The Einstein-Hilbert type action on foliations

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Abstract. The mixed gravitational field equations have been recently introduced for codimension one foliated spacetimes. These Euler-Lagrange equations for the total mixed scalar curvature (as analog of Einstein-Hilbert action) involve a new kind of Ricci curvature. In the work, based on variation formulas for the quantities of extrinsic geometry, we derive Euler-Lagrange equations of the action for arbitrary codimension foliations, in fact, for a closed Riemannian almost-product manifold and adapted variations of metric (i.e., preserving orthogonality of the distributions). Examples of critical metrics of the action are found among twisted products, isoparametric foliations and K-contact metrics.

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Introduction

The problem of minimizing geometric quantities has been studied for a long time: recall, for example, isoperimetric inequalities and estimates of total curvature of submanifolds. In the context of foliations (i.e., partitions of manifolds by submanifolds of a constant dimension), Gluck and Ziller [5] studied the problem of minimizing functions like volume and total bending defined for k-plane fields on manifolds. In the cases mentioned above, the authors consider a fixed Riemannian manifold and look for geometric objects (e.g. submanifolds, foliations) minimizing geometric quantities defined usually as integrals of curvatures of different types. In [10] the following approach to problems in geometry of codimension-one foliations is presented: given a foliated manifold and a property Q of a submanifold, depending on the principal curvatures of the leaves, study Riemannian metrics, which minimize the integral of Q in the class of variations of metrics such that the unit vector field orthogonal to the leaves is the same for all metrics of the variation family. Certainly (like in some of the cases mentioned before) such Riemannian structures may not exist, but if they do, they usually have interesting geometric properties.

Let M be a connected manifold endowed with a Riemannian metric g and complementary orthogonal distributions (subbundles of the tangent bundle TM) $\widetilde{\mathcal{D}}$ and

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$\mathcal{D}$ of ranks $\dim_{\mathbb{R}}\widetilde{\mathcal{D}}_x = n$ and $\dim_{\mathbb{R}}\mathcal{D}_x = p$ for every $x \in M$; the pair $(\widetilde{\mathcal{D}}, \mathcal{D})$ is called an *almost-product structure* on (M, g). The following convention is adopted for the range of indices: $1 \le a, b, \ldots \le n,\ 1 \le i, j, \ldots \le p$. In [2], a tensor calculus, adapted to the decomposition $TM = \widetilde{\mathcal{D}} \oplus \mathcal{D}$, is developed to study the geometry of both the distributions and the ambient manifold. Let $R^\nabla(X,Y) = \nabla_Y\nabla_X - \nabla_X\nabla_Y + \nabla_{[X,Y]}$ be the curvature tensor of the Levi-Civita connection ∇. A plane $\sigma \subset T_xM$ is mixed if $\sigma = \mathrm{span}(v, w)$ for some $v \in \widetilde{\mathcal{D}}_x$ and $w \in \mathcal{D}_x$. The *mixed scalar curvature* (that is an averaged mixed sectional curvature) is one of the simplest curvature invariants of an almost product manifold, see [4, 14], and [7] for generalized subbundles. It is

$$\mathrm{S}_{\mathrm{mix}} = \sum\nolimits_{a,i} K(E_a, \mathcal{E}_i) = \sum\nolimits_{a,i} g(R^\nabla(E_a, \mathcal{E}_i)E_a,\, \mathcal{E}_i)\,, \tag{0.1}$$

where $\{E_a \subset \widetilde{\mathcal{D}},\, \mathcal{E}_i \subset \mathcal{D}\}$ is a local g-orthonormal frame on M. If one of the distributions is spanned by a unit vector field N then $\mathrm{S}_{\mathrm{mix}}$ is the Ricci curvature in the N-direction. Our objectives are to deduce Euler–Lagrange equations of the action

$$J_{\mathrm{mix}} : g \mapsto \int_M \mathrm{S}_{\mathrm{mix}}(g)\,\mathrm{d}\,\mathrm{vol}_g \tag{0.2}$$

for *adapted* (i.e., preserving orthogonality of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$) variations g_t $(|t| < \varepsilon)$, of metric $g_0 = g$, and to characterize critical metrics in several classes of almost-product structures. Functional (0.2) is defined like the gravitational part of classical Einstein-Hilbert action, the difference being the fact that the scalar curvature is replaced by $\mathrm{S}_{\mathrm{mix}}$. The action and its physical meaning for a globally hyperbolic spacetime have been studied in [1], where the Euler-Lagrange equations (called the mixed gravitational field equations) were derived using variation formulas for the curvature tensor, then their linearization and solution for an empty space have been obtained, see also [3] for contact metric structures. As we shall see shortly, the Euler-Lagrange equations of (0.2) involve the partial Ricci tensor $r_{\mathcal{D}}$ (introduced in [9]) with $\mathrm{Tr}_g\, r_{\mathcal{D}} = \mathrm{S}_{\mathrm{mix}}$,

$$r_{\mathcal{D}}(X,Y) = \sum\nolimits_a g(R^\nabla(E_a,\, X^\perp)E_a,\, Y^\perp), \qquad X, Y \in \Gamma(TM), \tag{0.3}$$

and a new Ricci type curvature $\mathrm{Ric}_{\mathcal{D}}$ (the *mixed Ricci tensor*, introduced in [1] for codimension-one foliations), whose properties need to be further investigated.

In Section 1, we derive Euler-Lagrange equations of the action for a closed Riemannian almost-product manifold and adapted variations of metric (i.e., preserving orthogonality of the distributions). All of that done by discovering variation formulas for two types of variations (the second of which preserves the volume of M) for the quantities of extrinsic geometry, i.e., quantities depended on the second fundamental forms and integrability tensors, of an almost-product manifold. Section 2 is devoted to applications of the action to foliations, including flows and codimension-one case, and gives examples of critical metrics of (0.2) among twisted products, isoparametric foliations and K-contact metrics. Our work is restricted to the Riemannian case $(g \in \mathrm{Riem}(M))$ on a closed manifold, but the arbitrary (and Lorentzian) signature case on an open manifold will be on board in a separate forthcoming study.

1 Einstein-Hilbert type action on almost-product manifolds

1.1 Preliminaries

We will define several tensors for one of distributions (say, $\mathcal{D}$, similar tensors for $\widetilde{\mathcal{D}}$ are introduced using $\,\widetilde{}\,$ notation). For a section $X\in\Gamma(TM)$, let $X^\perp$ be the $\mathcal{D}$-component of X with respect to $(\widetilde{\mathcal{D}},\mathcal{D})$. Let $\mathrm{Sym}^2(M)$ be the space of all symmetric $(0,2)$-tensors tangent to M, and $\mathrm{Riem}(M)$ the subspace of positive definite tensors (i.e., Riemanian metrics). A tensor $B\in\mathrm{Sym}^2(M)$ is said to be *adapted* if $B(X^\top,Y^\perp)=0$ for any $X,Y\in\Gamma(TM)$. Let $\mathfrak{M}$ consist of all adapted symmetric tensors on $(M,\widetilde{\mathcal{D}},\mathcal{D})$. The domain of (0.2) is *a priori* the space $\mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})\equiv\mathrm{Riem}(M)\cap\mathfrak{M}$ of all adapted metrics. We say that $B\in\mathrm{Sym}^2(M)$ is $\mathcal{D}$-*truncated* if $B(X^\top,\,\cdot)=0$ for any $X\in\Gamma(TM)$. This notion can be extended to $(1,1)$-tensors. For $B\in\mathrm{Sym}^2(M)$ define the $\mathcal{D}$-truncated component $B^\perp\in\mathrm{Sym}^2(M)$ by setting $B^\perp(X,Y)=B(X^\perp,Y^\perp)$ for any $X,Y\in\Gamma(TM)$. Let $\mathfrak{M}_{\mathcal{D}}\subset\mathfrak{M}$ be the space of $\mathcal{D}$-truncated symmetric $(0,2)$-tensors. There is orthogonal decomposition

$$\mathfrak{M}=\mathfrak{M}_{\mathcal{D}}\oplus\mathfrak{M}_{\widetilde{\mathcal{D}}}\,, \tag{1.1}$$

with respect to the inner product g^* induced on $\mathfrak{M}$ by $g\in\mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})$. If $B\in\mathrm{Sym}^2(M)$ then $B\in\mathfrak{M}\iff B=B^\perp+\widetilde{B}$, see (1.1). Our aim is to compute the directional derivatives

$$D_gJ_{\mathrm{mix}}:T_g\,\mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})\equiv\mathfrak{M}\ \to\ \mathbb{R}$$

for any $g\in\mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})$, i.e., $g=g^\perp+\tilde g$, and study the critical points of J_{mix} with respect to adapted variations of metric. Certainly, in Section 1 we restrict ourselves to the case of $\mathcal{D}$-variations.

The "musical" isomorphisms $\sharp$ and $\flat$ are used for rank one tensors, e.g. if $\omega\in T^1_0(M)$ is a 1-form and $X\in\mathfrak{X}_M$ then $\omega(X)=g(\omega^\sharp,X)=X^\flat(\omega^\sharp)$. Moreover, if $B\in T^0_2M$ then the tensor $B^\sharp\in T^1_1M$ is defined by $g(B^\sharp X,Y)=B(X,Y)$ for $X,Y\in\Gamma(TM)$. For $(0,2)$-tensors A and B we have

$$\langle A,B\rangle=A^{ij}B_{ij}=\mathrm{Tr}\,_g(A^\sharp B^\sharp)=\langle A^\sharp,B^\sharp\rangle.$$

Let $T,h:\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}\to\mathcal{D}$ be the integrability tensor and the 2nd fundamental form of $\widetilde{\mathcal{D}}$,

$$T(X,Y)=(1/2)\,[X,\,Y]^\perp,\quad h(X,Y)\ =\ (1/2)\,(\nabla_XY+\nabla_YX)^\perp.$$

The mean curvature vector of $\widetilde{\mathcal{D}}$ is $H=\mathrm{Tr}\,_g\,h$. A distribution $\widetilde{\mathcal{D}}$ is called *totally umbilical*, *harmonic*, or *totally geodesic*, if $h=\frac1n H\,\tilde g$, $H=0$, or $h=0$, respectively. Let A_Z be the Weingarten operator of $\widetilde{\mathcal{D}}$ with respect to $Z\in\mathcal{D}$, i.e., $g(A_Z(X),Y)=g(h(X,Y),Z)$. The operator $T^\sharp_Z$ is given by $g(T^\sharp_Z(X),Y)=g(T(X,Y),Z)$.

The Divergence Theorem states that $\int_M(\mathrm{div}\,\xi)\,\mathrm{d}\,\mathrm{vol}_g\ =\ 0$ when M is closed. The $\mathcal{D}$-*divergence* of ξ is defined by $\mathrm{div}^\perp\xi=\sum_i g(\nabla_i\,\xi,E_i)$. Thus, $\mathrm{div}\,\xi=\mathrm{div}^\perp\xi+\widetilde{\mathrm{div}}\,\xi$. We have

$$\mathrm{div}^\perp X=\mathrm{div}\,X+g(X,\,H),\quad X\in\Gamma(\mathcal{D}). \tag{1.2}$$

For a $(1,2)$-tensor P, define $(0,2)$-tensor $(\operatorname{div} P)(X,Y) = \operatorname{div}^\perp P + \widetilde{\operatorname{div}}\, P$, where

$$(\operatorname{div}^\perp P)(X,Y) = \sum\nolimits_i g((\nabla_i P)(X,Y), \mathcal{E}_i).$$

For a $\widetilde{\mathcal{D}}$-valued P, we have $\sum_i g((\nabla_i P)(X,Y),\, \mathcal{E}_i) = -g(P(X,Y),\, \tilde H)$; hence,

$$\widetilde{\operatorname{div}}\, P = \operatorname{div} P + \langle P,\, \tilde H\rangle\,, \tag{1.3}$$

where $\langle P,\, \tilde H\rangle(X,Y) := g(P(X,Y),\, \tilde H)$ is a $(0,2)$-tensor. For example, $\operatorname{div}^\perp h = \operatorname{div} h + \langle h,\, H\rangle$. To study $r_{\mathcal{D}}$ (e.g. in Proposition 1.1) we introduce several tensors. The $\mathcal{D}$-deformation of a vector field Z is the symmetric part of ∇Z restricted to $\mathcal{D}$,

$$2\,\mathrm{Def}_{\mathcal{D}}\, Z(X,Y) = g(\nabla_X Z, Y) + g(\nabla_Y Z, X), \quad X, Y \in \Gamma(\mathcal{D}).$$

The antisymmetric part of ∇Z restricted to $\mathcal{D}$ is regarded as a 2-form $d_{\mathcal{D}}\, Z$.

Define the $\mathcal{D}$-truncated symmetric $(0,2)$-tensor Ψ by the identity

$$\Psi(X,Y) \;=\; \operatorname{Tr}_g(A_Y A_X + T^\sharp_Y T^\sharp_X), \quad X, Y \in \Gamma(\mathcal{D})\,.$$

Define $(1,1)$-tensors $\mathcal{A} := \sum_i A_i^2$ and $\mathcal{T} := \sum_i (T_i^\sharp)^2$. The *extrinsic curvature* of $\widetilde{\mathcal{D}}$:

$$R^{\,\mathrm{ex}}(X,\, Y,\, Z,\, W) = g(h(X^\top, Z^\top), h(Y^\top, W^\top)) - g(h(X^\top,\, W^\top), h(Z^\top,\, Y^\top))$$

is useful in the study of extrinsic geometry of foliations, see [8, 10]. The traces (on $\widetilde{\mathcal{D}}$)

$$\mathrm{Ric}^{\mathrm{ex}}(X,\, Y) = \operatorname{Tr}_g R^{\,\mathrm{ex}}(X,\, Y,\, \cdot,\, \cdot), \qquad \mathrm{S}_{\,\mathrm{ex}} = \operatorname{Tr}_g \mathrm{Ric}^{\mathrm{ex}}$$

are called the extrinsic Ricci and scalar curvature of $\widetilde{\mathcal{D}}$. Note that $\mathrm{S}_{\,\mathrm{ex}} = \|H\|^2 - \|h\|^2$.

Proposition 1.1. *Let* $g \in \mathrm{Riem}(M,\, \widetilde{\mathcal{D}},\, \mathcal{D})$. *Then the following identities hold:*

$$r_{\mathcal{D}} \;=\; \operatorname{div} \tilde h + \langle \tilde h,\, \tilde H\rangle - \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat - \Psi + \mathrm{Def}_{\mathcal{D}}\, H\,, \tag{1.4}$$

$$d_{\mathcal{D}}\, H \;=\; -\,\widetilde{\operatorname{div}}\, \tilde T + \sum\nolimits_a \big(\tilde A_a \tilde T^\sharp_a + \tilde T^\sharp_a \tilde A_a\big)^\flat. \tag{1.5}$$

Proof. For $X, Y \in \Gamma(\mathcal{D})$ and $U, V \in \Gamma(\widetilde{\mathcal{D}})$ we have, see [8, Lemma 2.25],

$$g(R^\nabla(U,\, X)V,\, Y) = g(((\nabla_U \tilde C)_V - \tilde C_V \tilde C_U)X, Y) + g(((\nabla_X C)_Y - C_Y\, C_X)U, V), \tag{1.6}$$

where conullity tensors $\tilde C : \Gamma(\widetilde{\mathcal{D}}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ and $C : \Gamma(\mathcal{D}) \times \Gamma(\widetilde{\mathcal{D}}) \to \Gamma(\widetilde{\mathcal{D}})$ are defined by

$$\tilde C_U(X) = -(\nabla_X\, U)^\perp, \qquad C_X(U) = -\,(\nabla_U\, X)^\top\,.$$

Assume $\nabla_X\, Y \in \widetilde{\mathcal{D}}_x$ and $\nabla_X\, E_a \in \mathcal{D}_x$ at a given point $x \in M$. Note that

$$\sum\nolimits_a g((\nabla_X\, C)_Y(E_a),\, E_a) = \nabla_X\big(g\,\big(\sum\nolimits_a h(E_a, E_a),\, Y\big)\big) = g(\nabla_X H,\, Y).$$

Write $\widetilde{\operatorname{div}}\, \tilde C = \sum_{a=1}^n (\nabla_a\, \tilde C)_a$. Thus, tracing (1.6) over $\widetilde{\mathcal{D}}_x$ yields

$$r_{\mathcal{D}}(X,Y) = g(\widetilde{\operatorname{div}}\, \tilde C(X), Y) - g(\sum\nolimits_a \tilde C_a^2(X), Y) + g(\nabla_X H, Y) - \operatorname{Tr}_g(C_Y C_X)\,. \tag{1.7}$$

Using $\operatorname{Tr}_g(A_Y T^\sharp_X) = 0 = \operatorname{Tr}_g(T^\sharp_Y A_X)$ (since h is symmetric and T is antisymmetric), we extract (1.4)–(1.5) as the symmetric and antisymmetric parts of (1.7). $\square$

Tracing (1.4) (over $\mathcal{D}$) and applying the equalities

$$\begin{aligned} \mathrm{Tr}_{\,g}\,\Psi &= \|h\|^2 - \|T\|^2, \quad \mathrm{Tr}\,\mathcal{A} = \|h\|^2, \quad \mathrm{Tr}\,\mathcal{T} = -\|T\|^2, \\ \mathrm{Tr}_{\,g}\,(\mathrm{div}\,h) &= \mathrm{div}\,H, \qquad \mathrm{Tr}_{\,g}\,(\mathrm{Def}_{\mathcal{D}}\,H) = \mathrm{div}\,H + \|H\|^2, \end{aligned}$$

yields (see also [14])

$$\mathrm{S}_{\mathrm{mix}} = \mathrm{S}_{\,\mathrm{ex}} + \widetilde{\mathrm{S}}_{\,\mathrm{ex}} + \|T\|^2 + \|\tilde T\|^2 + \mathrm{div}(H + \tilde H)\,. \tag{1.8}$$

1.2 Variation formulas

In order to apply the methods of variational calculus to J_{mix}, consider smooth 1-parameter variations $\big\{g_t \in \mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D}) : |t| < \varepsilon\big\}$ of an adapted metric $g_0 = g$. We adopt the notations

$$\partial_t \equiv \partial/\partial t, \quad B \equiv \{\partial_t g_t\}_{|t=0}.$$

Taking into account (1.1), it is sufficient to work with special curves $\{g_t\}_{|t|<\varepsilon}$ issuing at $g \in \mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})$ called $\mathcal{D}$-*variations*, as the associated infinitesimal variations B lie in $\mathfrak{M}_{\widetilde{\mathcal{D}}}$. An adapted variation of a metric $g \in \mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})$ has the form $\{g_t^\perp + \tilde g_t : |t| < \varepsilon\}$. The corresponding $\mathcal{D}$-variation of g is

$$\big\{g_t = g_t^\perp + \tilde g :\ |t| < \varepsilon\big\}. \tag{1.9}$$

For adapted variations and $X, Y, Z \in \Gamma(TM)$ we have, see for example [10, 13],

$$2\,g_t(\partial_t(\nabla^t_X\,Y), Z) \;=\; (\nabla^t_X\,B)(Y,Z) + (\nabla^t_Y\,B)(X,Z) - (\nabla^t_Z\,B)(X,Y). \tag{1.10}$$

Lemma 1.2. *Let a local $(\widetilde{\mathcal{D}},\,\mathcal{D})$-adapted frame $\{E_a,\,\mathcal{E}_i\}$ evolves by adapted variation g_t as*

$$\partial_t E_a = -(1/2)\,B^\sharp_t(E_a), \qquad \partial_t \mathcal{E}_i = -(1/2)\,B^\sharp_t(\mathcal{E}_i).$$

Then, for all t, $\{E_a(t), \mathcal{E}_i(t)\}$ is a g_t-orthonormal frame adapted to $(\widetilde{\mathcal{D}}, \mathcal{D})$.

Proof. For $\{E_a(t)\}$ (and similarly for $\{\mathcal{E}_i(t)\}$) we have

$$\begin{aligned} &\partial_t(g_t(E_a, E_b)) = g_t(\partial_t E_a(t), E_b(t)) + g_t(E_a(t), \partial_t E_b(t)) + (\partial_t g_t)(E_a(t), E_b(t)) \\ &= B_t(E_a(t), E_b(t)) - \frac{1}{2}\,g_t(B^\sharp_t(E_a(t)), E_b(t)) - \frac{1}{2}\,g_t(E_a(t), B^\sharp_t(E_b(t))) = 0 \end{aligned}$$

that completes the proof. □

Lemma 1.3 (see [11])**.** *For $\mathcal{D}$-variations of g we have*

$$\begin{aligned} 2\,\partial_t \tilde h(X,Y) &= (\tilde h - \tilde T)(B^\sharp(X), Y) + (\tilde h + \tilde T)(X, B^\sharp(Y)) - \widetilde{\nabla} B(X,Y), \\ 2\,\partial_t \tilde H &= -\widetilde{\nabla}(\mathrm{Tr}\ B^\sharp), \qquad \partial_t h = -B^\sharp \circ h, \qquad \partial_t H = -B^\sharp(H). \end{aligned}$$

Hence, $\mathcal{D}$-variations preserve total umbilicity, total geodesy and harmonicity of $\widetilde{\mathcal{D}}$.

Define symmetric $(0,2)$-tensor Φ_h and symmetric $(0,2)$-tensor Φ_T, which vanish when $n=1$, using the identities (with arbitrary $B\in\mathfrak{M}$)

$$\begin{aligned}\langle\Phi_h,\ B\rangle &= B(H,\,H)-\sum\nolimits_{a,\,b}B(h(E_a,E_b),h(E_a,E_b)),\\ \langle\Phi_T,\ B\rangle &= -\sum\nolimits_{a,\,b}B(T(E_a,E_b),T(E_a,E_b)).\end{aligned}$$

We have

$$\operatorname{Tr}_g\Phi_h=\mathrm{S}_{\rm ex},\quad \operatorname{Tr}_g\Phi_T=-\|T\|^2.$$

Define a self-adjoint $(1,1)$-tensor with zero trace

$$\mathcal{K}=\sum\nolimits_i[T_i^\sharp,A_i]=\sum\nolimits_i(T_i^\sharp A_i-A_iT_i^\sharp).$$

Observe that if $\widetilde{\mathcal{D}}$ is integrable then $\mathcal{K}=0$ (since $T_i^\sharp=0$). Also, if $\widetilde{\mathcal{D}}$ is totally umbilical, then every operator A_i is a multiple of identity and $\mathcal{K}$ vanishes as well.

Lemma 1.4. *For $\mathcal{D}$-variations of $g\in\operatorname{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})$ we have*

$$\begin{aligned}&\partial_t\,\widetilde{\mathrm{S}}_{\rm ex}(g)=\langle(\operatorname{div}\tilde H)\,g-\operatorname{div}\tilde h-\tilde{\mathcal{K}}^\flat,\ B\rangle+\operatorname{div}(\langle\tilde h,\,B\rangle-(\operatorname{Tr}_g B)\tilde H),\\ &\partial_t\,\mathrm{S}_{\rm ex}(g)=-\langle\Phi_h,\ B\rangle,\\ &\partial_t\,\|\,\tilde T\,\|^2=\langle 2\,\widetilde{\mathcal{T}}^\flat,\ B\rangle,\quad \partial_t\,\|\,T\,\|^2=-\langle\Phi_T,\ B\rangle\,.\end{aligned}\tag{1.11}$$

Proof. Assume $\nabla_a\,\mathcal{E}_i\in\widetilde{\mathcal{D}}_x$ at a point $x\in M$. In the calculations below we use (1.10) and Lemmas 1.2 and 1.3. First we obtain $(1.11)_3$:

$$\begin{aligned}\partial_t\,\|\,\tilde T\,\|^2 &= 2\sum\nolimits_{i,j,a}g(\tilde T(\mathcal{E}_i,\mathcal{E}_j),E_a)\,g\big(\tilde T(\partial_t\mathcal{E}_i,\mathcal{E}_j)+\tilde T(\mathcal{E}_i,\partial_t\mathcal{E}_j),E_a\big)\\ &= -\sum\nolimits_{i,j,a}g(\tilde T_a^\sharp(\mathcal{E}_i),\mathcal{E}_j)\,g((\tilde T_a^\sharp B^\sharp+B^\sharp\tilde T_a^\sharp)(\mathcal{E}_i),\mathcal{E}_j)\\ &= -\sum\nolimits_{i,a}g((\tilde T_a^\sharp B^\sharp+B^\sharp\tilde T_a^\sharp)(\mathcal{E}_i),\tilde T_a^\sharp(\mathcal{E}_i))\\ &= \sum\nolimits_{i,a}g(((\tilde T_a^\sharp)^2B^\sharp+\tilde T_a^\sharp B^\sharp\tilde T_a^\sharp)(\mathcal{E}_i),\,\mathcal{E}_i)\\ &= 2\sum\nolimits_a\operatorname{Tr}_g((\tilde T_a^\sharp)^2B^\sharp)=2\operatorname{Tr}_g(\widetilde{\mathcal{T}}B^\sharp)=\langle 2\,\widetilde{\mathcal{T}}^\flat,\ B\rangle.\end{aligned}$$

Next, by (1.3), we obtain

$$\begin{aligned}\partial_t\,\|\,\tilde h\,\|^2 &= \sum_{i,j,a}g(\tilde h(\mathcal{E}_i,\mathcal{E}_j),\,E_a)\,\big(g(\tilde T(\mathcal{E}_i,B^\sharp(\mathcal{E}_j))-\tilde T(B^\sharp(\mathcal{E}_i),\mathcal{E}_j),\,E_a)-\nabla_a\,B(\mathcal{E}_i,\mathcal{E}_j)\big)\\ &= \sum\nolimits_{i,j,a}\Big(g(\tilde A_a(\mathcal{E}_i),\,\mathcal{E}_j)\,g([B^\sharp,\tilde T_a^\sharp](\mathcal{E}_i),\,\mathcal{E}_j)-\nabla_a\,\big(g(B(\mathcal{E}_i,\mathcal{E}_j)\,\tilde h(\mathcal{E}_i,\mathcal{E}_j),\,E_a)\big)\\ &\quad-\ \nabla_a\,g(\tilde h(\mathcal{E}_i,\mathcal{E}_j),\,E_a)\,B(\mathcal{E}_i,\,\mathcal{E}_j)\Big)=\langle\widetilde{\operatorname{div}}\,\tilde h-\langle\tilde h,\,\tilde H\rangle+\tilde{\mathcal{K}}^\flat,\,B\rangle-\operatorname{div}(\langle\tilde h,\,B\rangle).\end{aligned}$$

Applying $B(\tilde H,\tilde H)=0$ (since B is $\mathcal{D}$-truncated) we get

$$\partial_t\,\|\,\tilde H\,\|^2\ =\ \partial_t g(\tilde H,\,\tilde H)=2\,g(\partial_t\tilde H,\,\tilde H)=-g(\nabla(\operatorname{Tr}\,B^\sharp),\,\tilde H).$$

Notice that $g(\nabla(\operatorname{Tr}\,B^\sharp),\,\tilde H)=\operatorname{div}((\operatorname{Tr}\,B^\sharp)\tilde H)-(\operatorname{div}\tilde H)\operatorname{Tr}\,B^\sharp$; hence, $(1.11)_1$ follows.

Next we have

$$\begin{aligned}\partial_t \| H \|^2 &= \partial_t g(H,\, H) = B(H,H) + 2\, g(\partial_t H,\, H)\\ &= B(H,H) - 2\, g(B^\sharp(H),\, H) = -B(H,H),\\ \partial_t \| h \|^2 &= \partial_t \sum\nolimits_{i,\,a,\,b} g(h(E_a,\, E_b),\, \mathcal{E}_i)^2 = -\sum\nolimits_{a,\,b} B(h(E_a,\, E_b),\, h(E_a,\, E_b))\,.\end{aligned}$$

From the above, $(1.11)_2$ follows. Finally, we have $(1.11)_4$:

$$\begin{aligned}\partial_t \| T \|^2 &= 2 \sum\nolimits_{i,\,a,\,b} g(T(E_a,\, E_b),\, \mathcal{E}_i)\, \partial_t (g(T(E_a,\, E_b),\, \mathcal{E}_i))\\ &= 2 \sum\nolimits_{i,\,a,\,b} g(T(E_a,\, E_b),\, \mathcal{E}_i) \big(B(T(E_a,\, E_b),\, \mathcal{E}_i) + g(T(E_a,\, E_b),\, \partial_t \mathcal{E}_i)\big)\\ &= \sum\nolimits_{i,\,a,\,b} g(T(E_a,\, E_b),\, \mathcal{E}_i)\, g(T(E_a,\, E_b),\, B^\sharp(\mathcal{E}_i))\\ &= \sum\nolimits_{a,\,b} B(T(E_a,\, E_b),\, T(E_a,\, E_b)).\end{aligned}$$

This completes the proof. □

1.3 Euler-Lagrange equations

In this section we compute directional derivatives of $J_{\rm mix}$ and then derive Euler-Lagrange equations of the variational principle $\delta J_{\rm mix}(g) = 0$ on a closed Riemannian almost-product manifold. For every $f \in L^1(M,\ {\rm d\,vol}_g)$, denote by

$$\overline{f}(M,g) = {\rm Vol}^{-1}(M,g) \int_M f \,{\rm d\,vol}_g$$

the mean value of f on M. Together with g_t of (1.9), consider the metrics

$$\bar g_t \;=\; \phi_t g_t^\perp + \tilde g, \quad \phi_t \equiv \big({\rm Vol}(M, g_t)/{\rm Vol}(M,g)\big)^{-2/p}\,, \quad |t| < \varepsilon. \tag{1.12}$$

We will show that ${\rm Vol}(M, \bar g_t) = {\rm Vol}(M, g)$ for all t.

Proposition 1.5. *The $\mathcal{D}$-variations of the action* (0.2)*, corresponding to $\bar g_t$ and g_t, are related by*

$$\frac{\rm d}{\rm dt}\big\{J_{\rm mix}(\bar g_t)\big\}_{|t=0} = \frac{\rm d}{\rm dt}\big\{J_{\rm mix}(g_t)\big\}_{|t=0} - \frac{1}{2}\, \overline{{\rm S}^*_{\rm mix}}(M,g) \int_M \big({\rm Tr}_{\,g}\, B\big)\, {\rm d\,vol}_g\,, \tag{1.13}$$

where ${\rm S}^*_{\rm mix} = {\rm S}_{\rm mix} - \frac{2}{p}\,({\rm S}_{\,\rm ex} + 2\,\|\tilde T\|^2 - \|T\|^2)$.

Proof. By (1.8) and the Divergence Theorem, we have

$$J_{\rm mix}(g) = \int_M Q(g)\ {\rm d\,vol}_g\,,$$

where $Q(g) := {\rm S}_{\rm mix} - {\rm div}(H + \tilde H)$ is represented using (1.8) as

$$Q(g) = {\rm S}_{\,\rm ex}(g) + \widetilde{\rm S}_{\,\rm ex}(g) + \|T\|^2_g + \|\tilde T\|^2_g\,. \tag{1.14}$$

Let us fix a $\mathcal{D}$-variation (1.9). The volume form evolves as (cf. [10])

$$\partial_t \left(\mathrm{d\,vol}_{g_t}\right) = \frac{1}{2}\,(\mathrm{Tr}_{\,g_t} B_t)\,\mathrm{d\,vol}_{g_t}\,. \tag{1.15}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{dt}}\big\{J_{\mathrm{mix}}(g_t)\big\}_{|t=0} = \int_M \Big\{\partial_t Q(g_t)_{|t=0} + \frac{1}{2}\,Q(g)\,\mathrm{Tr}_{\,g} B\Big\}\,\mathrm{d\,vol}_g\,. \tag{1.16}$$

As $\bar g_t$ are $\mathcal{D}$-conformal to g_t with constant scale ϕ_t, their volume forms are related as

$$\mathrm{d\,vol}_{\,\bar g_t} = \phi_t^{p/2}\mathrm{d\,vol}_{\,g_t}; \tag{1.17}$$

hence, $\mathrm{Vol}(M,\bar g_t) = \int_M \mathrm{d\,vol}_{\,\bar g_t} = \mathrm{Vol}(M,g)$. Let us differentiate (1.17) in order to obtain

$$\begin{aligned}\partial_t\,(\mathrm{d\,vol}_{\,\bar g_t}) &= (\phi_t^{p/2})'\,\mathrm{d\,vol}_{g_t} + \phi_t^{p/2}\,\partial_t\,(\mathrm{d\,vol}_{g_t})\\ &= \frac{1}{2}\Big(\mathrm{Tr}\;B_t^\sharp - \frac{1}{\mathrm{Vol}(M,g_t)}\int_M (\mathrm{Tr}_{\,g_t} B_t)\,\mathrm{d\,vol}_{\,g_t}\Big)\,\mathrm{d\,vol}_{\,\bar g_t}\,.\end{aligned}$$

We used (1.15) and the fact that $\phi_0 = 1$ and $\phi_t' = -\frac{\phi_t}{p\,\mathrm{Vol}(M,g_t)}\int_M (\mathrm{Tr}_{\,g_t} B_t)\,\mathrm{d\,vol}_{\,g_t}$. For the $\mathcal{D}$-scaling $\bar g = \phi\, g^\perp + \tilde g$ of $g = g^\perp + \tilde g$, using (1.11), we have

$$\begin{aligned}&\|T\|^2_{\bar g} = \phi\,\|T\|^2_g,\quad \|h\|^2_{\bar g} = \phi^{-1}\|h\|^2_g,\quad \|\tilde h\|^2_{\bar g} = \|\tilde h\|^2_g,\\ &\|\tilde T\|^2_{\bar g} = \phi^{-2}\|\tilde T\|^2_g,\quad \|H\|^2_{\bar g} = \phi^{-1}\|H\|^2_g,\quad \|\tilde H\|^2_{\bar g} = \|\tilde H\|^2_g.\end{aligned}$$

Introducing into $Q(\bar g)$ formula (1.14), we obtain

$$Q(\bar g) = Q(g) + (\phi^{-1}-1)\,\mathrm{S}_{\,\mathrm{ex}}(g) + (\phi^{-2}-1)\,\|\tilde T\|^2_g + (\phi-1)\,\|T\|^2_g\,.$$

(For example, $Q(\bar g) = Q(g)$ when $n=1$ and $\tilde T = 0$). Differentiating the above we get

$$\partial_t Q(\bar g_t)_{|t=0} = \partial_t Q(g_t)_{|t=0} - \phi_0'\left(\mathrm{S}_{\,\mathrm{ex}}(g) + 2\,\|\tilde T\|^2_g - \|T\|^2_g\right),$$

where $\phi_0' = -\frac{1}{p}\,\frac{1}{\mathrm{Vol}(M,g)}\int_M (\mathrm{Tr}_{\,g} B)\,\mathrm{d\,vol}_{\,g}$. From the above (1.13) follows, that completes the proof. □

Remark 1.1. (i) It should be stressed that we work with two types of variations, (1.9) and (1.12); the second of which preserves the volume of M. Formulas containing $\mathrm{S}^*_{\mathrm{mix}}$ correspond to 1-parameter variations (1.12). To obtain similar formulas, corresponding to 1-parameter variations of the form (1.9), one should merely delete the mean value terms $\mathrm{S}^*_{\mathrm{mix}}$ in the previous identities.

(ii) In general, (1.9) do not preserve the volume of M, and one may obtain only $J_{\mathrm{mix}}(g) = 0$ for critical metrics. Let $\widetilde{\mathcal{D}}$ be integrable of codimension-one. Define functions $\tau_1 = \mathrm{Tr}_{\,g} h$ and $\tau_2 = \|h\|^2_g$. For a $\mathcal{D}$-variation $g_t = g_t^\perp + \tilde g$ with $g_t^\perp = (1+t)\,g_0^\perp$ ($|t|<1$), we have $\partial_t g_t = \frac{1}{1+t}\,g_t^\perp$ and $\mathrm{d\,vol}_t = (1+t)^{1/2}\mathrm{d\,vol}$. Since $(\tau_1^2-\tau_2)_t = \frac{1}{1+t}\,(\tau_1^2-\tau_2)_0$, by (2.23) in Sect. 2.3, we find that

$$J_{\mathrm{mix}}(g_t) = \int_M (\tau_1^2-\tau_2)\,\mathrm{d\,vol}_t = (1+t)^{-\frac{1}{2}}J_{\mathrm{mix}}(g) \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{dt}}\big\{J_{\mathrm{mix}}(g_t)\big\}_{|t=0} = -\frac{1}{2}\,J_{\mathrm{mix}}(g).$$

Thus, if g is a $\mathcal{D}$-critical point of the action J_{mix}, then $J_{\mathrm{mix}}(g) = 0$.

Euler-Lagrange equations of the variational principle $\delta J_{\rm mix}(g) = 0$ on a closed almost-product manifold have a view $P = \lambda\, g^\perp$ or $P = \lambda\,\tilde g$ for certain tensor P and function λ on M.

Theorem 1.6 (Euler-Lagrange equations). *Let $\widetilde{\mathcal D}$ and $\mathcal D$ be complementary distributions on a closed manifold M. If $g \in {\rm Riem}(M,\,\widetilde{\mathcal D},\,\mathcal D)$ is a critical point of $J_{\rm mix}$ for $\mathcal D$-variations then*

$$\operatorname{div}\tilde h - 2\,\widetilde{\mathcal T}^\flat + \Phi_h + \Phi_T + \tilde{\mathcal K}^\flat = \frac12\,\big({\rm S}_{\rm mix} - {\rm S}^*_{\rm mix}(M,g) + \operatorname{div}(\tilde H - H)\big)\,g^\perp, \tag{1.18}$$

or equivalently, in terms of $r_{\mathcal D}$,

$$\begin{aligned} & r_{\mathcal D} - \langle \tilde h,\,\tilde H\rangle + \widetilde{\mathcal A}^\flat - \widetilde{\mathcal T}^\flat + \Phi_h + \Phi_T + \Psi - {\rm Def}_{\mathcal D}\,H + \tilde{\mathcal K}^\flat \\ & = \frac12\,\big({\rm S}_{\rm mix} - {\rm S}^*_{\rm mix}(M,g) + \operatorname{div}(\tilde H - H)\big)\,g^\perp. \end{aligned} \tag{1.19}$$

Proof. Applying Lemma 1.4 to (1.14), and using (1.3) and the Divergence Theorem, we get

$$\int_M \partial_t Q_{t|t=0}\,{\rm d\,vol}_g = \int_M \big\langle -\operatorname{div}(\tilde h - \tilde H\,g^\perp) + 2\,\widetilde{\mathcal T}^\flat - \Phi_h - \Phi_T - \tilde{\mathcal K}^\flat,\ B\big\rangle\,{\rm d\,vol}_g,$$

where $B = \partial_t g_{t\,|t=0} \in \mathfrak M_{\mathcal D}$. Notice that ${\rm Tr}_{\,g}\,B = \langle B,\,g^\perp\rangle$. By (1.16) and Proposition 1.5 we have

$$\begin{aligned} \frac{\rm d}{\rm dt}\,\big\{J_{\rm mix}(\bar g_t)\big\}_{|t=0} &= \int_M \Big\langle -\operatorname{div}(\tilde h - \tilde H\,g^\perp) + 2\,\widetilde{\mathcal T}^\flat - \Phi_h - \Phi_T - \tilde{\mathcal K}^\flat \\ &+\ \frac12\,\big({\rm S}_{\rm mix} - {\rm S}^*_{\rm mix}(M,g) - \operatorname{div}(\tilde H + H)\big)\,g^\perp,\ B\Big\rangle\,{\rm d\,vol}_g\,. \end{aligned} \tag{1.20}$$

If g is $\mathcal D$-critical for $J_{\rm mix}$ then the integrand in (1.20) is zero for any $B \in \mathfrak M_{\mathcal D}$; that yields (1.18). By Proposition 1.1 and replacing $\operatorname{div}\tilde h$ in (1.18) due to (1.4), we obtain (1.19). □

Remark 1.2. The above mixed field equations admit a number of solutions, see Section 2, which may find applications in theoretical physics (see discussion in [1]).

Example 1.3. Consider any of the Hopf fibrations $\pi: S^{\,2m+1} \to \mathbb{C}P^m$, $\pi: S^{4m+3} \to HP^m$, $\pi: CP^{2m+1} \to HP^m$, endowed with standard metrics g. Let $\widetilde{\mathcal D}$ be the distribution tangent to the fibers. Thus, $\widetilde{\mathcal D}$ is integrable. Since both distributions are totally geodesic, the critical metric g should satisfy

$$r_{\mathcal D} - \widetilde{\mathcal T}^\flat + \Phi_T + \Psi = \frac12\,\big({\rm S}_{\rm mix} - {\rm S}^*_{\rm mix}(M,g)\big)\,g^\perp$$

and its 'dual'. Certainly, the identities (1.4) and (1.8) yield: $r_{\mathcal D} = -\widetilde{\mathcal T}^\flat$, and $r_{\mathcal D} = \frac1p\,{\rm S}_{\rm mix}\,g^\perp$, where ${\rm S}_{\rm mix} = \|\tilde T\|^2 = {\rm const} > 0$. Thus, g is critical for $J_{\rm mix}$ for all adapted variations.

2 Einstein-Hilbert type action on foliations

In this section we study an n-dimensional foliation $\mathcal{F}$ (i.e., $\widetilde{\mathcal{D}} = T\mathcal{F}$) of a closed Riemannian manifold (M, g). There are many geometrically different types of foliations, e.g. totally geodesic ($h = 0$), Riemannian ($\tilde h = 0$), totally umbilical ($h = \frac{1}{n} H\, \tilde g$) and conformal ($\tilde h = \frac{1}{p}\, \tilde H\, g^\perp$). We write $r_{\,\mathcal{F}} = r_{\widetilde{\mathcal{D}}}$, thus, 'dual' to (1.4)–(1.5) equations read

$$r_{\,\mathcal{F}} = \operatorname{div} h + \langle h,\, H\rangle - \mathcal{A}^\flat - \widetilde\Psi + \operatorname{Def}_{\mathcal{F}} \tilde H\,, \qquad d_{\mathcal{F}}\, \tilde H = 0\,. \tag{2.1}$$

2.1 Critical adapted metrics

From Theorem 1.6, we obtain the following.

Corollary 2.1 (**Euler-Lagrange equations).** *Let $\mathcal{F}$ be a foliation with a transversal distribution $\mathcal{D}$ on a closed manifold M. If $g \in \operatorname{Riem}(M,\, T\mathcal{F},\, \mathcal{D})$ is a critical point of $J_{\rm mix}$ with respect to adapted variations, then*

$$\begin{aligned} & r_{\mathcal{D}} - \langle \tilde h,\, \tilde H\rangle + \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat + \Phi_h + \Psi - \operatorname{Def}_{\mathcal{D}} H + \tilde{\mathcal{K}}^\flat \\ & \quad = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(M,g) + \operatorname{div}(\tilde H - H)\big)\, g^\perp \quad (\text{for } \mathcal{D}\text{−variations}), \end{aligned} \tag{2.2}$$

$$\begin{aligned} & r_{\,\mathcal{F}} - \langle h,\, H\rangle + \mathcal{A}^\flat + \Phi_{\tilde h} + \Phi_{\tilde{\mathcal{T}}} + \widetilde\Psi - \operatorname{Def}_{\mathcal{F}} \tilde H \\ & \quad = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(M,g) + \operatorname{div}(H - \tilde H)\big)\, \tilde g \quad (\text{for } T\mathcal{F}\text{−variations}), \end{aligned} \tag{2.3}$$

where $\Psi(X,Y) = \operatorname{Tr}_g(A_Y A_X)$ and definition of $\mathrm{S}^*_{\rm mix}$ has the form

$$\mathrm{S}^*_{\rm mix} = \mathrm{S}_{\rm mix} - \begin{cases} \frac{2}{p}\big(\mathrm{S}_{\,\rm ex} + 2\,\|\tilde T\|^2\big) & \text{for } \mathcal{D}\text{−variations}, \\ \frac{2}{n}\big(\widetilde{\mathrm{S}}_{\,\rm ex} - \|\tilde T\|^2\big) & \text{for } T\mathcal{F}\text{−variations}\,. \end{cases} \tag{2.4}$$

Example 2.1. Let $\mathcal{F}$ be a totally umbilical foliation (i.e., $h = \frac{1}{n} H\tilde g$ and $T = 0$). Then

$$\Phi_h = \frac{n-1}{n}\, H^\flat \otimes H^\flat, \quad \mathcal{A}^\flat = \frac{1}{n^2}\, \|H\|^2\, \tilde g, \quad \Psi = \frac{1}{n}\, H^\flat \otimes H^\flat, \quad \mathrm{S}_{\,\rm ex} = \frac{n-1}{n}\, \|H\|^2.$$

Hence, (2.2) reads

$$\begin{aligned} & r_{\mathcal{D}} - \langle \tilde h,\, \tilde H\rangle + \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat + H^\flat \otimes H^\flat - \operatorname{Def}_{\mathcal{D}} H + \tilde{\mathcal{K}}^\flat \\ & \quad = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(M,g) + \operatorname{div}(\tilde H - H)\big)\, g^\perp \quad (\text{for } \mathcal{D}\text{−variations}). \end{aligned} \tag{2.5}$$

We find sufficient conditions for $\mathcal{D}$-critical (and similarly for $\widetilde{\mathcal{D}}$-critical) metrics.

Theorem 2.2. *Let a metric $g \in \operatorname{Riem}(M,\, T\mathcal{F},\, \mathcal{D})$ be $\mathcal{D}$-critical for (0.2), $n, p > 1$, and $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are tangent to totally umbilical foliations. Then the leaves of $\widetilde{\mathcal{D}}$ are totally geodesic and*

$$r_{\mathcal{D}} = (\mathrm{S}_{\rm mix}/p)\, g^\perp \ge 0, \quad \text{if } p = 2 \ \text{ then } \mathrm{S}_{\rm mix} = \text{const}. \tag{2.6}$$

Proof. We have the identity, see (1.4) with $\tilde T=0$ and $\tilde h=\frac1p\,\tilde H\,g^\perp$,

$$r_{\mathcal D}+\frac1n\,H^\flat\otimes H^\flat-\operatorname{Def}_{\mathcal D}H=\Big(\frac{p-1}{p^2}\,\|\tilde H\|^2+\frac1p\,\operatorname{div}\tilde H\Big)g^\perp\,. \tag{2.7}$$

Hence, or by (1.8), $\mathrm{S}_{\rm mix}=\frac{n-1}{n}\,\|H\|^2+\frac{p-1}{p}\,\|\tilde H\|^2+\operatorname{div}(H+\tilde H)$ and $J_{\rm mix}(g)\ge 0$.

Let the metric g be $\mathcal D$-critical. By (2.5) we have

$$r_{\mathcal D}+H^\flat\otimes H^\flat-\operatorname{Def}_{\mathcal D}H=\frac12\Big(\mathrm{S}_{\rm mix}-\mathrm{S}^*_{\rm mix}(M,g)+\frac{2(p-1)}{p^2}\|\tilde H\|^2+\operatorname{div}(\tilde H-H)\Big)g^\perp. \tag{2.8}$$

The difference of (2.8) and (2.7) is

$$\frac{n-1}{n}H^\flat\otimes H^\flat=\frac12\Big(\frac{n-1}{n}\|H\|^2+\frac{p-1}{p}\|\tilde H\|^2-\mathrm{S}^*_{\rm mix}(M,g)+\frac{2(p-1)}{p}\operatorname{div}\tilde H\Big)\,g^\perp.$$

As the symmetric $(0,2)$-tensor $H^\flat\otimes H^\flat$ has rank ≤ 1, and $g^\perp$ has rank p, then for $n,p>1$, we obtain $H=0$; hence, the leaves of $\widetilde{\mathcal D}$ are totally geodesic, and by (2.7), $r_{\mathcal D}$ is $\mathcal D$-conformal and $\mathrm{S}_{\rm mix}=\mathrm{S}^*_{\rm mix}(M,g)-\frac{p-2}{p}\,\operatorname{div}\tilde H$ holds. □

Example 2.2. Let $M=M_1\times M_2$ be a product of Riemannian manifolds $(M_i\,,\,g_i)$ $(i\in\{1,2\})$, and let $\pi_i:M\to M_i$ and $d\pi_i:TM\to TM_i$ be the canonical projections. Given twisting functions $f_i\in C^\infty(M)$, a *double-twisted product* $M_1\times_{(f_1,f_2)}M_2$ is $M_1\times M_2$ with the metric $g=e^{f_1}\,\pi_1^*g_1+e^{f_2}\,\pi_2^*g_2$. If $f_1=\mathrm{const}$ then we have a twisted product (a warped product if, in addition, $f_2=F\circ\pi_1$ for some $F\in C^\infty(M_1)$). The leaves $M_1\times\{y\}$ (tangent to $\widetilde{\mathcal D}$) and the fibers $\{x\}\times M_2$ (tangent to $\mathcal D$) are totally umbilical in (M,g) and this property characterizes double-twisted products (cf. [6]). For any double-twisted product, we have $T=\tilde T=0$ and

$$A_Y\;=\;-Y(f_1)\,\widetilde{\rm id}\,,\qquad h=-(\nabla^\perp f_1)\,\tilde g,\quad H=-n\nabla^\perp f_1,$$

where $Y\in\mathcal D$ is a unit vector. Define the $\mathcal D$-Laplacian of f_1 by $\Delta^\perp f_1=\operatorname{div}^\perp((\nabla f_1)^\perp)$. The identity $\operatorname{div}(\phi\,\xi)=\phi\operatorname{div}\xi+\xi(\phi)$ together with (1.2) for $\xi=\nabla^\perp f_1$ imply

$$\Delta^\perp f_1=\operatorname{div}(\,\nabla^\perp f_1+f_1H)-(\,\operatorname{div}H)f_1.$$

In our case, $\operatorname{div}H=-n\,\Delta^\perp f_1-n^2\|\nabla^\perp f_1\|^2$. By (1.8),

$$\mathrm{S}_{\rm mix}=\operatorname{div}(H+\tilde H)+\frac{n-1}{n}\,\|H\,\|^2+\frac{p-1}{p}\,\|\tilde H\|^2,\quad J_{\rm mix}(g)\ge 0.$$

If M is closed then, by Theorem 2.2, $H=0$, the leaves are totally geodesic, and (2.6) holds. Summarizing, we have: a double-twisted product metric on a closed manifold $M=M_1^n\times M_2^p$ $(n,p>1)$ is $\mathcal D$-critical for (0.2) if and only if f_1 does not depend on M_2 i.e., (M,g) the twisted product of $(M_1\,,\,e^{f_1}g_1)$ and $(M_2\,,\,g_2)$, and (2.6) holds.

The following theorem continues Example 1.3.

Theorem 2.3. *Let a metric $g\in\mathrm{Riem}(M,\,T\mathcal F,\,\mathcal D)$ be $\mathcal D$-critical for* (0.2)*, and $\widetilde{\mathcal D}$ be tangent to a totally geodesic Riemannian foliation. Then*

$$r_{\mathcal D}=(\mathrm{S}_{\rm mix}/p)\,g^\perp\ge 0,\quad \text{if } p\ne 4\ \text{ then } \mathrm{S}_{\rm mix}=\mathrm{const}. \tag{2.9}$$

Proof. Since $h = 0 = T$, then (1.4) reads

$$r_{\mathcal{D}} = -\widetilde{\mathcal{T}}^{\flat}. \tag{2.10}$$

By (1.8), $\mathrm{S}_{\rm mix} = \|\tilde T\|^2$ is nonnegative. From Euler-Lagrange (2.2) we obtain

$$r_{\mathcal{D}} - \widetilde{\mathcal{T}}^{\flat} = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(M,g)\big)\, g^{\perp}. \tag{2.11}$$

Adding (2.10) and (2.11), we obtain $r_{\mathcal{D}} = \frac14\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(M,g)\big)\, g^{\perp}$, see $(2.9)_1$. Tracing this, we find $(p-4)\,\mathrm{S}_{\rm mix} = p\,\mathrm{S}_{\rm mix}(M,g)$, that proves $(2.9)_2$. □

Remark 2.3. From the Euler-Lagrange equations (2.2)–(2.3) we observe the following: if $\mathcal{F}$ is a totally geodesic foliation of a closed Riemannian manifold (M,g) with integrable normal bundle, and

$$\operatorname{div}\big(\tilde h - (\tilde H/p)\, g^{\perp}\big) = 0, \qquad \Phi_{\tilde h} = (\widetilde{\mathrm{S}}_{\rm ex}/n)\,\tilde g\,, \tag{2.12}$$

then g is critical for the action $J_{\rm mix}$ with respect to all adapted variations of (M,g). Note that $(2.12)_1$ is satisfied, if $\mathcal{D} = (T\mathcal{F})^{\perp}$ is totally umbilical, i.e., $\tilde h = \frac1p\,\tilde H\, g^{\perp}$ with $\tilde H \ne 0$. Then $(2.12)_2$ reduces itself to $(p-1)[\tilde H^{\flat}\otimes\tilde H^{\flat} - \frac1n\,\|\tilde H\|^2\tilde g\,] = 0$, hence, is identically satisfied only when $p=1$ or $n=1$.

2.2 Flows

let $\widetilde{\mathcal{D}}$ be spanned by a nonsingular vector field N, then N defines a flow (a one-dimensional foliation). An example is provided by a circle action $S^1\times M\to M$ without fixed points. In this case, $r_{\widetilde{\mathcal{D}}} = \mathrm{Ric}_N\ \tilde g$ and $r_{\mathcal{D}} = (R_N)^{\flat}$, where $R_N = R^{\nabla}(N,\,\cdot\,)N$. Then (0.2) reduces itself to

$$J_{\rm mix}(g) = \int_M \mathrm{Ric}_N\ \mathrm{d\,vol}_g\,. \tag{2.13}$$

We have $\tilde h = \tilde h_{sc}N$ and $g(\tilde A_N X, Y) = g(\tilde h_{sc}(X,Y), N)$, where $\tilde h_{sc} = \langle \tilde h, N\rangle$ is the scalar second fundamental form and $\tilde A_N$ the Weingarten operator of $\mathcal{D}$. Define the functions $\tilde\tau_i = \mathrm{Tr}\,(\tilde A_N^{\,i})$ $(i\ge 0)$. It is easy to check that $\widetilde{\mathrm{S}}_{\rm ex} = \tilde\tau_1^2 - \tilde\tau_2$ and

$$\operatorname{div} N = -\tilde\tau_1, \qquad \operatorname{div}(\tilde\tau_1 N) = N(\tilde\tau_1) - \tilde\tau_1^2.$$

From Theorem 1.6 (of Corollary 2.1), we obtain the following.

Corollary 2.4 (Euler-Lagrange equations). *Let a distribution $\widetilde{\mathcal{D}}$ be spanned by a unit vector field N on a closed manifold M. If $g\in\mathrm{Riem}(M,\,\widetilde{\mathcal{D}},\,\mathcal{D})$ is a critical point of* (2.13) *with respect to adapted variations then*

$$\begin{aligned}&\big(R_N + \tilde A_N^2 - (\tilde T_N^{\sharp})^2 + [\tilde T_N^{\sharp}, \tilde A_N]\big)^{\flat} - \tilde\tau_1\tilde h_{sc} + H^{\flat}\otimes H^{\flat} - \mathrm{Def}_{\mathcal{D}}\, H\\ &\quad = \frac12\big(\,\mathrm{Ric}_N - \mathrm{S}^*_{\rm mix}(M,g) + \operatorname{div}(\tilde\tau_1 N - H)\big)\, g^{\perp} \quad (\text{for } \mathcal{D}\text{−variations}),\end{aligned} \tag{2.14}$$

$$\mathrm{Ric}_N = -\mathrm{S}^*_{\rm mix}(M,g) + 4\,\|\tilde T\|^2 + \operatorname{div}(\tilde\tau_1 N + H) \quad (\text{for } \widetilde{\mathcal{D}}\text{−variations}). \tag{2.15}$$

Proof. An easy computation shows that

$$
\begin{aligned}
\widetilde{\mathcal{A}} &= \tilde{A}_N^2, \quad \langle \tilde{h}_{sc}N,\, \tilde{H}\rangle = \tilde{\tau}_1 \tilde{h}_{sc}, \quad \Psi = H^\flat \otimes H^\flat, \quad \widetilde{\Psi} = (\tilde{\tau}_2 - \|\tilde{T}\|^2)\,\tilde{g},\\
\mathcal{A} &= \|H\|^2\, \widetilde{\mathrm{id}}\,, \quad \mathcal{T} = 0, \quad \langle h,\, H\rangle = \|H\|^2 \tilde{g},\\
H &= \nabla_N N, \quad h = H\,\tilde{g}, \quad \|h\| = \|H\|,\\
\tilde{H} &= \tilde{\tau}_1 N, \quad \tilde{\tau}_1 = \mathrm{Tr}_{\,g}\, \tilde{h}_{sc}, \quad \|\tilde{h}\|^2 = \tilde{\tau}_2, \quad \mathrm{Def}_{\widetilde{\mathcal{D}}}\, \tilde{H} = N(\tilde{\tau}_1)\,\tilde{g}\,.
\end{aligned}
\tag{2.16}
$$

Notice that $(H^\flat \otimes H^\flat)(X,Y) = g(H,X)\, g(H,Y)$. Introducing the values

$$
\begin{gathered}
\Phi_h = 0 = \mathrm{S}_{\,\mathrm{ex}}, \quad \widetilde{\mathrm{S}}_{\,\mathrm{ex}} = \tilde{\tau}_1^2 - \tilde{\tau}_2, \quad \widetilde{\mathcal{T}} = \tilde{T}_N^{\sharp\, 2},\\
h = H\,\tilde{g}, \quad \Phi_{\tilde{h}} = (\tilde{\tau}_1^2 - \tilde{\tau}_2)\,\tilde{g}, \quad \Phi_{\tilde{T}} = -\|\tilde{T}\|^2 \tilde{g}
\end{gathered}
$$

and (2.16) into (1.19) and its 'dual', yield (2.14) and (2.15). □

By (1.3), we have $\mathrm{div}(\tilde{h}_{sc}\, N) = \nabla_N\, \tilde{h}_{sc} - \tilde{\tau}_1 \tilde{h}_{sc}$ and $\mathrm{div}\, h = (\mathrm{div}\, H)\,\tilde{g}$. Then, see (1.4) and (1.8),

$$
\begin{aligned}
\big(R_N + \tilde{A}_N^2 + (\tilde{T}_N^\sharp)^2\big)^\flat &= \nabla_N\, \tilde{h}_{sc} - H^\flat \otimes H^\flat + \mathrm{Def}_{\mathcal{D}}\, H,\\
\mathrm{Ric}_N &= \mathrm{div}(\tilde{\tau}_1 N + H) + \tilde{\tau}_1^2 - \tilde{\tau}_2 + \|\tilde{T}\|^2.
\end{aligned}
\tag{2.17}
$$

Remark that the known formula $(2.17)_2$ is simply the trace of $(2.17)_1$.

A flow of a unit vector field N is geodesic if the orbits are geodesics (i.e., $h = 0$), and a flow is Riemannian if the metric is bundle-like (i.e., $\tilde{h} = 0$). A nonsingular Killing vector field clearly defines a Riemannian flow; moreover, a Killing vector field of constant (nonzero) length generates a geodesic Riemannian flow, see [12, Ch. 10].

Proposition 2.5. *Let a unit vector field N generates a geodesic Riemannian flow on a closed Riemannian manifold (M^{p+1}, g). If g is a $\mathcal{D}$-critical point of the functional* (2.13) *then*

$$
R_N = (1/p)\ \mathrm{Ric}_N\, \mathrm{id}^\perp, \quad \text{where} \quad \mathrm{Ric}_N = \mathrm{const} \quad \text{when} \quad p \ne 4;
\tag{2.18}
$$

and g is $\widetilde{\mathcal{D}}$-critical, too. Furthermore, $K_{\mathrm{mix}} \ge 0$ is a function of a point only: if p is odd then $K_{\mathrm{mix}} = 0$ and M splits, and if $K_{\mathrm{mix}} > 0$ then p is even.

Proof. From Theorem 2.3, (2.18) follows, and (2.10) reads $R_N = -(\tilde{T}_N^\sharp)^2$. For such N-flows, (2.14)–(2.15) reduces to

$$
R_N - (\tilde{T}_N^\sharp)^2 = \frac{1}{2}\left(\mathrm{Ric}_N - \mathrm{S}^*_{\mathrm{mix}}(M,g)\right) g^\perp \quad (\text{for } \mathcal{D}\text{−variations}),
\tag{2.19}
$$

$$
\mathrm{Ric}_N = -\mathrm{S}^*_{\mathrm{mix}}(M,g) + 4\,\|\tilde{T}\|^2 \qquad (\text{for } \widetilde{\mathcal{D}}\text{−variations});
\tag{2.20}
$$

moreover, (2.20) is the trace of (2.19). Thus, g is critical for $\widetilde{\mathcal{D}}$-variations, too. If p is odd then the skew-symmetric operator $\tilde{T}_N^\sharp$ has zero eigenvalue; hence, $R_N = 0$ and $\tilde{T} = 0$; using de Rham Decomposition Theorem completes the proof. □

Recall [3] that a manifold M^{2n+1} with a 1-form η such that $d\eta(\xi,\,\cdot) = 0$ and $\eta(\xi) = 1$ is called a *contact manifold*, and ξ is called the characteristic vector field. A Riemannian metric g on (M, η) is *associated* if there exists a $(1,1)$-tensor ϕ such that

$$
\eta = g(\xi,\,\cdot), \quad d\eta(X,Y) = g(X, \phi(Y)), \quad \phi^2 = -I + \eta \otimes \xi \quad (X, Y \in TM).
$$

The above (ϕ, ξ, η, g) is called a *contact metric structure* on M, and the integral curves of ξ are geodesics. A contact metric structure for which ξ is Killing is called *K-contact*. In [3], the action (2.13) has been studied on the set of metrics associated to a given contact form: *An associated metric g on a contact manifold (M, η) is critical for the action* (2.13) *considered on the set of metrics associated to η if and only if it is K-contact.* For a K-contact structure, ξ spans a geodesic Riemannian flow. We can use Proposition 2.5 to show that K-contact structures are still critical points of (2.13) on a set of adapted metrics.

Proposition 2.6. *Any K-contact metric g is critical for the action* (2.13) *considered on the set of adapted metrics.*

Proof. By [3, Theorem 7.2], if (M, g) is K-contact then (2.18) holds for $\mathrm{Ric}_N = p$. $\square$

Proposition 2.6 and [3, Theorem 10.12] yield the following characteristic of some critical metrics on contact manifolds.

Corollary 2.7. *Let g be an associated metric on a contact manifold (M, η). Then g is critical for the action* (2.13) *considered on the set of adapted metrics on M if and only if g is K-contact.*

2.3 Codimension-one foliations

Examples of codimension-one foliations are the level surfaces of a function $u : M \to \mathbb{R}$ without critical points. The structure theory and dynamics of codimension-one foliations on closed manifolds are fairly well understood. Let $\mathcal{F}$ be a codimension-one foliation with a unit normal $N \in \Gamma(TM)$ on a closed Riemannian manifold (M^{n+1}, g). We have, see (0.3) and its 'dual',

$$r_{\mathcal{D}} = \mathrm{Ric}_N\, g^{\perp}, \quad r_{\mathcal{F}} = (R_N)^{\flat}.$$

Let $h_{\rm sc}$ be the scalar second fundamental form, and A_N the Weingarten operator of $\mathcal{F}$. We have $T = 0 = \tilde{T}$ and

$$h_{\rm sc}(X, Y) = g(\nabla_X Y, N), \quad A_N(X) = -\nabla_X N, \quad (X, Y \in T\mathcal{F}).$$

Power sums of the eigenvalues of A_N are given by $\tau_\alpha = \mathrm{Tr}\,(A_N^\alpha)$ $(\alpha \geq 0)$, cf. [10]. For example, $\tau_1 = \mathrm{Tr}\ A_N = -\operatorname{div} N$, $\tau_2 = \mathrm{Tr}\,(A_N^2)$, and $H = \tau_1 N$. It is easy to see that (2.4) takes the form

$$\mathrm{S}^*_{\rm mix} = \begin{cases} \mathrm{Ric}_N - 2\,(\tau_1^2 - \tau_2) & \text{for } \mathcal{D}-\text{variations}, \\ \mathrm{Ric}_N & \text{for } T\mathcal{F}-\text{variations}. \end{cases} \tag{2.21}$$

Notice that $\mathcal{A} = A_N^2$ and $\widetilde{\mathcal{A}} = \|\tilde{H}\|^2 N$, where $\tilde{H} = \nabla_N N$ is the curvature vector of N-curves. By $(2.1)_1$ and $\widetilde{\Psi} = \tilde{H}^\flat \otimes \tilde{H}^\flat$, we obtain

$$(R_N + A_N^2)^\flat = \nabla_N\, h_{\rm sc} - \tilde{H}^\flat \otimes \tilde{H}^\flat + \mathrm{Def}_{\mathcal{F}}\, \tilde{H}. \tag{2.22}$$

Then we find, taking trace of (2.22), that (cf. also [10, 14])

$$\mathrm{Ric}_N = \operatorname{div}(\tau_1 N + \tilde{H}) + \tau_1^2 - \tau_2. \tag{2.23}$$

Since M is a closed manifold, we have $J_{\rm mix}(g) = \int_M (\tau_1^2 - \tau_2)\,\mathrm{d}\,\mathrm{vol}_g$.

From Theorem 1.6 or Corollary 2.1 we obtain the following.

Corollary 2.8. *Let $\mathcal{F}$ be a codimension-one foliation of a closed manifold M^{n+1}, whose transversal distribution $\mathcal{D}$ is spanned by a nonzero vector field N. If $g \in$* $\mathrm{Riem}(M,\, T\mathcal{F},\, \mathcal{D})$ *is a critical point of the functional* (2.13) *with respect to adapted variations, then the following Euler-Lagrange equations are satisfied:*

$$\mathrm{Ric}_N + \mathrm{S}^*_{\rm mix}(M,g) - \mathrm{div}(\tau_1 N + \tilde H) = 0 \quad (\text{for } \mathcal{D}\text{−variations}), \tag{2.24}$$

$$(R_N + A_N^2)^\flat - \tau_1 h_{\,\rm sc} + \tilde H^\flat \otimes \tilde H^\flat - \mathrm{Def}_{\mathcal{F}}\,\tilde H = \frac12\big(\mathrm{Ric}_N - \mathrm{S}^*_{\rm mix}(M,g) + \mathrm{div}(\tau_1 N - \tilde H)\big)\,\tilde g \quad (\text{for } T\mathcal{F}\text{−variations}). \tag{2.25}$$

Remark 2.4. Using (2.22)–(2.23), one may rewrite (2.24)–(2.25) as

$$\tau_1^2 - \tau_2 \;=\; -\mathrm{S}^*_{\rm mix}(M,g) \quad (\text{for } \mathcal{D}\text{−variations}), \tag{2.26}$$

$$\mathrm{div}(h_{\,\rm sc}N) - \mathrm{div}(\tau_1 N)\,\tilde g = \frac12\big(\tau_1^2 - \tau_2 - \mathrm{S}^*_{\rm mix}(M,g)\big)\,\tilde g \quad (\text{for } T\mathcal{F}\text{−variations}). \tag{2.27}$$

(i) By (2.26), if $n=1$ then $J_{\rm mix}(g)=0$, and if $n>1$, then $\tau_1^2-\tau_2=\mathrm{const}$, (respectively, $\tau_1^2-\tau_2=0$ when the variations do not preserve the volume of M).

(ii) Equation (2.27) is equivalent to result in [10, Example 2.5], where notation $J_{\rm mix} = 2\,E_N$ is used and the Euler-Lagrange equations are given in the form

$$-2\ \mathrm{div}(T_1(h_{\,\rm sc})N) = (\tau_1^2 - \tau_2 - \mathrm{S}^*_{\rm mix}(M,g))\,\tilde g \quad (\text{for } T\mathcal{F}\text{−variations}). \tag{2.28}$$

Here, $T_1(h_{\,\rm sc}) = \tau_1\tilde g - h_{\,\rm sc}$ corresponds to the first Newton transformation. Indeed, by the above,

$$-\,\mathrm{div}(T_1(h_{\,\rm sc})N) = \nabla_N\, h_{\,\rm sc} - \tau_1 h_{\,\rm sc} - \mathrm{div}(\tau_1 N)\,\tilde g.$$

By (1.3), $\mathrm{div}(h_{\,\rm sc}N) = \nabla_N\, h_{\,\rm sc} - \tau_1 h_{\,\rm sc}$ is valid. Hence (2.28) reduces itself to (2.27).

(iii) Note that adapted variations provide the same Euler-Lagrange equations as in [1], where the action (2.13) was examined in a foliated globally hyperbolic space-time. There, $\mathcal{D}$ was spanned by a unit (for initial metric g), time-like vector field N with integrable orthogonal distribution $\widetilde{\mathcal{D}}$. Equations (2.26) and (2.27) are there formulated in terms of a newly introduced tensor $\mathrm{Ric}_{\mathcal{D}}(g)$,

$$\begin{aligned}
&\mathrm{Ric}_{\mathcal{D}}(g)(X,Y) = \big(\nabla_N\, h_{sc} - \tau_1 h_{sc}\big)(X,Y),\\
&\mathrm{Ric}_{\mathcal{D}}(g)(X,N) = \mathrm{div}(A_N(X)), \quad \mathrm{Ric}_{\mathcal{D}}(g)(N,X) = -\,\mathrm{div}(A_N(X)),\\
&\mathrm{Ric}_{\mathcal{D}}(g)(N,N) = -\,\mathrm{div}\,\tilde H,
\end{aligned}$$

whose trace is denoted by $\mathrm{Scal}_{\mathcal{D}}(g)$. The Euler-Lagrange equations for the action (2.13) are presented in the following form [1]:

$$\mathrm{Ric}_{\mathcal{D}}(g) - \frac12\,\mathrm{Scal}_{\mathcal{D}}(g)\,g - \mathrm{Ric}_N\,\big(N^\flat\otimes N^\flat + \frac12\,g\big) = 0\,, \tag{2.29}$$

where one should actually use only the symmetric part of $\mathrm{Ric}_{\mathcal{D}}(g)$ in (2.29). Also, (2.29) reduces to (2.26) when evaluated on $\mathcal{D}$ (with $\mathrm{S}^*_{\rm mix}(\Omega,g)=0$, because in [1] the volume preserving variations are not considered), while evaluating (2.29) on $T\mathcal{F}$ yields (2.27).

Theorem 2.9. *Let* (M^{n+1}, g) $(n > 1)$ *be a codimension-one foliated closed Riemannian manifold with a unit normal vector field* N*. Then* g *is a critical point of* $J_{\rm mix}$ *with respect to adapted variations preserving the volume of* (M, g) *if and only if*

$$\tau_1 = 0, \quad \tau_2 = -\operatorname{Ric}_N(M, g), \quad \nabla_N h_{\rm sc} = 0. \tag{2.30}$$

In particular, conditions (2.30) *are satisfied trivially when* $\mathcal{F}$ *is totally geodesic.*

Proof. By (2.23), $J_{\rm mix} = \int_M (\tau_1^2 - \tau_2)\,{\rm d\,vol}_g$. Taking trace of (2.27), we obtain

$$(1 - n)\big(N(\tau_1) - \tau_1^2\big) = \frac{n}{2}\,(\tau_1^2 - \tau_2 - {\rm S}^*_{\rm mix}(M, g)) \quad (\text{for } T\mathcal{F}\text{−variations}). \tag{2.31}$$

By (2.26), $\tau_1^2 - \tau_2 = \text{const}$. Hence, and due to (2.21), rhs of (2.31) vanishes. Since $n > 1$, we get the ODE $N(\tau_1) = \tau_1^2$ along complete N-curves, whose solution is $\tau_1 = 0$. Thus, (2.26) and (2.21) yield $\tau_2 = -\operatorname{Ric}_N(M, g)$. By (2.27), we find ${\rm div}(h_{\rm sc} N) = \nabla_N h_{\rm sc} = 0$. □

The next example shows that there are many solutions to (2.30), but assuming that a critical metric is bundle-like we obtain isoparametric foliations.

Definition 2.5 (see Chap. 8 in [12])**.** A smooth function $f : M \to \mathbb{R}$ without critical points on a Riemannian manifold (M, g) is called *isoparametric* if for any vector X tangent to a level hypersurface of f the following conditions are satisfied:

$$X(g(\nabla f, \nabla f)) = 0, \qquad X(\Delta f) = 0.$$

Example 2.6 (Isoparametric foliations). Let $(x_0 = t, x_1, \ldots x_n)$ be biregular foliated coordinates on M^{n+1}, see for example [10, Sect. 2.2.1], with the leaves $\{x_0 = c\}$, $g_{ij} = 0$ for $i \ne j$ and $N = (g_{00})^{-1/2}\,\partial_t$. Let g be a critical point of the action (2.13) with respect to adapted variations, then $\tau_1 = 0$. Then (2.27) becomes the system of n independent equations:

$$g_{ii,\,00} - \frac{1}{g_{ii}}\,(g_{ii,\,0})^2 - \frac{1}{2}\,g_{ii,\,0}\,(\log g_{00})_{,\,0} = 0 \quad (i = 1, \ldots n). \tag{2.32}$$

We seek solutions of (2.32) in the following form:

$$g_{ii} = f_i(x_1, \ldots x_n)\, e^{-2\int \sqrt{g_{00}}\, y_i(t)\,{\rm d}\,t}, \tag{2.33}$$

where f_i $(i = 1, \ldots n)$ are positive functions. It follows that the Weingarten operator has diagonal form and $y_1, \ldots, y_n$ are the principal curvatures. Hence,

$$y_1(t) + \ldots + y_n(t) = \tau_1, \quad y_1^2 + \ldots + y_n^2 = \tau_2 = -\operatorname{Ric}_N(M, g).$$

The metric (2.33) is critical for adapted variations if and only if $y_i'(t) = 0$ and $\tau_1 = 0$; hence, all y_i are constant with zero sum. By $(2.31)_3$, $\operatorname{Ric}_N(M, g) \le 0$ (and if $\operatorname{Ric}_N(M, g) = 0$ then the only solution is a totally geodesic foliation). For a function of the view $g_{00} = P(t) > 0$, we have $\operatorname{Ric}_N = \text{const} \le 0$. Recall (see [12]), that *for a foliation* $\mathcal{F}$ *of* (M, g) *by the level hypersurfaces of a function* f *without critical points on* M *the following conditions are equivalent:* (i) $\mathcal{F}$ *is a Riemannian foliation, and every its leaf has constant mean curvature;* (ii) f *is an isoparametric function.* Thus, our foliation is given by an isoparametric function x_0.

The reader can find more examples for codimension-one foliations in [1].

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