Variational problems of some second order Lagrangians given by Pfaff forms

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Abstract. In this paper we study the dynamics of some second order Lagrangians that come from Pfaff forms, i.e. differential forms on tangent bundles. In the non-singular case, mainly considered in the paper, the generalized Euler-Lagrange equation is a third order differential equation. We prove that the solutions of the differential equations of motion of a charge in a field and the Euler equations of a rigid body can be obtained as particular solutions of suitable Pfaff forms, with non-negative second variations along their solutions. A non-standard Hamiltonian approach is also considered in the non-singular case, using energy functions associated with suitable semi-sprays.

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1 Introduction

The Euler-Lagrange equation of a first order Lagrangian is one of the most known and widely used variational equation in mathematics, mechanics and physics. Its solutions are the critical curves of the action defined by the Lagrangians on curves; in the case when the Lagrangian comes from a Riemannian, a non-Riemannian or a Finslerian metric, these solutions are known as geodesics, since they locally minimise the distance. The second variation decides if the solution is an extreme (see [11, Ch.1, Sect.2]). The local expression of the first order Euler-Lagrange equation contains the second derivatives and, in the case of a hyperregular Lagrangian, its solutions are integral curves of a global second order differential equation.

In this paper we consider an other type of dynamics, where the local expression of a generalized Euler-Lagrange equation contains the third derivatives and, in the regular case, the solutions are integral curves of a global third order differential equation. The generalized Euler-Lagrange equation is obtained by a variational method on an action of a second order Lagrangian defined by a Pfaff form (i.e. a differential 1-form on the tangent space of a manifold).

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According to [3, Sect. 6.3], some special regularity conditions can be considered in this case and they are studied in detail in our paper. The second order Lagrangian is linear in the second order velocities (accelerations), as in [8]. The Euler-Lagrange equation of a higher order Lagrangian was firstly described by Ostrogradski and studied, for example, in the monographs [13] or in [5]. The second order Lagrangian considered in our paper, comes from a Pfaff form and has a null Hessian. Besides its Euler-Lagrange equation, we are interested also in the second derivative of the variation, deduced in a classical variational way. We prove that the solutions of the differential equations of motion of a charge in a field (formulas (17) in [4, Section 17]) and the Euler equations of a rigid body [6] can be obtained as particular solutions of suitable Pfaff forms, with some non-negative second variations.

A dual Hamiltonian approach to higher order Lagrangians, particularly on second order, was first used also by Ostrogradski (see, for example [13, 5]). But, technically, this approach can not be used in the particular case when the Lagrangian is linear in the second order velocities (or accelerations); the second order momenta can not be related to the accelerations, since the Legendre transformation is degenerated on fibers. We use here an unusual Hamiltonian approach, as in [8, 10] in the case of a second and higher order Lagrangian, linear in the second or higher order velocities. The line in [8] uses a general Hamiltonian duality that is extended in [9] to the higher order case. More exactly, we consider energy functions of the Pfaff form, each associated with a suitable semi-spray; then we obtain the local solutions of the generalized Euler-Lagrange equation in some particular but relevant cases, given by the integral curves of the second order differential equations coming from the semi-sprays (see Propositions 3.3 and 3.4).

2 Actions on curves given by Pfaff forms

A *Pfaff form* is a differentiable 1-form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$:

$$\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i. \tag{2.1}$$

We define the action I_0 of ω on a curve $\gamma : I = [a, b] \to TM$ by the formula

$$I_0(\gamma) = \int_a^b (\omega_0 + \omega_i \frac{dx^i}{dt} + \bar{\omega}_i \frac{d^2 x^i}{dt^2}) dt.$$

If $\omega_i = \bar{\omega}_i \equiv 0$, $\omega_0 = L : \mathbb{R} \times TM \to \mathbb{R}$, then it is easy to see that I_0 is the action I for the Lagrangian L.

Denoting

$$L'(t, x^i, y^{(1)i}, y^{(2)i}) = \omega_0(t, x^i, y^{(1)i}) + \omega_i(t, x^i, y^{(1)i}) y^{(1)i} + \bar{\omega}_i(t, x^i, y^{(1)i}) y^{(2)i},$$

we obtain that I_0 is in fact the variation of a second order Lagrangian, affine in the second order velocities (accelerations), as studied in [8].

Let us consider two points x, $y \in M$ and $\gamma_0 = (x_0^i(t))$ a curve joining x and y, i.e. $x_0^i(0) = x$ and $x_0^i(1) = y$. Let us consider *allowed variations* of γ_0, as variations by curves joining x and y, having the local form $\gamma_\varepsilon = (x_\varepsilon^i(t))$, where $x_\varepsilon^i(t) = x_0^i(t) + \varepsilon h^i(t)$ and satisfying the conditions

$$h^i(a) = h^i(b) = 0, \frac{dh^i}{dt}(a) = \frac{dh^i}{dt}(b) = 0. \tag{2.2}$$

A Pfaff form can be related to a second order dynamical form considered in [3]. According to [3, Section 2], a *first order dynamical form* on the bundle $Y = \mathbb{R} \times M \to M$ is a one contact and horizontal two form ν on $J^1(Y)$, having the local form $\nu = \nu_i(t, x^j, y^j) dx^i \wedge dt + \bar{\nu}_i(t, x^j, y^j) dy^i \wedge dt$. Obviously a first order dynamical form is equivalent to give a Pfaff form with $\omega_0 = 0$. An advantage to use Pfaff forms is having the Lagrangian forms (i.e. $\omega_i = \bar{\omega}_i = 0$) in the same setting. An other motivation to study Pfaff forms is given by the possibility to use their actions on curves through some second order Lagrangians that vertical Hessians are null.

One have:

$$\begin{aligned}
&\frac{d}{d\varepsilon} I_0(\gamma_\varepsilon)\,|_{\varepsilon=0} \\
&= \int_a^b \left(\frac{\partial \omega_0}{\partial x^i} h^i + \frac{\partial \omega_0}{\partial y^i} \frac{dh^i}{dt}\right) dt + \int_a^b \left(\frac{\partial \omega_j}{\partial x^i} h^i + \frac{\partial \omega_j}{\partial y^i} \frac{dh^i}{dt}\right) \frac{dx_0^j}{dt} dt + \\
&\int_a^b \omega_i \frac{dh^i}{dt} dt + \int_a^b \left(\frac{\partial \bar{\omega}_j}{\partial x^i} h^i + \frac{\partial \bar{\omega}_j}{\partial y^i} \frac{dh^i}{dt}\right) \frac{d^2 x_0^j}{dt^2} dt + \int_a^b \bar{\omega}_i \frac{d^2 h^i}{dt^2} dt.
\end{aligned}$$

2.1 The case of non-singular Pfaff forms

A Pfaff form ω given locally by (2.1) is *regular* if the vertical 2-form

$$\left(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j}\right) dy^i \wedge dy^j \tag{2.3}$$

is regular, i.e. the matrix $\left(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j}\right)_{i,j}$ is non-singular. If the vertical 2-form 2.3 is not vanishing, i.e. its matrix is only non-null, we say that the Pfaff form is *non-singular*.

Let us consider now, for a singular curve γ, a variation that satisfies the conditions (2.2). We have:

$$\begin{aligned}
&\frac{d}{d\varepsilon} I_0(\gamma_\varepsilon)\,|_{\varepsilon=0} \\
&= \frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_j}{\partial x^i} \frac{dx_0^j}{dt} + \frac{\partial \bar{\omega}_j}{\partial x^i} \frac{d^2 x_0^j}{dt^2} - \frac{d}{dt}\left(\frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} \frac{dx_0^j}{dt} + \omega_i + \frac{\partial \bar{\omega}_j}{\partial y^i} \frac{d^2 x_0^j}{dt^2}\right) + \frac{d^2}{dt^2} \bar{\omega}_i = 0.
\end{aligned} \tag{2.4}$$

If the Pfaff form ω is non-singular, then the equation is of third order. For a regular Pfaff form one can prove the following result.

Proposition 2.1. *If the Pfaff form ω is regular, then the solutions of the generalized Euler-Lagrange equation (2.4) are the same with the solutions of a third order equation given by a global second order semi-spray $S : T^2 M \to T^3 M$.*

We omit the proof, since it is not relevant for the rest of the paper. Some important classes of Pfaff forms are when $\omega_0 = 0$ (for example, the case of time independent Lagrangians $L = L(x^i, y^i)$) and when $\omega_0 = \omega_i = 0$ (for example, this is the case of $L = L(y^i)$);

If $\omega = \bar{\omega}_j(y^i)dy^j$, then there are constants c_i so that the equation (2.4) has the form

$$(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_j}{\partial y^i})\frac{d^2 x_0^j}{dt^2} = c_i.$$

Example 1. Let us consider coordinates (x, y) on $\mathbb{R}^2$ and (x, y, X, Y) on $\mathbb{R}^4 = T\mathbb{R}^2$. Let $\omega = YdX - XdY$. The equations (2.4) have the form:

$$-\frac{d}{dt}\left(\frac{d^2 y}{dt^2}\right) - \frac{d^2}{dt^2}\left(\frac{dy}{dt}\right) = 0,$$

that simply implies $\frac{d^3 y}{d^3 t} = 0$, and $\frac{d^3 x}{d^3 t} = 0$. The general solution is: $x(t) = C_1 + C_2 t + C_3 t^2$, $y(t) = C_4 + C_5 t + C_6 t^2$.

Example 2. In $\mathbb{R}^2$, as in Example 1. above, let $\omega = -ydx + xdy + YdX - XdY$. The equations (2.4) have the form

$$\frac{dy}{dt} + \frac{d^3 y}{d^3 t} = 0, \frac{dx}{dt} + \frac{d^3 x}{d^3 t} = 0.$$

The general solution is $x(t) = c_1 \cos t + c_3 \sin t + c_5$, $x(t) = c_2 \cos t + c_4 \sin t + c_6$, thus the integral curves are ellipses and straight lines. If $t_1 < t_2 < t_3$ are given, then for every three distinct points $A_\alpha(x_\alpha, y_\alpha) \in \mathbb{R}^2$, $\alpha = \overline{1, 3}$, there is a unique integral curve in the family that contains the three points, i.e. $t \to (x(t), y(t))$, $x(t_\alpha) = x_\alpha$, $y(t_\alpha) = y_\alpha$, $\alpha = \overline{1, 3}$. Notice that this feature characterizes the dynamics generated by a third order differential equation, while in general, an integral curve is determined by three distinct points. Let us notice that for a second order differential equation, an integral curve is determined, in general, by two distinct points.

Let us consider now the case $\dim M = 1$. In this case, since the only sqew-symmetric matrix of first order is the null matrix, the equation (2.4) is always of second order, for every Pfaff form $\tilde{\omega} = \omega_0 dt + \omega dx + \bar{\omega} dy$.

In the case when the local functions ω_0, ω and $\bar{\omega}$ do not depend on y, the Euler equation has the form

$$2\frac{\partial \bar{\omega}}{\partial x}\frac{d^2 x_0}{dt^2} + \frac{\partial^2 \bar{\omega}}{\partial x^2}\left(\frac{dx_0}{dt}\right)^2 + 2\frac{\partial^2 \bar{\omega}}{\partial x \partial t}\frac{dx_0}{dt} + \frac{\partial \omega_0}{\partial x} - \frac{\partial \omega}{\partial t} + \frac{\partial^2 \bar{\omega}}{\partial t^2} = 0. \tag{2.5}$$

According to [2, Section 2.], a *standard* Lagrangian has the form

$$L(t, x, y) = \frac{1}{2}P(x, t)y^2 + Q(x, t)y + R(x, t). \tag{2.6}$$

Its Euler-Lagrange equation is $2Px'' + P_x (x')^2 + 2P_t x' + 2(Q_t - R_x) = 0$, where subscripts x, t denote partial derivatives and $x' = \frac{dx}{dt}$, $x'' = \frac{d^2 x}{dt^2}$. In [2, Proposition 2.1.] one prove that there is a standard Lagrangian description (2.6) for a second order equation

$$x'' + a(t, x)(x')^2 + b(t, x)x' + c(t, x) = 0$$

iff $b_x = 2a_t$; then $P = \exp(2\int^x a(t,\ s)ds)$ and $R = \int^x (Q_t(t, s) - c(t, s)P(t, s))ds$, where $Q = Q(x, t)$ is an arbitrary function. The following result can be proved by a straightforward verification.

Proposition 2.2. *The equation (2.5) admits a standard Lagrangian description.*

We calculate now the second derivative of $I(\gamma_\varepsilon)$ (the second variation). Taking into account of the conditions (2.2), we obtain:

$$\begin{aligned}
&\frac{d^2}{d\varepsilon^2} I_0(\gamma_\varepsilon)\,|_{\varepsilon=0} \\
= \;& \frac{1}{2}\int_a^b \Big(\frac{\partial^2\omega_0}{\partial x^i\partial x^j}h^ih^j - \frac{d}{dt}\Big(\frac{\partial^2\omega_0}{\partial x^i\partial y^j} + \frac{\partial^2\omega_0}{\partial y^i\partial x^j}\Big)h^ih^j + \frac{\partial^2\omega_0}{\partial y^i\partial y^j}\frac{dh^i}{dt}\frac{dh^j}{dt}\Big)dt + \\
& \frac{1}{2}\int_a^b \Big(\frac{\partial^2\omega_k}{\partial x^i\partial x^j}\frac{dx_0^k}{dt}h^ih^j - \frac{d}{dt}\Big(\Big(\frac{\partial^2\omega_k}{\partial x^i\partial y^j} + \frac{\partial^2\omega_k}{\partial y^i\partial x^j}\Big)\frac{dx_0^k}{dt}\Big)h^ih^j + \frac{\partial^2\omega_k}{\partial y^i\partial y^j}\frac{dx_0^k}{dt}\frac{dh^i}{dt}\frac{dh^j}{dt}\Big)dt + \\
& \frac{1}{2}\int_a^b \Big(-\frac{d}{dt}\Big(\frac{\partial\omega_j}{\partial x^i} + \frac{\partial\omega_i}{\partial x^j}\Big)h^ih^j + \Big(\frac{\partial\omega_j}{\partial y^i} + \frac{\partial\omega_i}{\partial y^j}\Big)\frac{dh^i}{dt}\frac{dh^j}{dt}\Big)dt + \\
& \frac{1}{2}\int_a^b \Big(\frac{\partial^2\bar\omega_k}{\partial x^i\partial x^j}\frac{d^2x_0^k}{dt^2}h^ih^j - \frac{d}{dt}\Big(\Big(\frac{\partial^2\bar\omega_k}{\partial x^i\partial y^j} + \frac{\partial^2\bar\omega_k}{\partial y^i\partial x^j}\Big)\frac{d^2x_0^k}{dt^2}\Big)h^ih^j + \frac{\partial^2\bar\omega_k}{\partial y^i\partial y^j}\frac{d^2x_0^k}{dt^2}\frac{dh^i}{dt}\frac{dh^j}{dt}\Big)dt + \\
& \frac{1}{2}\int_a^b \Big(-\frac{d^2}{dt^2}\Big(\frac{\partial\bar\omega_j}{\partial x^i} + \frac{\partial\bar\omega_i}{\partial x^j}\Big)h^ih^j - 2\Big(\frac{\partial\bar\omega_j}{\partial x^i} + \frac{\partial\bar\omega_i}{\partial x^j}\Big)\frac{dh^i}{dt}\frac{dh^j}{dt} - \frac{d}{dt}\Big(\frac{\partial\bar\omega_j}{\partial y^i} + \frac{\partial\bar\omega_i}{\partial y^j}\Big)\frac{dh^i}{dt}\frac{dh^j}{dt}\Big)dt.
\end{aligned}$$

We particularize below this long formula in some important particular cases.

2.2 Some particular cases

If $\omega = \bar\omega_j(y^i)dy^j$, then there are the constants c_i such that the equation (2.4) becomes

$$\left(\frac{\partial\bar\omega_j}{\partial y^i} - \frac{\partial\bar\omega_i}{\partial y^j}\right)\frac{d^2x_0^j}{dt^2} = c_i. \tag{2.7}$$

In this case, the second variation is

$$\frac{d^2}{d\varepsilon^2} I_0(\gamma_\varepsilon)\,|_{\varepsilon=0} = \Big(\frac{\partial^2\bar\omega_k}{\partial y^i\partial y^j} - \frac{\partial^2\bar\omega_i}{\partial y^k\partial y^j} - \frac{\partial^2\bar\omega_j}{\partial y^k\partial y^i}\Big)\frac{d^2x_0^k}{dt^2}\frac{dh^i}{dt}\frac{dh^j}{dt}.$$

If $m = \dim M$, then we obtain the quadratic form

$$\left(\left(\frac{\partial^2\bar\omega_k}{\partial y^i\partial y^j} - \frac{\partial^2\bar\omega_i}{\partial y^k\partial y^j} - \frac{\partial^2\bar\omega_j}{\partial y^k\partial y^i}\right)\frac{d^2x_0^k}{dt^2}\right)_{i,j=\overline{1,m}}. \tag{2.8}$$

Let us consider now two examples. Even the equations of motion used in the following examples have the second order, their integral curves are obtained from some suitable equations of Pfaff forms. First, we consider a system that has the form

$$\begin{cases} \frac{dy^1}{dt} = c_1 + cy^2 - by^3 \\ \frac{dy^2}{dt} = c_2 + ay^3 - cy^1 \\ \frac{dy^3}{dt} = c_3 + by^1 - ay^2 \end{cases} \tag{2.9}$$

where the coefficients are constants. Notice that the equations of motion of a charge in a field (formulas (17) in [4, Section 17]) have this form.

Proposition 2.3. *There is a Pfaff form $\omega = \bar{\omega}_j(y^i)dy^j$ on $\mathbb{R}^3$ such that the solutions of the system (2.9) are also solutions of the generalized Euler-Lagrange equation (2.7).*

Proof. One can check that we can consider the Pfaff form $\omega = \omega_1 dy^1 + \omega_2 dy^2 + \omega_3 dy^3$, where

$$\begin{aligned}
(2.10)\qquad \omega_1(y^j) &= c_3 y^2 + b y^1 y^2 + a y^1 y^3, \omega_2(y^j) = c_1 y^3 + a y^1 y^2 + c y^2 y^3,\\
(2.11)\qquad \omega_3(y^j) &= -c_2 y^1 + c y^1 y^3 + b y^2 y^3 . \Box
\end{aligned}$$

One can find other Pfaff forms with the property asked by Proposition 2.3, looking for $\bar{\omega}_i = B_{ij} y^j + \frac{1}{2} C_{ijk} y^i y^j$, $C_{ijk} = C_{ikj}$ (all constants).

Let us investigate the second derivative of the variation. The matrix (2.8) has in this case the form

$$diag(2a\frac{dy^1}{dt} - 2ac_1 - 2cc_3 - 2bc_2, 2b\frac{dy^2}{dt} - 2bc_2 - 2cc_3 - 2ac_1, 2c\frac{dy^3}{dt} - 2cc_3 - 2bc_2 - 2ac_1).$$

Let us take, as in [4, Section 17], $(a, b, c) = (0, 0, 1)$ (the z-axis) and $(c_1, c_2, c_3) = (0, c_2, c_3)$ (in the YZ plane), we obtain $z'' = c_3$, thus the above matrix becomes $diag(-2c_3, -2c_3, 0)$. It follows that along the solutions $(\frac{d^2}{d\varepsilon^2} I(\gamma_\varepsilon)|_{\varepsilon=0})$ has a non-positive or non-negative sign, according to $-c_3$; we can find a suitable c_3 according to $z(t) = \frac{1}{2}c_3 t^2 + \alpha t + \beta$.

The second example is constructed using the Euler equations of a rigid body, as follows. Let us consider a system of second order equations having the form

$$x'' = \beta_1 y' z', y'' = \beta_2 z' x', z'' = \beta_3 x' y'. \tag{2.12}$$

According to [6] the Euler equations of a rigid body have the above form (2.12), where

$$\beta_1 = \frac{I_2 - I_3}{I_1}, \beta_2 = \frac{I_3 - I_1}{I_2}, \beta_3 = \frac{I_1 - I_2}{I_3}. \tag{2.13}$$

Proposition 2.4. *There is a Pfaff form $\omega = \bar{\omega}_j(y^i)dy^j$ on $\mathbb{R}^3$ such that the solutions of the system (2.12) are solutions of the generalized Euler-Lagrange equation (2.7) too.*

Proof. One can check that we can consider the Pfaff form with

$$\begin{aligned}
\omega_1(y^j) &= \tfrac{\beta_3}{4} y^1 \left(y^2\right)^2 - \tfrac{\beta_2}{4} y^1 \left(y^3\right)^2, \omega_2(y^j) = \tfrac{\beta_1}{4} y^2 \left(y^3\right)^2 - \tfrac{\beta_3}{4} y^2 \left(y^1\right)^2,\\
\omega_3(y^j) &= \tfrac{\beta_2}{4} y^3 \left(y^1\right)^2 - \tfrac{\beta_1}{4} y^3 \left(y^2\right)^2 . \Box
\end{aligned}$$

One can find other Pfaff forms with the property asked by Proposition 2.3, looking for

$$\bar{\omega}_1 = A_1 y^1 (y^2)^2 + B_1 y^1 (y^3)^2, \bar{\omega}_2 = A_2 y^2 (y^3)^2 + B_2 y^2 (y^1)^2, \bar{\omega}_3 = A_3 y^3 (y^1)^2 + B_3 y^3 (y^2)^2,$$

where A_i and B_i are constants, and $2(A_2 - B_3) = \beta_1$, $2(A_3 - B_1) = \beta_2$ and $2(A_1 - B_2) = \beta_3$.

In the explicit case of the Euler equation of the rigid body, when (2.13) holds, one can take

$$\begin{aligned}
\omega_1(y^1,y^2,y^3) &= \tfrac{I_1}{2I_3} y^1 \left(y^2\right)^2 + \tfrac{I_1}{2I_2} y^1 \left(y^3\right)^2 + \tfrac{\delta_1 (I_3-I_2)}{6} \left(y^1\right)^3, \\
\omega_2(y^1,y^2,y^3) &= \tfrac{I_2}{2I_1} y^2 \left(y^3\right)^2 + \tfrac{I_2}{2I_3} y^2 \left(y^1\right)^2 + \tfrac{\delta_2 (I_1-I_3)}{6} \left(y^2\right)^3, \\
\omega_3(y^1,y^2,y^3) &= \tfrac{I_3}{2I_2} y^3 \left(y^1\right)^2 + \tfrac{I_3}{2I_1} y^3 \left(y^2\right)^2 + \tfrac{\delta_3 (I_2-I_1)}{6} \left(y^3\right)^3.
\end{aligned}$$

Let us investigate the second derivative of the variation. The matrix (2.8) has the form

$$\frac{1}{I_1 I_2 I_3} \begin{pmatrix} \frac{y_1y_2y_3a_1}{I_1I_2I_3} & \frac{y_3b_1}{I_1I_2I_3} & \frac{y_2b_2}{I_1I_2I_3} \\ \frac{y_3b_1}{I_1I_2I_3} & \frac{y_1y_2y_3a_2}{I_1I_2I_3} & \frac{y_1b_3}{I_1I_2I_3} \\ \frac{y_2b_2}{I_1I_2I_3} & \frac{y_1b_3}{I_1I_2I_3} & \frac{y_1y_2y_3a_3}{I_1I_2I_3} \end{pmatrix},$$

where $a_1 = (I_2 - I_3)\left(I_1^2 + \delta_1 I_2^2 I_3 - \delta_1 I_2 I_3^2\right)$, $a_2 = (I_3 - I_1)\left(\delta_2 I_1 I_3^2 - \delta_2 I_1^2 I_3 + I_2^2\right)$, $a_3 = (I_1 - I_2)\left(\delta_3 I_1^2 I_2 - \delta_3 I_1 I_2^2 + I_3^2\right)$, $b_1 = I_1^3 y_1^2 - I_3 I_1^2 y_1^2 - I_2^3 y_2^2 + I_3 I_2^2 y_2^2$, $b_2 = I_1^3 y_1^2 - I_2 I_1^2 y_1^2 - I_3^3 y_3^2 + I_2 I_3^2 y_3^2$, $b_3 = I_2^3 y_2^2 - I_1 I_2^2 y_2^2 - I_3^3 y_3^2 + I_1 I_3^2 y_3^2$. It is easy to see that for δ_i large enough, the above matrix comes from a positively or negatively definite quadratic form, according to the sign of $y_1 y_2 y_3$.

Proposition 2.5. *Let us consider the system (2.12) with the coefficients (2.13) coming from the Euler equation of the rigid body, in an bounded domain U, where $y^1y^2y^3 \neq 0$.*

Then there is a Pfaff form $\omega = \bar{\omega}_j(y^i)dy^j$ defined for $(y^i) \in U$, such that the solutions of the system (2.12) are extremal solutions for the generalized Euler-Lagrange equation (2.7), i.e. the second variation has a constant sign along these solutions.

3 A Hamiltonian description of non-singular Pfaff forms

In this section we study the solutions of the generalized Euler-Lagrange equations for non-singular Pfaff forms, considering an energy function associated with a Pfaff form and a semi-spray.

A section $S : \mathbb{R} \times TM \to \mathbb{R} \times T^2M$ of the affine bundle $\mathbb{R} \times T^2M \overset{\pi_2}{\to} \mathbb{R} \times TM$ is called a (first order) *semi-spray* on TM. It can be regarded as well as a (time dependent) vector field Γ_0 on the manifold TM, since $T^2M \subset TTM$.

Let ω be a Pfaff form as in (2.1) and $S : \mathbb{R} \times TM \to \mathbb{R} \times T^2M$, $(t, x^i y^i) \overset{S}{\to} (t, x^i, y^i, S^i(t, x^j, y^j))$ be a semi-spray. We consider *the energy*:

$$\mathcal{E}_S : T^*TM \to \mathbb{R}, \mathcal{E}(x^i, y^i, q_i, p_i) = (q_i + \omega_i)y^i + 2(p_i + \bar{\omega}_i)S^i + \omega_0.$$

Proposition 3.1. *The energy $\mathcal{E}_S$ is a global function on T^*TM.*

Proof. One have $(q_{i'} + \omega_{i'})y^{i'} + 2(p_{i'} + \bar{\omega}_{i'})S^{i'} + \omega_0 = q_i y^i - y^i \frac{\partial y^{i'}}{\partial x^i} p_{i'} + y^i(\omega_i - \frac{\partial y^{i'}}{\partial x^i}\bar{\omega}_{i'}) + 2(p_i + \bar{\omega}_i)S^i + \frac{1}{2}(p_{i'} + \bar{\omega}_{i'})\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} \overset{i}{y}\, y^j + \omega_0 = (q_i + \omega_i)y^i + 2(p_i + \bar{\omega}_i)S^i + \omega_0$.
□

Let

$$X_{\mathcal{E}_S}=\frac{\partial \mathcal{E}_S}{\partial q_i}\frac{\partial}{\partial x^i}+\frac{\partial \mathcal{E}_S}{\partial p_i}\frac{\partial}{\partial y^i}-\frac{\partial \mathcal{E}_S}{\partial x^i}\frac{\partial}{\partial q_i}-\frac{\partial \mathcal{E}_S}{\partial y^i}\frac{\partial}{\partial p_i} \tag{3.1}$$

be the Hamiltonian vector field of the Hamiltonian $\mathcal{E}_S$ according to the canonical symplectic form on T^*TM and let $\mathcal{M}\subset T^*TM$ be the submanifold defined by $p_i+\bar{\omega}_i=0$.

Proposition 3.2. *If an integral curve of $X_{\mathcal{E}_S}$ is tangent to the submanifold $\mathcal{M}$, then the curve projects to an integral curve of the generalized Euler-Lagrange equation of the Pfaff form ω.*

Proof. Let $t\to(x^i(t),y^i(t),q_i(t),p_i(t))$ be an integral curve of $X_{\mathcal{E}_S}$. By (3.1):

$$\begin{aligned}
&\frac{dx^i}{dt}=y^i,\ \frac{dy^i}{dt}=2S^i,\\
&\frac{dq_i}{dt}=-\frac{\partial\omega_j}{\partial x^i}y^j-2\frac{\partial\bar{\omega}_j}{\partial x^i}S^j-2(p_j+\bar{\omega}_j)\frac{\partial S^j}{\partial x^i}-\frac{\partial\omega_0}{\partial x^i},\\
&\frac{dp_i}{dt}=-\frac{\partial\omega_j}{\partial y^i}y^j-q_i-\omega_i-2\frac{\partial\bar{\omega}_j}{\partial y^i}S^j-2(p_j+\bar{\omega}_j)\frac{\partial S^j}{\partial y^i}-\frac{\partial\omega_0}{\partial y^i}.
\end{aligned} \tag{3.2}$$

Since $p_i=-\bar{\omega}_i$, $\frac{dx^i}{dt}=y^i$ and $\frac{d^2x^i}{dt^2}=2S^i$, one have

$$\frac{d^2\bar{\omega}_i}{dt^2}=\frac{d}{dt}(\frac{\partial\omega_j}{\partial y^i}\frac{dx^j}{dt}+\omega_i+\frac{\partial\bar{\omega}_j}{\partial y^i}\frac{d^2x^j}{dt^2}+\frac{\partial\omega_0}{\partial y^i})-\frac{\partial\omega_j}{\partial x^i}\frac{dx^j}{dt}-\frac{\partial\bar{\omega}_j}{\partial x^i}\frac{d^2x^j}{dt^2}-\frac{\partial\omega_0}{\partial x^i}.$$

Then the Euler-Lagrange equation (2.4) holds. □

Using the definition of the submanifold $\mathcal{M}$, then its tangent subspace is generated by the local frame of vectors

$$\left\{\frac{\partial}{\partial x^i}-\frac{\partial\bar{\omega}_j}{\partial x^i}\frac{\partial}{\partial p_j},\frac{\partial}{\partial y^i}-\frac{\partial\bar{\omega}_j}{\partial y^i}\frac{\partial}{\partial p_j},\frac{\partial}{\partial q_j}\right\}.$$

Thus the vector $X_{\mathcal{E}_S}$ is tangent to $\mathcal{M}$ in a certain point of T^*TM having as local coordinates $((x^i,y^i,q_i,p_i=-\bar{\omega}_i(x^j,y^j))$ iff

$$2S^i\left(\frac{\partial\bar{\omega}_j}{\partial y^i}-\frac{\partial\bar{\omega}_i}{\partial y^j}\right)+y^i\frac{\partial\bar{\omega}_j}{\partial x^i}-\frac{\partial\omega_i}{\partial y^j}y^i-q_j-\omega_j-\frac{\partial\omega_0}{\partial y^j}=0. \tag{3.3}$$

Example 3. Let us consider the setting of Example 1. and $\omega=(X+Y)dX+YdY$. We use below the notations $x=x^1$, $y=x^2$, $X=y^1$, $Y=y^2$. Let $\{S^i(x^j,y^j)\}_{i=\overline{1,2}}$ be some real functions considered as the components of a semi-spray S on $\mathbb{R}^2$. Then $\mathcal{E}_S(x^i,y^i,q_i,p_i)=(q_i+\omega_i)y^i+2(p_i+\bar{\omega}_i)S^i$ and the differential system that gives the integral curves of $X_{\mathcal{E}_S}$ has the form

$$\frac{dx^i}{dt}=y^i,\ \frac{dy^i}{dt}=2S^i,\ \frac{dq_i}{dt}=0,\ \frac{dp_i}{dt}=-q_i-2\frac{\partial\bar{\omega}_j}{\partial y^i}S^j-2(p_j+\bar{\omega}_j)\frac{\partial S^j}{\partial y^i}.$$

We have $q_i(t)=q_i^0$ ($=const.$) along the solution. The condition (3.3) gives $S^1=\frac{1}{4}q_2^0$ and $S^2=-\frac{1}{4}q_1^0$. Taking a semi-spray on $\mathbb{R}^2$ with constant coefficients, we obtain the equations $\frac{dx^i}{dt}=y^i$, $\frac{\partial y^i}{dt}=c^i$. It follows that $\frac{d^3x^i}{dt^2}=0$. Thus all the solutions of (2.4) are obtained in this case for a suitable semi-spray on $\mathbb{R}^2$ with constant coefficients.

The above example can be extended as follows.

Proposition 3.3. *Let us suppose that there are some coordinates such that the local coefficients of a regular Pfaff form ω depend only on (y^i). Then there are some local semi-sprays whose local coefficients depend only on (y^i), such that:*

1. *An integral curve Γ of $X_{\mathcal{E}_S}$ that intersects $\mathcal{M}$ is included in $\mathcal{M}$.*
2. *An integral curve Γ projects on M on a curve γ that is a solution of the generalized Euler-Lagrange equation of the Pfaff form ω.*
3. *The curve γ is obtained as an integral curve of a semi-spray whose local coefficients depend only on (y^i).*

Proof. If the local coefficients of ω depend only on (y^i), then the second equation (3.2) implies $q_i(t) = q_i^0$ $(= const.)$ and the compatibility condition (3.3) becomes

$$S^i\left(\frac{\partial\bar{\omega}_j}{\partial y^i} - \frac{\partial\bar{\omega}_i}{\partial y^j}\right) - \frac{\partial\omega_i}{\partial y^j}y^i - q_j^0 - \omega_j - \frac{\partial\omega_0}{\partial y^j} = 0.$$

Since the Pfaff form is regular, it follows that the matrix $\left(\frac{\partial\bar{\omega}_j}{\partial y^i} - \frac{\partial\bar{\omega}_i}{\partial y^j}\right)$ is non-singular and also the resulting solution (S^i) has each component depending only on (y^i). $\square$

Example 4. Consider setting of Example 2. and the notations $x = x^1$, $y = x^2$, $X = y^1$, $Y = y^2$. Let $\{2S^i(x^j, y^j)\}_{i=\overline{1,2}}$ be some real functions considered as the components of a semi-spray S on $\mathbb{R}^2$. Then the differential system that gives the integral curves of $X_{\mathcal{E}_S}$ reads

$$\frac{dx^i}{dt} = y^i,\ \frac{dy^i}{dt} = 2S^i,\ \frac{dq_i}{dt} = -\frac{\partial\omega_j}{\partial x^i}y^j,\ \frac{dp_i}{dt} = -q_i - \omega_i - 2\frac{\partial\bar{\omega}_j}{\partial y^i}S^j - 2(p_j + \bar{\omega}_j)\frac{\partial S^j}{\partial y^i}.$$

Considering $2S^1(x^i, y^i) = -x^1 + c_1$ and $2S^2(x^i, y^i) = -x^2 + c_2$, then taking into account Example 2., the integral curves of all semi-sprays S having this form give all the solutions of the generalized Euler-Lagrange equation (2.4) of ω.

The above example can be extended as follows.

Proposition 3.4. *Let us suppose that there are some coordinates such that the local form of a regular Pfaff form is $\omega = \omega_i(x^j)dx^i + \bar{\omega}_i(y^j)dy^i$ and also $\frac{\partial\omega_i}{\partial x^j} + \frac{\partial\omega_j}{\partial x^i} = 0$. Then there are some local semi-sprays S such that:*

1. *An integral curve Γ of $X_{\mathcal{E}_S}$ that intersects $\mathcal{M}$ is included in $\mathcal{M}$.*
2. *An integral curve Γ projects on M on a curve γ that is a solution of the generalized Euler-Lagrange equation of the Pfaff form ω.*

Proof. Let us consider a semi-spray given by the condition

$$2S^j g_{ij} = 2\omega_i + 2c_i,$$

where c_i are constants. Let $t \xrightarrow{\gamma} (x^i(t), y^i(t))$ be an integral curve of the semi-spray S and denote by Γ the curve $t \xrightarrow{\Gamma} (x^i(t), y^i(t), q_i(t) = \omega_i(x^j(j)) + 2c_i, p_i(t) = -\bar{\omega}_i(y^j(t)))$. Then $\Gamma \subset \mathcal{M}$ is an integral curve of the Hamiltonian vector field $X_{\mathcal{E}_S}$ (3.2), as it can be checked by a straightforward verification. The assertion 1. follows from the construction of Γ, taking into account the uniqueness of the integral curve of $X_{\mathcal{E}_S}$ passing through a given point of $\mathcal{M}$. The assertion 2. follows using Proposition 3.2. $\square$

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