

Curvatures of left invariant Randers metrics on the five-dimensional Heisenberg group

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Abstract. A left invariant Z -Randers metric on the five-dimensional Heisenberg group is a left invariant Randers metric with deformation vector from the center of the Heisenberg algebra. In this note we prove that for every left invariant Z -Randers metric on the five-dimensional Heisenberg group there exist flags of strictly negative and there exist flags of strictly positive curvatures.

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1 Introduction

The geometry of any Lie group N with left invariant Riemann metric reflects strongly the algebraic structure of its Lie algebra \mathcal{N} . Many of the results in [3] illustrate this principle. We mention here just two well-known theorems:

Theorem 1.1 (Milnor [6]). *If Z belongs to the center of the Lie algebra \mathcal{N} , then for any left invariant metric the inequality for the sectional curvature $K(Z, X) \geq 0$ is satisfied for all X .*

Theorem 1.2 (Wolf [10]). *Any nonabelian nilpotent Lie group with left invariant metric must admit both positive and negative sectional curvatures.*

The purpose of this paper is to develop the above results for a special type of five-dimensional Randers spaces. A general study of Berwald-type Randers metric on two-step homogeneous nilmanifolds of dimension five can be found in [7], where the author shows that the only space which admits left-invariant Randers metric of Berwald type has three-dimensional center. In that case the author gives explicit formula for the flag curvature, and the sign of the flag curvature is studied. In the present paper we study strictly non-Berwald Randers metrics on the two-step nilmanifold with one-dimensional center (usually called Heisenberg manifold). The

three-dimensional case is treated in [5]. We use basically a local calculus. In [5] we use the Berwald-Moór frame for the computations (see e.g. [8]), while here the Homolya-Kowalski basis is used (see [4]).

2 Preliminaries

Through this paper we use [2] as a basic reference for foundations of Finsler geometry. A Finsler manifold (N, F) is a differentiable manifold N equipped with a Finsler metric $F: TN \rightarrow \mathbf{R}$. The Finsler geometry counterpart of the Riemannian sectional curvature is the *flag curvature*, which can be introduced by considering the osculating Riemannian metric

$$(2.1) \quad \langle X, Y \rangle_W = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} F^2(W + sX + tY),$$

where $X, Y \in T_x N$, and $W \in T_x N \setminus \{0\}$. A flag $\sigma(W, X)$ consists of the flag pole W and a two-dimensional subspace spanned by W and a nonzero transverse vector X . The *flag curvature* for $X \in T_x N$ is defined by

$$(2.2) \quad K(\sigma(W, X)) = K(W, X) = \frac{\langle R^W(X, W)W, X \rangle_W}{\|W\|_W^2 \|X\|_W^2 - \langle X, W \rangle_W^2},$$

where

$$R^W(X, Y)Z = \nabla_X^W \nabla_Y^W Z - \nabla_Y^W \nabla_X^W Z - \nabla_{[X, Y]}^W Z$$

is the curvature of the Chern-Rund connection ∇^W for F . The Chern-Rund connection for the nowhere vanishing vector field W is the torsion free, almost metric affine connection $\nabla^W: \mathfrak{X}(N) \times \mathfrak{X}(N) \rightarrow \mathfrak{X}(N)$, defined by the generalized Koszul formula

$$(2.3) \quad \begin{aligned} 2\langle \nabla_X^W Y, Z \rangle_W &= X\langle Y, Z \rangle_W + Y\langle Z, X \rangle_W - Z\langle X, Y \rangle_W + \\ &+ \langle [X, Y], Z \rangle_W - \langle [Y, Z], X \rangle_W + \langle [Z, X], Y \rangle_W - \\ &- 2\langle \nabla_X^W W, Y, Z \rangle_W - 2\langle \nabla_Y^W W, Z, X \rangle_W + 2\langle \nabla_Z^W W, X, Y \rangle_W, \end{aligned}$$

where

$$\langle X, Y, Z \rangle_W = \frac{1}{4} \frac{\partial^3}{\partial r \partial s \partial t} \Big|_{r,s,t=0} F^2(W + rX + sY + tZ)$$

is the $(0, 3)$ -type Cartan tensor ([9]). ‘Almost metric’ here means

$$X\langle Y, Z \rangle_W = \langle \nabla_X^W Y, Z \rangle_W + \langle Y, \nabla_X^W Z \rangle_W + 2\langle \nabla_X^W W, Y, Z \rangle_W.$$

Hereafter let N be the five-dimensional Heisenberg group, which is up to isomorphism, the only two-step nilpotent Lie group with a 1-dimensional center ([4]). Let \mathcal{N} denote the five-dimensional real Lie algebra of N , with center $\mathcal{Z} = \text{span } Z$, spanned by the element Z . We assume that \mathcal{N} is equipped with the Euclidean scalar product $\langle \cdot, \cdot \rangle$, and suppose that $\|Z\| = 1$.

For $X_0 \in \mathcal{N}$ with property $\|X_0\| < 1$ the function

$$(2.4) \quad f: \mathcal{N} \rightarrow \mathbf{R}, \quad X \mapsto f(X) = \sqrt{\langle X, X \rangle} + \langle X_0, X \rangle$$

defines a Minkowski functional on \mathcal{N} ; therefore, it can be extended to a *left-invariant Randers type Finsler metric* F on the Lie group N of \mathcal{N} by left translations. From now on elements of \mathcal{N} are regarded as left invariant vector fields on N . For left invariant vector fields the first three terms of the right hand side of (2.3) vanish.

Excluding the case $X_0 = 0$, the remaining Randers metrics are non-Riemannian [2, p. 283]. In this paper $X_0 = \xi Z$ for a real number $0 < \xi < 1$. This choice gives a geometric relationship between the Lie algebra and the Randers metric, and we call such type of Randers metric *Z-Randers metric*. Moreover, the condition $Z \in \mathcal{Z}$ guarantees that the Randers metric is not Berwald. Namely, the Randers metric based on (2.4) is Berwald if and only if X_0 is parallel with respect to the Levi-Civita connection of \langle, \rangle (see e.g. [1, Theorem 3.1.4.1.]). The Levi-Civita connection has the form

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle, \quad X, Y, Z \in \mathcal{N}.$$

Thus, for all $U \in \mathcal{N}$: $\nabla_U X_0 = 0$ if and only if

$$\forall U \in \mathcal{N} \forall V \in \mathcal{N} : \langle [U, V], X_0 \rangle = 0.$$

Consequently, if $X_0 \in \mathcal{Z}$, then the Randers metric is not Berwald.

The osculating scalar product can be calculated from the Euclidean scalar product by

$$(2.5) \quad \langle U, V \rangle_W = \langle U, V \rangle + \langle X_0, U \rangle \langle X_0, V \rangle - \langle X_0, W \rangle \langle W, U \rangle \langle W, V \rangle \\ + \langle X_0, U \rangle \langle W, V \rangle + \langle X_0, W \rangle \langle U, V \rangle + \langle X_0, V \rangle \langle W, U \rangle,$$

where $W \in \mathcal{N}$ and $\langle W, W \rangle = 1$; moreover, the (0, 3)-type symmetric Cartan tensor is

$$(2.6) \quad \langle U, V, X \rangle_W = \frac{1}{2} \sum_{[U, V, X]} \{ \langle X_0, W \rangle \langle W, U \rangle \langle W, V \rangle \langle W, X \rangle \\ - \langle X_0, W \rangle \langle X, V \rangle \langle U, W \rangle - \langle X_0, X \rangle \langle W, V \rangle \langle W, U \rangle + \langle X_0, U \rangle \langle X, V \rangle \},$$

see [5].

3 The main theorem

Theorem 3.1. *For every left invariant Z-Randers metric on the five-dimensional Heisenberg group there exist flags of strictly negative and there exist flags of strictly positive curvatures.*

Proof. Table 1 outlines the proof, where we summarized the flag curvature of the Z-Randers metric for special flags $\sigma(W, X)$. In Table 1 we use the Homolya–Kowalski basis. In [4] the authors construct an orthonormal basis (e_1, e_2, e_3, e_4, Z) in \mathcal{N} such that

$$(3.1) \quad [e_1, e_2] = -[e_2, e_1] = \lambda Z, \quad [e_3, e_4] = -[e_4, e_3] = \mu Z, \quad \lambda \geq \mu > 0,$$

and all the other Lie brackets are zero. The inner product here is the Euclidean scalar product \langle, \rangle .

	flag pole W	transverse vector X ($X \neq W$)	$K(X, W)$
(1.1)	$W \in \mathcal{Z}$	$X \in \text{span}(e_1, e_2)$	$\frac{\lambda^2}{4} > 0$
(1.2)		$X \in \text{span}(e_3, e_4)$	$\frac{\mu^2}{4} > 0$
(2.1)	$W \in \text{span}(e_1, e_2)$	$X \in \mathcal{Z}$	$\frac{1-\xi^2}{4}\lambda^2 > 0$
(2.2)		$X \in \text{span}(e_1, e_2)$	$\frac{\xi^2-3}{4}\lambda^2 < 0$
(2.3)		$X \in \text{span}(e_3, e_4)$	$\frac{\mu^2-\lambda^2}{4}\xi^2 \leq 0$
(3.1)	$W \in \text{span}(e_3, e_4)$	$X \in \mathcal{Z}$	$\frac{1-\xi^2}{4}\mu^2 > 0$
(3.2)		$X \in \text{span}(e_1, e_2)$	$\frac{\lambda^2-\mu^2}{4}\xi^2 \geq 0$
(3.3)		$X \in \text{span}(e_3, e_4)$	$\frac{\xi^2-3}{4}\mu^2 < 0$

Table 1: Flag curvatures for special flags

We calculate the Chern–Rund connection from the metric using method described in [9, Theorem 3.10] directly. Also, this method is used in [5] for the three-dimensional case. In that paper a more detailed description of the algorithm can be found. We record in Tables 2–6 the explicit formulæ for the Chern–Rund connection, restricted to the demand of the proof.

Cases (1.1) and (1.2) If $W \in \mathcal{Z}$ then (2.6) gives $\langle U, V, X \rangle_Z = 0$ for all U, V, X , and from (2.5) we get

$$\begin{aligned} \langle U, V \rangle_Z &= (1 + \xi)\langle U, V \rangle, \quad U, V \in \mathcal{V} \\ \langle U, V \rangle_Z &= 0, \quad U \in \mathcal{V}, \quad V \in \mathcal{Z} \\ \langle U, V \rangle_Z &= (1 + \xi)(\langle U, V \rangle + \xi\langle Z, U \rangle\langle Z, V \rangle), \quad U, V \in \mathcal{Z}. \end{aligned}$$

The generalized Koszul formula (2.3) simplifies to

$$(3.2) \quad 2\langle \nabla_U^Z V, X \rangle_Z = \langle [U, V], X \rangle_Z - \langle [V, X], U \rangle_Z + \langle [X, U], V \rangle_Z.$$

From (3.2) it is easy to determine the components of the Chern–Rund connection in the basis $(e_1, e_2, e_3, e_4, e_5 = Z/(\xi + 1))$, see Table 2. Substituting expressions from Table 2 into (2.2) we get

$$K(Z, e_1) = \frac{\lambda^2}{4}, \quad K(Z, e_2) = \frac{\lambda^2}{4}, \quad K(Z, e_3) = \frac{\mu^2}{4}, \quad K(Z, e_4) = \frac{\mu^2}{4},$$

which gives cases (1.1) and (1.2).

In what follows, $W = w_1e_1 + w_2e_2 + w_3e_3 + w_4e_4 \in \mathcal{V}$, i.e. W is from the Euclidean orthogonal complement of the center.

	e_1	e_2	e_3	e_4	e_5
∇_{e_1}	0	$\frac{\lambda}{2}Z$	0	0	$-\frac{\lambda}{2}e_2$
∇_{e_2}	$-\frac{\lambda}{2}Z$	0	0	0	$\frac{\lambda}{2}e_1$
∇_{e_3}	0	0	0	$\frac{\mu}{2}Z$	$-\frac{\mu}{2}e_4$
∇_{e_4}	0	0	$-\frac{\mu}{2}Z$	0	$\frac{\mu}{2}e_3$
∇_{e_5}	$-\frac{\lambda}{2}e_2$	$\frac{\lambda}{2}e_1$	$-\frac{\mu}{2}e_4$	$\frac{\mu}{2}e_3$	0

Table 2: Local components of the Chern–Rund connection, $W = Z$

From (2.5) we get in this case

$$(3.3) \quad \langle X, Y \rangle_W = \langle X, Y \rangle + \xi^2 \langle Z, X \rangle \langle Z, Y \rangle \quad (X, Y \in \mathcal{Z}), \text{ spec. } \langle Z, Z \rangle_W = 1 + \xi^2$$

$$(3.4) \quad \langle X, Y \rangle_W = \langle X, Y \rangle \quad (X, Y \in \mathcal{V}), \text{ spec. } \langle W, W \rangle_W = 1$$

$$(3.5) \quad \langle X, Y \rangle_W = \xi \langle Z, Y \rangle \langle W, X \rangle \quad (X \in \mathcal{V}, Y \in \mathcal{Z}), \text{ spec. } \langle Z, W \rangle_W = \xi.$$

Moreover, for the Cartan tensor we have

$$(3.6) \quad \langle U, V, X \rangle_W = \frac{1}{2} \sum_{[U, V, X]} \{-\langle X_0, X \rangle \langle W, V \rangle \langle W, U \rangle + \langle X_0, U \rangle \langle X, V \rangle\}.$$

Applying the Gram–Schmidt process we get that

$$(3.7) \quad (e_1, e_2, e_3, e_4, e_5 = Z - \xi W)$$

is an orthonormal basis w.r.t. the osculating scalar product $\langle \cdot, \cdot \rangle_W$.

Let

$$(3.8) \quad W^\perp = \lambda w_2 e_1 - \lambda w_1 e_2 + \mu w_4 e_3 - \mu w_3 e_4.$$

For W^\perp we have

$$(3.9) \quad \langle W, W^\perp \rangle = \langle W, W^\perp \rangle_W = 0.$$

Moreover, the generalized Koszul formula (2.3) becomes

$$2\langle \nabla_W^W W, e_i \rangle_W = 2\langle [e_i, W], W \rangle_W = 2\xi \langle Z, [e_i, W] \rangle = \begin{cases} 2\xi w_2 \lambda, & i = 1 \\ -2\xi w_1 \lambda, & i = 2 \\ 2\xi w_4 \mu, & i = 3 \\ -2\xi w_3 \mu, & i = 4. \end{cases}$$

Using (2.3) once more

$$2\langle \nabla_W^W W, e_5 \rangle_W = 2\langle [e_5, W], W \rangle_W = 0,$$

hence

$$(3.10) \quad \nabla_W^W W = \xi W^\perp.$$

	W	W^\perp	Z
∇_W^W	ξW^\perp	$-\frac{1}{2}\lambda^2(\xi W + Z)$	$\frac{1}{2}W^\perp$
$\nabla_{W^\perp}^W$	$\frac{1}{2}\lambda^2(Z - \xi W)$	$-\frac{1}{4}\xi\lambda^2 W^\perp$	$\frac{1}{4}\lambda^2((\xi^2 - 2)W - \xi Z)$
∇_Z^W	$\frac{1}{2}W^\perp$	$\frac{1}{4}\lambda^2((\xi^2 - 2)W - \xi Z)$	$\frac{1}{4}\xi W^\perp$

Table 3: $W \in \text{span}(e_1, e_2)$

	W	W^\perp
$\nabla_{e_3}^W$	$-\frac{1}{2}\mu\xi e_4$	$-\frac{1}{4}\xi\lambda^2 e_3$
$\nabla_{e_4}^W$	$\frac{1}{2}\mu\xi e_3$	$-\frac{1}{4}\xi\lambda^2 e_4$

Table 4: $W \in \text{span}(e_1, e_2)$

Case (2.1). Using Table 3 we get

$$\begin{aligned} R(Z, W)W &= \nabla_Z^W \nabla_W^W W - \nabla_W^W \nabla_Z^W W = \xi \nabla_Z^W W^\perp - \frac{1}{2} \nabla_W^W W^\perp \\ &= \frac{\lambda^2}{4} (\xi^2 - 1) (\xi W - Z). \end{aligned}$$

It follows that

$$\begin{aligned} K(W, Z) &= \frac{\langle R(Z, W)W, Z \rangle_W}{\|W\|_W \|Z\|_W - \langle Z, W \rangle_W^2} \\ &= \frac{\lambda^2}{4} \cdot \frac{\xi^2 - 1}{(1 + \xi^2) - \xi^2} (\xi^2 - (1 + \xi^2)) = \frac{\lambda^2}{4} (1 - \xi^2). \end{aligned}$$

Case (2.2). While $\sigma(W, X) = \sigma(W, W^\perp)$ in this case, there is no loss of generality in choosing W^\perp as the transverse edge. The required components of the Chern-Rund connection are computed from the generalized Koszul formula, and they are in Table 3.

$$\begin{aligned} R^W(W^\perp, W)W &= \nabla_{W^\perp}^W \nabla_W^W W - \nabla_W^W \nabla_{W^\perp}^W W - \nabla_{[W^\perp, W]}^W W \\ &= \xi \nabla_{W^\perp}^W W^\perp - \frac{1}{2}\lambda^2 (\nabla_W^W Z - \xi \nabla_W^W W) - \lambda^2 \nabla_Z^W W = -\frac{1}{4}\xi\lambda^2 W^\perp, \end{aligned}$$

hence

$$K(W, W^\perp) = \frac{1}{4}\lambda^2(\xi^2 - 3) \frac{\|W^\perp\|_W^2}{\|W\|_W^2 \|W^\perp\|_W^2} = \frac{1}{4}\lambda^2(\xi^2 - 3).$$

Case (2.3). Since $[e_3, W] = 0$ in this case,

$$\begin{aligned} R^W(e_3, W)W &= \nabla_{e_3}^W \nabla_W^W W - \nabla_W^W \nabla_{e_3}^W W \\ &= \xi \nabla_{e_3}^W W^\perp + \frac{1}{2}\xi\mu \nabla_W^W e_4 = \frac{1}{4}\xi^2(\mu^2 - \lambda^2)e_3. \end{aligned}$$

	W	W^\perp	Z
∇_W^W	ξW^\perp	$-\frac{1}{2}\mu^2(\xi W + Z)$	$\frac{1}{2}W^\perp$
$\nabla_{W^\perp}^W$	$\frac{1}{2}\mu^2(Z - \xi W)$	$-\frac{1}{4}\xi\mu^2 W^\perp$	$\frac{1}{4}\mu^2((\xi^2 - 2)W - \xi Z)$
∇_Z^W	$\frac{1}{2}W^\perp$	$\frac{1}{4}\mu^2((\xi^2 - 2)W - \xi Z)$	$\frac{1}{4}\xi W^\perp$

Table 5: $W \in \text{span}(e_3, e_4)$

	W	W^\perp
$\nabla_{e_1}^W$	$-\frac{1}{2}\lambda\xi e_2$	$-\frac{1}{4}\xi\mu^2 e_1$
$\nabla_{e_2}^W$	$\frac{1}{2}\lambda\xi e_1$	$-\frac{1}{4}\xi\mu^2 e_2$

Table 6: $W \in \text{span}(e_3, e_4)$

Consequently,

$$K(W, e_3) = \frac{\langle R^W(e_3, W)W, e_3 \rangle_W}{\|e_3\|_W \|W\|_W - \langle e_3, W \rangle_W} = \frac{1}{4}\xi^2(\mu^2 - \lambda^2).$$

Analogously, $R^W(e_4, W)W = \frac{1}{4}\xi^2(\mu^2 - \lambda^2)e_4$ and $K(W, e_4) = \frac{1}{4}\xi^2(\mu^2 - \lambda^2)$. Hence, for all $X \in \text{span}(e_3, e_4)$ we have $K(W, X) = \frac{1}{4}\xi^2(\mu^2 - \lambda^2)$.

Proof of statements in the rows (3.1), (3.2) and (3.3) of Table 1 are completely analogous, and we give only the form of the Chern–Rund connection, see Tables 5 and 6. \square

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