Geometry of compact shrinking Ricci solitons

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Abstract. Einstein manifolds are trivial examples of gradient Ricci solitons with constant potential function and thus they are called trivial Ricci solitons. In this paper, we prove two characterizations of compact shrinking trivial Ricci solitons.

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1 Introduction

A smooth vector field ξ on a Riemannian manifold (M, g) is said to define a *Ricci soliton* if it satisfies

$$\frac{1}{2}\pounds_\xi g + Ric = \lambda g,$$

where $\pounds_\xi g$ is the Lie-derivative of the metric tensor g with respect to ξ, Ric is the Ricci tensor of (M, g) and λ is a constant. We shall denote a Ricci soliton by (M, g, ξ, λ) and call the vector field ξ the potential field of the Ricci soliton. The Ricci soliton (M, g, ξ, λ) is called *shrinking, steady* or *expanding* according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

It is well-known that if (M, g, ξ, λ) is a compact Ricci soliton, then the potential field ξ is a gradient of some smooth function f up to the addition of a Kiling field and thus a compact Ricci soliton is a gradient Ricci soliton (cf. [16]). We shall denote a gradient Ricci soliton by (M, g, f, λ) and call the smooth function f the *potential function* of the gradient Ricci soliton. For a gradient Ricci soliton (M, g, f, λ) it is always possible to choose the potential function f satisfying

$$2\lambda f = ||\nabla f||^2 + S,$$

where S denotes the scalar curvature of M (cf. section 2 for details). A gradient Ricci soliton (M, g, f, λ) with such a potential function is simply called *a gradient Ricci soliton with normalized potential.*

In the following, we denote by λ_1 the first nonzero eigenvalue of the Laplace operator Δ on a gradient Ricci soliton (M, g, f, λ).

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Recall that Hamilton conjectured in [9,10] that a compact gradient Ricci soliton with positive curvature operator is an Einstein manifold (a trivial Ricci soliton), which is settled in [1]. The next important question is to find conditions under which a compact gradient Ricci soliton is an Einstein manifold.

Since the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Grigory Perelman solved the Poincaré conjecture and observed that on compact simply connected Riemannian manifolds, the Ricci solitons as solutions of Ricci flow, are gradient Ricci solitons (cf. [15,16]). An Einstein manifold is a trivial example of a gradient Ricci soliton with constant potential function and therefore it is called a *trivial Ricci soliton*. There exist many non-trivial examples of Ricci solitons compact as well as non-compact (cf. [2]-[4], [11]-[13]).

There are two aspects of the study of Ricci solitons, one looking at the influence on the topology by the Ricci soliton structure of the Riemannian manifold (cf. [5,14,18]) and the other looking at its influence on its geometry (cf. [1,6,7]). In this paper, we are interested in finding characterizations of trivial Ricci solitons among compact gradient Ricci solitons.

On a compact Riemannian manifold (M, g) and a smooth function $\varphi : M \to R$, the average value of φ, denoted by φ_{av}, is a real number defined by

$$\varphi_{av} = \frac{1}{\mathrm{Vol}(M)} \int_M \varphi.$$

We prove the following characterization of trivial Ricci solitons.

Theorem 1.1. *An n-dimensional compact connected shrinking gradient Ricci soliton (M, g, f, λ) with normalized potential is trivial if and only if*

$$(fS)_{av} \leq \frac{1}{2} n^2 \lambda,$$

where S denotes the scalar curvature of (M, g).

The *Poisson equation* on a Riemannian manifold (M, g) is

$$\Delta \varphi = \sigma,$$

where Δ is the Laplace operator, σ is a given function, and φ is the solution to be determined. The Poisson equation plays a fundamental role in Physics; also well known for its importance in Electrostatics, Biophysics and Engineering.

It is known that the Poisson equation $\Delta\varphi = \sigma$ on a compact Riemannian manifold (M, g) has a unique solution up constants if and only if the integral of σ is equal to zero (cf. [8]). Also, in order to use a Poisson equation to study the geometry of a compact gradient Ricci soliton (M, g, f, λ), we need to construct a function σ whose integral is equal to zero.

On a compact gradient Ricci soliton (M, g, f, λ), the function $\sigma = \lambda(n\lambda - S)$ satisfies the property $\int_M \sigma = 0$ (see equation (2.8)). In the next theorem, we study the Poisson equation $\Delta\varphi = \sigma$ with $\sigma = \lambda(n\lambda - S)$ on the gradient Ricci soliton (M, g, f, λ).

Our second result is the following.

Theorem 1.2. *Let (M, g, f, λ) be an n-dimensional compact connected shrinking gradient Ricci soliton and let $\sigma = \lambda(n\lambda - S)$. If the scalar curvature S is a solution of the Poisson equation*

$$\Delta\varphi = \sigma, \tag{1.1}$$

then either M is trivial or the first nonzero eigenvalue λ_1 of the Laplace operator Δ of M satisfies $\lambda_1 \leq \lambda$.

The significance of this theorem is the following immediate consequence, which provides another characterization of trivial Ricci solitons.

Corollary 1.3. *An n-dimensional compact connected shrinking gradient Ricci soliton (M, g, f, λ) with $\lambda < \lambda_1$ is trivial if and only if the scalar curvature S satisfies the following Poisson equation*

$$\Delta\varphi = \lambda(n\lambda - S).$$

In the last section of this paper, we observe that the requirement that the scalar curvature S of a compact shrinking gradient Ricci soliton satisfies the Poisson equation (1.1) in Theorem 1.2 is dictated by the behavior of the Ricci curvature in the direction of the gradient vector field ∇S of S (cf. Theorem 5.1); and it gives yet another characterization of trivial Ricci solitons (cf. Corollary 5.2).

2 Preliminaries

Let (M, g, f, λ) be an n-dimensional compact gradient Ricci soliton and let $\mathfrak{X}(M)$ denote the Lie algebra of smooth vector fields on M. Then we have (cf. [2,3,4])

$$H_f(X, Y) + Ric(X, Y) = \lambda g(X, Y), \qquad X, Y \in \mathfrak{X}(M), \tag{2.1}$$

where $H_f(X, Y) = g(\nabla_X \nabla f, Y)$ is the Hessian and ∇f is the gradient of the potential function f.

Since S is the scalar curvature of (M, g), equation (2.1) yields

$$\Delta f = n\lambda - S, \tag{2.2}$$

where $\Delta f = \mathrm{Trace}\,(H_f)$ is the Laplacian of f. The Ricci operator Q is defined by

$$Ric\,(X, Y) = g\,(Q\,(X)\,, Y)\,, \ \ X, Y \in \mathfrak{X}(M)\,. \tag{2.3}$$

It is well known that the Ricci operator Q satisfies

$$\sum_i (\nabla Q)\,(e_i, e_i) = \frac{1}{2}\nabla S, \tag{2.4}$$

where $\{e_1, .., e_n\}$ is a local orthonormal frame and ∇Q is the covariant derivative of Q defined by

$$(\nabla Q)\,(X, Y) = \nabla_X\,(QY) - Q\,(\nabla_X Y)\,.$$

We define the symmetric operator A_f by

$$H_f(X, Y) = g(A_f X, Y), \quad X, Y \in \mathfrak{X}(M)\,.$$

Then, by using the definition of curvature tensor field R, we have

$$(\nabla A_f)(X,Y) - (\nabla A_f)(Y,X) = R(X,Y)\nabla f.$$

After applying the above equation, $\Delta f = \mathrm{Trace}\,(A_f)$, and the symmetry of A_f, we obtain

$$\begin{aligned} X(\Delta f) &= \sum_i g((\nabla A_f)(X,e_i),e_i) \\ &= \sum_i g((\nabla A_f)(e_i,X) + R(X,e_i)\nabla f, e_i) \\ &= -Ric(X,\nabla f) + \sum_i g((\nabla A_f)(e_i,e_i),X) \end{aligned} \tag{2.5}$$

for $X \in \mathfrak{X}\,(M)$. Also, it follows from equation (2.1) that

$$(\nabla A_f)(X,Y) = -(\nabla Q)(X,Y).$$

By substituting this into equation (2.5) and using equations (2.2) and (2.4), we find

$$-X(S) = -Ric(X,\nabla f) - \frac{1}{2}X(S),$$

which implies

$$Q(\nabla f) = \frac{1}{2}\nabla S. \tag{2.6}$$

Note that on a connected gradient Ricci soliton (M,g,f,λ), it follows from equations (2.1) and (2.6) that

$$\frac{1}{2}X\big(\,\|\nabla f\|^2 + S\big) = H_f(X,\nabla f) + Ric(X,\nabla f) = \lambda g(X,\nabla f),$$

that is

$$X\big(\,\|\nabla f\|^2 + S - 2\lambda f\big) = 0, \quad X \in \mathfrak{X}(M).$$

This shows that

$$\frac{1}{2}\big(\,\|\nabla f\|^2 + S\big) - \lambda f = c$$

for a constant c. Now, after replacing the potential function f of the connected shrinking gradient Ricci soliton (M,g,f,λ) by $f - \frac{c}{\lambda}$, we see that the gradient shrinking Ricci soliton (M,g,f,λ) satisfies

$$2\lambda f = \|\nabla f\|^2 + S. \tag{2.7}$$

On a compact gradient Ricci soliton, equation (2.2) gives

$$\int_M (n\lambda - S) = 0. \tag{2.8}$$

3 Proof of Theorem 1.1

Let (M, g, f, λ) be an n-dimensional compact and connected shrinking gradient Ricci soliton. Then, it follows from equations (2.2) and (2.7) that

$$\begin{aligned} \frac{1}{2}\Delta f^2 &= f\Delta f + \|\nabla f\|^2 \\ &= (n+2)\lambda f - fS - S, \end{aligned} \tag{3.1}$$

which together with equation (2.8) gives

$$\int_M fS = \lambda(n+2)\int_M \left(f - \frac{n}{n+2}\right). \tag{3.2}$$

Note that equations (2.7) and (2.8) imply

$$\int_M \left(f - \frac{n}{2}\right) = \frac{1}{2\lambda}\int_M \|\nabla f\|^2,$$

which together with equation (3.1) gives

$$\int_M fS = \frac{1}{2}n^2\lambda \operatorname{Vol}(M) + \frac{n+2}{2}\int_M \|\nabla f\|^2. \tag{3.3}$$

If the condition $(fS)_{av} \le \frac{1}{2}n^2\lambda$ holds, then we shall have

$$\int_M fS \le \frac{1}{2}n^2\lambda \operatorname{Vol}(M). \tag{3.4}$$

By combining (3.3) and (3.4), we obtain $\int_M \|\nabla f\|^2 = 0$, which implies that the potential function f is a constant. Consequently, it follows from (2.1) that M is an Einstein manifold. Thus the Ricci soliton is trivial.

Conversely, if an n-dimensional compact and connected shrinking gradient Ricci soliton is trivial, then $S = n\lambda$ and f is a constant. Therefore, by equation (2.7) we obtain $f = S/(2\lambda)$. Consequently, we have $(fS)_{av} = \frac{1}{2}n^2\lambda$. This completes the proof of Theorem 1.1. □

Remark 3.1. Theorem 1.1 can also be proved by using the techniques in [17].

4 Proof of Theorem 1.2

Let (M, g, f, λ) be an n-dimensional compact and connected shrinking gradient Ricci soliton. Suppose that the scalar curvature S satisfies the Poisson equation

$$\Delta\varphi = \sigma, \tag{4.1}$$

with $\sigma = \lambda(n\lambda - S)$. Note that the function $\psi = \frac{1}{2}(\|\nabla f\|^2 + S)$ satisfies

$$\psi = \lambda f \tag{4.2}$$

due to equation (2.7). Combining this with equation (2.2) gives

$$\Delta\psi = \lambda(n\lambda - S) = \sigma.$$

Therefore, both S and ψ are the solutions of the Poisson equation (4.1). Hence, we have $S = \psi + c$ for some constant c (cf. [8]). Consequently, we obtain

$$\nabla S = \nabla\psi = \lambda\nabla f. \tag{4.3}$$

Now, by applying equation (2.8) and the well known minimum principle of λ_1, we have

$$\int_M \|\nabla S\|^2 \geq \lambda_1 \int_M (n\lambda - S)^2. \tag{4.4}$$

On the other hand, it follows from equation (2.8) that

$$\int_M (n\lambda - S)^2 = \int_M (S^2 - n^2\lambda^2).$$

Consequently, inequality (4.4) takes the form

$$\int_M \|\nabla S\|^2 \geq \lambda_1 \int_M (S^2 - n^2\lambda^2). \tag{4.5}$$

Because the scalar curvature S is a solution of the Poisson equation (4.1) with $\sigma = \lambda(n\lambda - S)$, we have

$$S\Delta S = \lambda(n\lambda S - S^2).$$

By applying integration by parts to the last equation and by using equation (2.8), we obtain

$$\int_M \|\nabla S\|^2 = \lambda \int_M (S^2 - n^2\lambda^2),$$

which together with the inequality (4.5) gives

$$(\lambda_1 - \lambda) \int_M (n^2\lambda^2 - S^2) \geq 0.$$

Note that using equation (2.2), we have

$$n^2\lambda^2 - S^2 = (n\lambda + S)\Delta f = n\lambda\Delta f + S\Delta f,$$

which on insertion in the above inequality gives

$$(\lambda_1 - \lambda) \int_M (S\Delta f) \geq 0. \tag{4.6}$$

From (2.2), (4.6), $\Delta S = \lambda(n\lambda - S)$, and integration by parts, we get

$$\begin{aligned} 0 &\leq (\lambda_1 - \lambda) \int_M (S\Delta f) \\ &= (\lambda_1 - \lambda) \int_M S(n\lambda - S) \\ &= \frac{\lambda_1 - \lambda}{\lambda} \int_M (S\Delta S) \\ &= \frac{\lambda - \lambda_1}{\lambda} \int_M \|\nabla S\|^2 . \end{aligned}$$

By combining this with equation (4.3), we obtain

$$\lambda(\lambda_1 - \lambda) \int_M \|\nabla f\|^2 \leq 0,$$

which implies that either $\lambda_1 \leq \lambda$ holds or (M, g, f, λ) is trivial. This completes the proof of the theorem. $\square$

Remark 4.1. Notice that Corollary 1.3 follows immediately from Theorem 1.2. As for the converse, we have $S = n\lambda$ for any trivial (M, g, f, λ), which satisfies the given Poisson equation.

5 A remark

Observe that if (M, g, f, λ) is an n-dimensional compact connected shrinking trivial Ricci soliton, then the scalar curvature S is a constant equal to $n\lambda$. Thus it satisfies the Poisson equation in Theorem 1.2 trivially. It is interesting to point out that the condition that the scalar curvature satisfying this Poisson equation is dictated by the behavior of certain Ricci curvature of the Ricci soliton, as seen in the following.

Theorem 5.1. *Let (M, g, f, λ) be an n-dimensional compact connected shrinking gradient Ricci soliton of positive Ricci curvature. If the Ricci curvature Ric and the scalar curvature S of (M, g) satisfy*

$$Ric(\nabla S, \nabla S) \leq \lambda \left(\|\nabla S\|^2 + \frac{\lambda}{2}\left(n^2\lambda^2 - S^2\right) \right), \tag{5.1}$$

then S is a solution of the Poisson equation $\Delta\varphi = \sigma$ with $\sigma = \lambda(n\lambda - S)$.

Proof. Let (M, g, f, λ) be an n-dimensional compact connected shrinking gradient Ricci soliton of positive Ricci curvature. Then equation (2.6) gives

$$Ric(\nabla f, \nabla S) = \frac{1}{2}\|\nabla S\|^2 \tag{5.2}$$

and

$$Ric(\nabla f, \nabla f) = \frac{1}{2} g(\nabla f, \nabla S). \tag{5.3}$$

Now, using equation (2.2), we find

$$g(\nabla f, \nabla S) = \nabla f(S) = \operatorname{div}(S\nabla f) - S(n\lambda - S).$$

By substituting this value into equation (5.3), integrating the resulting equation and applying divergence theorem, we get

$$\int_M Ric(\nabla f, \nabla f) = \frac{1}{2}\int_M \left(S^2 - n^2\lambda^2\right), \tag{5.4}$$

where we have used equation (2.8). Clearly, we have

$$\begin{aligned} &Ric(\nabla S - \lambda\nabla f, \nabla S - \lambda\nabla f) \\ &\qquad = Ric(\nabla S, \nabla S) - 2\lambda Ric(\nabla f, \nabla S) + \lambda^2 Ric(\nabla f, \nabla f). \end{aligned}$$

After integrating the above equation and applying equations (5.2) and (5.4), we arrive at

$$\begin{aligned}
&\int_M Ric\,(\nabla S - \lambda \nabla f, \nabla S - \lambda \nabla f) \\
&\quad = \int_M \left\{ Ric(\nabla S, \nabla S) - \lambda \|\nabla S\|^2 + \frac{\lambda^2}{2}\left(S^2 - n^2\lambda^2\right) \right\} \\
&\quad\quad = \int_M \left\{ Ric\,(\nabla S, \nabla S) - \lambda \left(\|\nabla S\|^2 + \frac{\lambda}{2}\left(n^2\lambda^2 - S^2\right) \right) \right\}.
\end{aligned}$$

Now, by applying condition (5.1) and the positiveness of Ricci curvature on M from the hypothesis, we may conclude from the above equation that

$$\nabla S = \lambda \nabla f.$$

After combining this equation with equation (2.2), we obtain

$$\Delta S = \lambda(n\lambda - S) = \sigma,$$

which implies the theorem. □

Combining Theorem 5.1 and Corollary 1.3 gives the following.

Corollary 5.2. *An n-dimensional compact connected shrinking gradient Ricci soliton (M, g, f, λ) of positive Ricci curvature with $\lambda < \lambda_1$ is trivial if and only if the scalar curvature S satisfies*

$$Ric(\nabla S, \nabla S) \leq \lambda \left(\|\nabla S\|^2 + \frac{\lambda}{2}\left(n^2\lambda^2 - S^2\right) \right). \tag{5.5}$$

Proof. If condition (5.5) holds, then Theorem 5.1 implies that the scalar curvature satisfies the Poisson equation $\Delta S = \sigma$. Therefore, Theorem 1.2 together with $\lambda < \lambda_1$ implies that the Ricci soliton is trivial.

Conversely, if (M, g, f, λ) is trivial, then $S = n\lambda$ satisfies the condition in the statement. □

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