# Conformal vector fields on a Riemannian manifold

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**Abstract.** In this paper, we use a non-Killing conformal vector field on an $n$-dimensional compact Riemannian manifold $(M, g)$ to find a characterization of a $n$-sphere $S^n(c)$. We also use a non-Killing conformal vector field on an $n$-dimensional complete connected Riemannian manifold to find a characterization of the Euclidean space $\mathbb{R}^n$.

**M.S.C. 2010**: 53C21, 58J05, 53C42.
**Key words**: Conformal vector fields; Obata's theorem; Laplacian; rough Laplacian.

## 1 Introduction

A smooth vector field $\xi$ on a Riemannian manifold $(M, g)$ is said to a conformal vector field if there exists a smooth function $f$ on $M$ that satisfies $\pounds_\xi g = 2fg$, where $\pounds_\xi g$ is the Lie derivative of $g$ with respect $\xi$, that is the flow of the vector field $\xi$ consists of conformal transformations of the Riemannian manifold $(M, g)$, the function $f$ is called the potential function of the conformal vector field $\xi$. We say $\xi$ a nontrivial conformal vector field if $\xi$ is a non-Killing conformal vector field. If the conformal vector field $\xi$ is a closed vector field, then $\xi$ is said to be a closed conformal vector field. Riemannian manifolds admitting closed conformal vector fields or conformal gradient vector fields have been investigated in (cf. [3], [4], [7], [9]-[10]) and it has been observed that there is a close relationship between the potential functions of conformal vector fields and Obata's differential equation. In [2], conformal vector fields those are also eigenvectors of the Laplacian operator have been studied on a compact Riemannian manifold of constant scalar curvature and under a suitable restriction on the Ricci curvature of this manifold it is shown that the Riemannian manifold must be isometric to a sphere. Note that the scalar curvature of the Riemannian manifold being constant (or the manifold is an Einstein manifold) gives a convenient combination with the presence of a conformal vector field to study the geometry of the manifold, specially in getting the characterizations of spheres using conformal vector field. However, if the scalar curvature of the Riemannian manifold is not a constant, then finding such characterizations is a difficult task and we do not find results in the existing literature studying the geometry of Riemannian manifolds of non-constant scalar curvature admitting a conformal vector field.

Balkan Journal of Geometry and Its Applications, Vol.19, No.2, 2014, pp. 86-93.

Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold that admits a nontrivial conformal vector field $\xi$ with potential function $f$. We denote by $\lambda_1$ the first nonzero eigenvalue of the Laplacian operator $\Delta$ acting on smooth functions of $M$ and by $Ric$ and $S$ the Ricci tensor field and the scalar curvature of $M$ respectively. In this short note, we attempt to study the geometry of a compact Riemannian manifold of non-constant scalar curvature that admits a nontrivial conformal vector field, with a mild condition that the scalar curvature is constant along the integral curves of the conformal vector field. Such a condition together with an upper bound on the scalar curvature and a lower bound on the Ricci curvature in certain direction gives a characterization of a $n$-sphere, as seen the following theorem, which we intend to prove in this paper.

**Theorem 1.1.** *Let $\xi$ be a nontrivial conformal vector field with potential function $f$ on an $n$-dimensional compact and connected Riemannian manifold $(M, g)$. Let $\lambda_1$ be the first nonzero eigenvalue of the Laplacian operator $\Delta$ on $M$. If the scalar curvature $S$ satisfies*

$$\xi(S) = 0, \quad S \leq (n-1)\lambda_1,$$

*and the Ricci curvature in the direction of the gradient vector field $\nabla f$ of the potential function $f$ is bounded below by $n^{-1}S$, then $M$ is isometric to a $n$-sphere $S^n(c)$, for a constant $c$.*

Note that there are several nontrivial conformal vector fields on a $n$-sphere $S^n(c)$ and all the conditions of the above theorem are satisfied for $S^n(c)$ and thus the above theorem gives a necessary and sufficient condition for an $n$-dimensional compact and connected Riemannian manifold to be isometric to a $S^n(c)$, that is it gives a characterization of a $n$-sphere.

Next, consider the Euclidean space $\mathbb{R}^n$, the position vector field $\xi$ on $\mathbb{R}^n$ is a gradient conformal non-Killing vector field, that is $\xi = \nabla\rho$, where $\rho = \frac{1}{2}\|\xi\|^2$. Moreover, the vector field $\xi$ satisfies $\Delta\xi = 0$, where $\Delta$ is the rough Laplacian operator acting on the smooth vector fields on $\mathbb{R}^n$, that is the vector field $\xi$ is a harmonic gradient conformal vector field. The natural question arises as to whether such a vector field characterizes the Euclidean space. We show that the answer to this question is in affirmative without assuming that the nontrivial conformal vector field being a gradient conformal vector field and with the flatness of $\mathbb{R}^n$ being replaced by the condition that the vector field $\xi$ annihilates the Ricci operator $Q$, which is the symmetric $(1,1)$-tensor field associated to the Ricci tensor $Ric$ of the Riemannian manifold $(M, g)$ by $Ric(X, Y) = g(QX, Y)$ for smooth vector fields $X$ and $Y$. Indeed, we prove the following :

**Theorem 1.2.** *An $n$-dimensional complete and connected Riemannian manifold $(M, g)$, $(n \geq 3)$, admits a nontrivial harmonic conformal field $\xi$ that annihilates the Ricci operator and satisfies $d\eta(X, \xi) = 0$ for smooth vector fields $X$ on $M$, where $\eta$ is the 1-form dual to $\xi$, if and only if $M$ is isometric to the Euclidean space $\mathbb{R}^n$.*

# 2 Preliminaries

Let $(M, g)$ be an $n$-dimensional Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on $M$. A vector field $\xi \in \mathfrak{X}(M)$ is said to be a conformal vector field if

$$\pounds_\xi g = 2fg, \tag{2.1}$$

for a smooth function $f \in C^\infty(M)$, called the potential function, where $\pounds_\xi$ is the Lie derivative with respect to $\xi$. Using Koszul's formula (cf. [1]), we immediately obtain the following for a vector field $\xi$ on $M$

$$2g(\nabla_X \xi, Y) = (\pounds_\xi g)(X, Y) + d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{2.2}$$

where $\eta$ is the 1-form dual to $\xi$ that is $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$. Define a skew symmetric tensor field $\varphi$ of type $(1,1)$ on $M$ by

$$d\eta(X, Y) = 2g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.3}$$

Then using equations (2.1), (2.2) and (2.3), we immediately get the following

$$\nabla_X \xi = fX + \varphi X, \quad X \in \mathfrak{X}(M). \tag{2.4}$$

Recall that a conformal vector field $\xi$ is said to be a nontrivial conformal vector field if $\xi$ is not a Killing vector field. For example, consider the $n$-sphere $S^n(c)$ of constant curvature $c$ (that is of radius $\sqrt{\frac{1}{c}}$) as hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ with unit normal vector field $N$ and take a constant vector field $Z$ on $\mathbb{R}^{n+1}$, which can be expressed as $Z = \xi + \rho N$, where $\xi$ is the tangential component of $Z$ to $S^n(c)$ and $\rho = \langle Z, N\rangle$ is the smooth function on $S^n(c)$, $\langle , \rangle$ being the Euclidean metric on $\mathbb{R}^{n+1}$. Then it is easy to show that $\pounds_\xi g = -2\sqrt{c}\rho g$, that is $\xi$ is a conformal vector field on $S^n(c)$ with potential function $f = -\sqrt{c}\rho$ and it is easy to show that it is a nontrivial conformal vector field.

We shall denote by $\Delta$ the Laplacian operator acting on smooth functions on $M$ and by $\lambda_1$ the first nonzero eigenvalue of the Laplacian operator $\Delta$. For a smooth function $h \in C^\infty(M)$ on the Riemannian manifold $(M, g)$, we denote by $\nabla h$ the gradient of $h$ and by $A_h$ the Hessian operator $A_h : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $A_h(X) = \nabla_X \nabla h$. On an $n$-dimensional compact Riemannian manifold $(M, g)$ that admits a conformal vector field $\xi$, using the skew symmetry of the tensor field $\varphi$ the equation (2.4) gives $\operatorname{div} \xi = nf$ and consequently, we have

$$\int_M f = 0, \tag{2.5}$$

which gives

$$\int_M \|\nabla f\|^2 \geq \lambda_1 \int_M f^2. \tag{2.6}$$

Also, we have $\operatorname{div}(f\xi) = \xi(f) + nf^2$, which gives

$$\int_M g(\nabla f, \xi) = -n \int_M f^2. \tag{2.7}$$

Note that the smooth 2-form given by $g(\varphi X, Y)$ is closed and therefore, we have

$$g\left((\nabla\varphi)(X,Y),Z\right)+g\left((\nabla\varphi)(Y,Z),X\right)+g\left((\nabla\varphi)(Z,X),Y\right)=0, \tag{2.8}$$

where the covariant derivative $(\nabla\varphi)(X,Y) = \nabla_X\varphi Y - \varphi(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$. Moreover, we compute the curvature tensor field $R(X,Y)\xi$, using the equation (2.4) to arrive at

$$R(X,Y)\xi = X(f)Y - Y(f)X + (\nabla\varphi)(X,Y) - (\nabla\varphi)(Y,X).$$

Using the above equation in the equation (2.8) and the skew-symmetry of the tensor field $\varphi$, we get

$$g\left(R(X,Y)\xi + Y(f)X - X(f)Y, Z\right) + g\left((\nabla\varphi)(Z,X),Y\right) = 0,$$

that is

$$(\nabla\varphi)(X,Y) = R(X,\xi)Y + Y(f)X - g(X,Y)\nabla f, \quad X, Y \in \mathfrak{X}(M). \tag{2.9}$$

The Ricci operator $Q$ is a symmetric $(1,1)$-tensor field that is defined by $g(QX,Y) = Ric(X,Y)$, $X, Y \in \mathfrak{X}(M)$, where $Ric$ is the Ricci tensor of the Riemannian manifold. Choosing a local orthonormal frame $\{e_1, ..., e_n\}$ on $M$, and using

$$Q(X) = \sum R(X, e_i)e_i,$$

in the equation (2.9), we compute

$$\sum(\nabla\varphi)(e_i, e_i) = -Q(\xi) - (n-1)\nabla f. \tag{2.10}$$

The operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ on a Riemannian manifold $(M,g)$ defined by

$$\Delta X = \sum\left(\nabla_{e_i}\nabla_{e_i}X - \nabla_{\nabla_{e_i}e_i}X\right),$$

where $\{e_i, ..., e_n\}$ is a local orthonormal frame on $M$ is called the rough Laplacian operator acting on smooth vector fields on $M$, and a smooth vector field $\xi$ on $M$ is said to be a harmonic vector field if $\Delta\xi = 0$.

# 3 Proof of the Theorems

**Lemma 3.1.** *Let $\xi$ be a conformal vector field on a compact Riemannian manifold $(M,g)$ with potential function $f$. Then,*

$$\int_M \left((n-1)\left\|\nabla f\right\|^2 + \frac{n-2}{2}Sf^2 + \frac{S}{2}g(\nabla f, \xi)\right) = 0,$$

*where $\nabla f$ is the gradient of the function $f$ and $S$ is the scalar curvature.*

*Proof.* Recall by a well known formula (cf. [1]), we have

$$\sum_{i=1}^{n}(\nabla Q)(e_i,e_i)=\frac{1}{2}\nabla S, \tag{3.1}$$

where $\{e_1,...,e_n\}$ is a local orthonormal frame on $M$ and $(\nabla Q)(X,Y)=\nabla_X QY-Q(\nabla_X Y)$, $X,Y\in\mathfrak{X}(M)$. We use a point wise constant local orthonormal frame $\{e_1,...,e_n\}$ and the equations (2.4), (3.1) to compute the divergence of the vector field $Q(\xi)$ as

$$\begin{aligned}\operatorname{div}Q(\xi) &= \sum_{i=1}^{n}g(\nabla_{e_i}Q(\xi),e_i)=\sum_{i=1}^{n}e_i g(\xi,Q(e_i))\\ &= \sum_{i=1}^{n}g(fe_i+\varphi e_i,Qe_i)+\frac{1}{2}\xi(S)\\ &= fS+\frac{1}{2}\xi(S)+\sum_{i=1}^{n}g(\varphi e_i,Qe_i).\end{aligned}$$

Choosing a local orthonormal frame that diagonalizes the symmetric operator $Q$ and using the skew-symmetry of the tensor $\varphi$, we conclude that $\sum_{i=1}^{n}g(\varphi e_i,Qe_i)=0$, which together with above equation gives

$$\operatorname{div}Q(\xi)=fS+\frac{1}{2}\xi(S).$$

Using the above equation we get

$$\operatorname{div}(fQ(\xi))=Ric(\nabla f,\xi)+Sf^2+\frac{1}{2}f\xi(S).$$

Also, we have $\operatorname{div}(fS\xi)=f\xi(S)+S\operatorname{div}(f\xi)=f\xi(S)+S\xi(f)+nSf^2$, which together with the above equation gives

$$\operatorname{div}Q(\xi)=Ric(\nabla f,\xi)+Sf^2+\frac{1}{2}\left(\operatorname{div}(fS\xi)-S\xi(f)-nSf^2\right). \tag{3.2}$$

Now, we use the equation (2.9), to compute the divergence of the vector field $\varphi(\nabla f)$ and get

$$\begin{aligned}\operatorname{div}(\varphi(\nabla f)) &= 0-g\left(\nabla f,-Q(\xi)-(n-1)\nabla f\right)\\ &= Ric(\nabla f,\xi)+(n-1)\left\|\nabla f\right\|^2,\end{aligned}$$

where we have used $\sum g(A_f e_i,\varphi e_i)=0$, which follows by the fact that the Hessian operator $A_f$ is symmetric and the tensor field $\varphi$ is skew-symmetric. Using the above equation in the equation (3.2) and integrating the resulting equation, we get the Lemma. □

Now, we proceed to prove the Theorem 1.1. Note that the condition $\xi(S)=0$, gives

$$\begin{aligned}Sg(\nabla f,\xi) &= S\xi(f)=\operatorname{div}(fS\xi)-f\operatorname{div}(S\xi)\\ &= \operatorname{div}(fS\xi)-f(0+nfS).\end{aligned}$$

Inserting this value in the Lemma 3.1, we get

$$\int_M \left((n-1)\|\nabla f\|^2 - Sf^2\right) = 0,$$

which together with the inequality (2.6) gives

$$\int_M \left((n-1)\lambda_1 - S\right) f^2 \leq 0.$$

Now, using the upper bound on the scalar curvature $S$ in the statement, we conclude that

$$\left((n-1)\lambda_1 - S\right) f^2 = 0$$

Note that the above equation on the connected $M$ implies either $S = (n-1)\lambda_1$ or else $f = 0$. However, the second option together with the equation (2.1) implies that $\xi$ is a Killing vector field, which is contradictory to the fact that $\xi$ is a nontrivial conformal vector field. Hence, we have $S = (n-1)\lambda_1$, that is the scalar curvature $S$ is a constant. Now, the Lemma 3.1, together with the equation (2.7) gives

$$\int_M \|\nabla f\|^2 = \lambda_1 \int_M f^2, \tag{3.3}$$

that is the inequality in (2.6) is the equality, which hold if and only if $\Delta f = -\lambda_1 f$. Now, we have the following Bochner's formula

$$\int_M \left(Ric(\nabla f, \nabla f) + \|A_f\|^2 - (\Delta f)^2\right) = 0,$$

which gives

$$\int_M \left(\left(Ric(\nabla f, \nabla f) - \frac{S}{n}\|\nabla f\|^2\right) + \left(\|A_f\|^2 - \frac{1}{n}(\Delta f)^2\right)\right) = 0, \tag{3.4}$$

where we used the equality $\Delta f = -\lambda_1 f$ and the equation (3.3). As the trace $TrA_f = \Delta f$, we know that $\|A_f\|^2 \geq \frac{1}{n}(\Delta f)^2$ and the equality holds if and only if $A_f = \left(\frac{\Delta f}{n}\right) I$. Thus, using the lower bound on the Ricci curvature $R(\nabla f, \nabla f)$, in the equation (3.4), we have

$$A_f = \left(\frac{\Delta f}{n}\right) I = -\frac{\lambda_1}{n} f I,$$

that is

$$\nabla_X \nabla f = -\frac{\lambda_1}{n} f X, \quad X \in \mathfrak{X}(M). \tag{3.5}$$

Note that if the potential function $f$ is a constant, then the equation (2.5) gives $f = 0$ and that will imply $\xi$ is a Killing vector field which is not allowed by the hypothesis.

Hence $f$ is a non-constant function which satisfies the Obata's equation (3.5) (cf. [5], [6]), and consequently, $M$ is isometric to a $n$-sphere $S^n(c)$.

Finally, to prove the Theorem 1.2, first observe that the position vector field $\xi$ on the Euclidean space $\mathbb{R}^n$ is a nontrivial gradient conformal vector field that satisfies

$$\Delta\xi = 0 \quad \text{and} \quad Q\xi = 0. \tag{3.6}$$

Since $\xi$ is a gradient of a smooth function, we have $d\eta = 0$, where $\eta$ is smooth 1-form dual to $\xi$, we see that the vector field $\xi$ satisfies the requirement of the hypothesis of the Theorem. Conversely, suppose $\xi$ is a nontrivial conformal vector field on an $n$-dimensional complete and connected Riemannian manifold $(M, g)$ that satisfies the equation (3.6) and that the condition $d\eta(X, \xi) = 0$, $X \in \mathfrak{X}(M)$ holds. Then the equation (2.3) gives

$$\varphi\xi = 0. \tag{3.7}$$

Let $\{e_1, ..., e_n\}$ be a local orthonormal frame on $M$. Then the equation (2.10) gives

$$\sum (\nabla\varphi)(e_i, e_i) = -Q(\xi) - (n-1)\nabla f = -(n-1)\nabla f \tag{3.8}$$

Now, we use the equation (2.4) to compute

$$\nabla_X \nabla_X \xi - \nabla_{\nabla_X X}\xi = X(f)X + (\nabla\varphi)(X, X),$$

which gives

$$\Delta\xi = \nabla f + \sum (\nabla\varphi)(e_i, e_i) = 0. \tag{3.9}$$

Combining the equations (3.8) and (3.9), we get $\nabla f = 0$ on the connected $M$, that is $f$ is a constant. Define a smooth function $h$ by $h = \frac{1}{2}\|\xi\|^2$, which on using the equation (2.4) has the gradient

$$\nabla h = f\xi - \varphi\xi = f\xi. \tag{3.10}$$

We claim that the function $h$ is a non-constant function, for otherwise, we have $f\xi = 0$, which gives either the constant $f = 0$ or that $\xi = 0$. In both cases we get that $\xi$ is a Killing vector field, which contradicts our assumption that $\xi$ is a nontrivial conformal vector field. Using the equations (2.4) and (3.10) and the fact that $f$ is a constant, we get

$$\nabla_X \nabla h = f(fX + \varphi X), \quad X \in \mathfrak{X}(M).$$

The above equation gives

$$H_h(X, Y) = f^2 g(X, Y) + fg(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{3.11}$$

where $H_h(X, Y) = g(A_h X, Y)$ is the Hessian of the smooth function $h$. Now, using the symmetry of the Hessian $H_h$ and the skew-symmetry of $\varphi$ in the equation (3.11), we get $f\varphi = 0$, and as argued above the constant $f \neq 0$, and consequently, $\varphi = 0$. Hence the equation (3.11) takes the form

$$H_h(X, Y) = f^2 g(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

for a nonzero constant $f$, which proves that $M$ is isometric to the Euclidean space $\mathbb{R}^n$ (cf. [8]).

**Acknowledgement.** This work is supported by King Saud University, Deanship of Scientific Research.

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