n m_i$ and $|\alpha| = \sum_{i=1}^n \alpha_i$ is the length of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$.
□

4 Application

4.1 Compact Riemann surfaces

Using the symplectic chain complex, in [16] we proved the formula relating the Reidemeister torsion of closed Riemann surfaces with their period matrices. More precisely, let Σ_g be a closed oriented Riemann surface of genus $g \geq 2$. Let $\Gamma^g = \{\gamma_1, \ldots, \gamma_g, \gamma_{1+g}, \ldots, \gamma_{2g}\}$ be a canonical basis for $H_1(\Sigma_g)$, i.e. γ_r intersects γ_{r+g} once positively and does not intersect others. By applying Theorem 3.2, we have the following result as an application.

Theorem 4.1. *For $i = 1, \ldots, n$, let Σ_{g_i} be a closed oriented Riemann surface of genus $g_i \geq 2$, and let Γ^{g_i} be a canonical basis for $H_1(\Sigma_{g_i})$. Let $\Sigma = \times_{i=1}^n \Sigma_{g_i}$. For $p = 0, 1, 2$, and $i = 1, \ldots, n$, let $\mathbf{h}_{p,i}$ be a basis of $H_p(\Sigma_{g_i})$. Then,*

$$\left|\mathbb{T}(\Sigma, \{\oplus_{|\alpha|=p}\mathbf{h}_{\alpha_1,1} \otimes \cdots \otimes \mathbf{h}_{\alpha_n,n}\}_{p=0}^{2n})\right| = \prod_{i=1}^{n} \left|\frac{\det H_{0,2}(\Sigma_{g_i})}{\det \wp(\mathbf{h}^{1,i}, \Gamma^{g_i})}\right|^{\chi(\Sigma)/\chi(\Sigma_i)},$$

where $\mathbf{h}^{1,i}$ is the Poincaré dual basis of $H^1(\Sigma_{g_i})$ corresponding to the basis $\mathbf{h}_{1,i}$ of $H_1(\Sigma_{g_i})$. □

4.1.1 Grassmannian $G(d, N)$

Let $G(d, N)$ be the Grassmannian of $d-$dimensional linear subspaces of $\mathbb{C}^N$. As is well known that $G(d, N)$ is a smooth algebraic variety of complex dimension dn(with $n = N - d$), and that the Schubert cells stratify $G(d, N)$. The *Schubert varieties* are the closures of these cells. To be more precise, let $F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_N = \mathbb{C}^N$ be a complete flag of subspaces of $\mathbb{C}^N$ with $\dim F_i = i$, $i = 0, \ldots, N$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0)$ be a decreasing sequence of non-negative integers with $\lambda_1 \leq n$. Then, the Young diagram of the partition λ fits inside a $d \times n$ rectangle and this is denoted as $\lambda \subset (n^d)$.

The Schubert variety $X_\lambda(F_\bullet)$ associated to the complete flag $F_\bullet$ and the partition λ is

$$X_\lambda(F_\bullet) = \{\Lambda \in G(d, N) : \dim(\Lambda \cap F_{n+i-\lambda_i}) \geq i,\ i = 1, \ldots, d\}.$$

$X_\lambda(F_\bullet)$ is a codimension $|\lambda|$ closed subvariety of $G(d, N)$, where $|\lambda| = \sum \lambda_i$ is the weight of λ. From Poincaré duality it follows that $X_\lambda(F_\bullet)$ is associated to the Schubert class $\sigma_\lambda = [X_\lambda(F_\bullet)] \in H^{2|\lambda|}(G(d, N); \mathsf{Z})$. By the transitive action of $GL_N(\mathbb{C})$ on $G(d, N)$ and on the flags in $\mathbb{C}^N$, σ_λ does not depend on the flag $F_\bullet$ used to define X_λ.

$H^*(G(d, N); \mathbb{Z}) = \bigoplus_{\lambda \subset (n^d)} \mathbb{Z} \cdot \sigma_\lambda$ is a free abelian group generated by the Schubert classes. All odd dimensional cohomologies are zero and the Euler characteristic $\chi(G(d, N)) = \left(\ Nd \ \right)$. It follows from Schubert Duality theorem that for any λ and μ with $|\lambda| + |\mu| = dn$, we have $\int_{G(d,N)} \sigma_\lambda \sigma_\mu = \delta_{\hat{\lambda},\mu}$, where $\hat{\lambda} = (\lambda_{N-d-\lambda_d}, \ldots, \lambda_{N-d-\lambda_1})$ is the dual partition of λ.

In [16], we proved the formula for computing the Riedemeister torsion of Grassmannians.

Theorem 4.2. *Let $M = G(d, N)$ be the Grassmannian of d-dimensional linear subspaces of $\mathbb{C}^N$. For $p = 0, \ldots, 2m$, let $\mathbf{h}_p$ be a basis of $H_p(M)$, where $m = d(N - d)$. Then,*

(i) $\left|\mathbb{T}(M, \{\mathbf{h}_p\}_0^{2m})\right| = \prod_{p \in E_{m-1}} |\det H_{p,2m-p}(M)|$ *for m odd,*

(ii) $\left|\mathbb{T}(M, \{\mathbf{h}_p\}_0^{2m})\right| = \prod_{p \in E_{m-1}} |\det H_{p,2m-p}(M)| \sqrt{|\det H_{m,m}(M)|}$ *for m even,*

where E_{m-1} is the set of even numbers in $\{0,\dots,m-1\}$. □

Consider the complex projective space $\mathbb{CP}^m$. It is well known that for p even $H^p(\mathbb{CP}^m)$ is generated by ω_{FS}^p, where ω_{FS} is the Fubini-Study metric of $\mathbb{CP}^m$ and ω_{FS}^p is the p times wedge product of ω_{FS}. By applying Theorem 3.1, we have the following result for product of complex projective spaces.

Theorem 4.3. *Let $M = \times_{i=1}^n \mathbb{CP}^{m_i}$ be the cartesian product of $\mathbb{CP}^{m_i}$, $i=1,\dots,n$. For $i=1,\dots,n$, and $p=0,\dots,2m_i$, let $\mathbf{h}_{p,i}$ be a basis of $H_p(\mathbb{CP}^{m_i})$. Then,*

$$\begin{aligned}&\left|\mathbb{T}(M,\{\oplus_{|\alpha|=p}\mathbf{h}_{\alpha_1,1}\otimes\cdots\otimes\mathbf{h}_{\alpha_n,n}\}_{p=0}^{2m})\right|\\ &=\prod_{i=1}^n\left|\mathbb{T}(\mathbb{CP}^{m_i},\{\mathbf{h}_{p,i}\}_{p=0}^{2m_i})\right|^{(1+m_1)\cdots(1+m_n)/(1+m_i)}\end{aligned}$$

where $m=\sum_{i=1}^n m_i$. □

4.2 Quantum entanglement and Reidemeister torsion of manifolds of pure states

For this section the fundamental reference is [18].

Quantum entanglement constitutes the most important resource in quantum information processing such as quantum teleportation, dense coding, quantum cryptography, quantum error correction and quantum repeater. In this section we will discuss the geometry of quantum states and we will calculate Reidemeister torsion of manifolds of pure bipartite states with Schmidt ranks.

Let $\mathcal{H}$ be an n-dimensional complex Hilbert space. The space of density matrices on $\mathcal{H}$, $\mathcal{D}(\mathcal{H})$, is naturally stratified manifold with the stratification induced by the rank of state. The space of all density matrices with rank r, $\mathcal{D}^r(\mathcal{H})$, $r=1,2,\dots,n$, is a smooth and connected manifold of real dimension $2nr-r^2-1$. In particular, $\mathcal{D}^1(\mathcal{H})$ is the set of pure states. Every element of $\mathcal{D}(\mathcal{H})$ is a convex combination of points from $\mathcal{D}^1(\mathcal{H})$ which is a complex manifold. It is diffeomorphic to the $(n-1)$-dimensional complex projective space $\mathbb{CP}^{n-1}$ with a metric g determined by the inner product $\langle M,N\rangle=\frac{1}{2}\,\text{Tr}\,MN$ for density matrices M and N. So we can define the Hermitian structure h on $\mathcal{D}^1(\mathcal{H})$ by means of g. By straightforward calculation, we have

$$h^{(\alpha)}=\sum_{k,j}{h_{kj}}^{(\alpha)}dz_k\otimes d\overline{z_j},\quad h^{(\alpha)}=h_{|D_\alpha},\quad \alpha=1,2,\dots,n,$$

where

$${h_{kj}}^{(\alpha)}=\frac{(1+\sum_{l=1,l\neq\alpha}^n|z_l|^2)\delta_{kj}-z_j\overline{z_k}}{(1+\sum_{l=1,l\neq\alpha}^n|z_l|^2)},$$

D_α is the α-th coordinate chart with local complex coordinates z and $\bar{z}$. Hence it is clear that h differs from the Fubini-Study form on $\mathbb{CP}^{n-1}$ by a constant multiple.

The quantum entanglement concerns composite systems. Now we will give manifold structures and classification of pure bipartite states. Let $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ be the product Hilbert space, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are respectively n and m $(n\leq m)$ dimensional complex Hilbert spaces. We present the manifold constituted by the states with certain Schmidt ranks or with given Schmidt coefficients.

For any state $|x>= x \in \mathcal{H}$, x can be written as the sum of tensor products,

$$x = (x_1 \otimes y_1) + (x_2 \otimes y_2) + \cdots + (x_k \otimes y_k), \quad k \in \mathbb{N},$$

where $x_i \in \mathcal{H}_1$, $y_i \in \mathcal{H}_2$.

The last expression is linear independent if $x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k$ are linearly independent vectors, respectively.

Definition 4.1. We say that *length* of x is k if the expansion of x above is linearly independent.

The length is just the Schmidt rank because the Schmidt decomposition is a special expression of a linearly independent one. So the length of x in all linear independent expansions is the same. Let ${\mathcal{D}_k}^1(\mathcal{H})$, a submanifold of $\mathcal{D}^1(\mathcal{H})$, be the set of all mormalized pure states with length k,

$${\mathcal{D}_k}^1(\mathcal{H}) = \{x \in \mathcal{H} : \text{the length of} \quad x \quad \text{is} \quad k, \|x\|^2 = 1\}.$$

Then we have the following diffeomorphism from [18].

$${\mathcal{D}_k}^1(\mathcal{H}) \simeq \mathbb{G}(n,k) \times (\mathbb{CP}^{k^2-1} \backslash \overline{M}) \times \mathbb{G}(m,k),$$

where $\overline{M}$ is a hypersurface of $\mathbb{CP}^{k^2-1}$, $\mathbb{G}(n,k)$ is the Grassmannian manifold.

Definition 4.2. For any pure state $[e] \in {\mathcal{D}_k}^1(\mathcal{H})$, in the Schmidt representation $e = \mu_1 a_1 \otimes b_1 + \cdots \mu_k a_k \otimes b_k$, where a_i, b_i are orthonormal vectors in $\mathcal{H}_1, \mathcal{H}_2$, respectively, μ_i are called *Schmidt coefficients* of e.

We have another result from [18]. Let ${\mathcal{D}_k}^1(\mu_1, \ldots, \mu_k)$ of ${\mathcal{D}_k}^1(\mathcal{H})$ of pure states with the Schmidt coefficients $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ is a submanifold of real dimension $2k(m+n-k) - k - 1$, which is diffeomorphically equivalent to a manifold

$$(\mathbb{CP}^{n-1} \times \mathbb{CP}^{m-1}) \times \cdots \times (\mathbb{CP}^{n-k} \times \mathbb{CP}^{m-k}) \times \mathbb{T}^{k-1},$$

where $\mathbb{T}^{k-1}$ is a torus of real dimension $k-1$. Let $S = (\mathbb{CP}^{n-1} \times \mathbb{CP}^{m-1}) \times \cdots \times (\mathbb{CP}^{n-k} \times \mathbb{CP}^{m-k}) \times \mathbb{T}^{k-1}$ and $S' = \mathbb{G}(n,k) \times (\mathbb{CP}^{k^2-1} \backslash M) \times \mathbb{G}(m,k)$.

Then we can give our result.

Theorem 4.4. *The Reidemeister torsion of the product manifolds S and S' is equal to 1 with respect to any homological bases.*

Proof. Since $\chi(\mathbb{T}^{k-1}) = 0$ then the Reidemeister torsion of S is 1 for any homological basis. On the other hand $(\mathbb{CP}^{k^2-1} \backslash M)$ is one dimensional and its Euler characteristics is 0 so the Reidemeister torsion of S' is 1. □

The manifolds S and S' represent separable cases of pure states. We know that entanglement measure of separable cases is zero. We claim that there is a relation between Reidemeister torsion and entanglement measure of bi-partite states. So the Reidemeister torsion of manifolds S and S' is equal to exponential of minus multiple of entanglement measure of states.

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