An inequality for generalized normalized δ-Casorati curvatures of slant submanifolds in generalized complex space form

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Abstract. In this paper, we prove an inequality between the normalized scalar curvature and the generalized normalized δ-Casorati curvatures for slant submanifolds of generalized complex space form. Moreover, we characterize those submanifolds for which the equality case holds.

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1 Introduction

The theory of Chen invariants, one of the most interesting research area of differential geometry is to establish the simple relationships between the main intrinsic invariants and the main extrinsic invariants of the submanifolds started in 1993 by Chen [6]. In the initial paper Chen established inequalities between the scalar curvature and the sectional curvature(intrinsic invariants) and the squared norm of the mean curvature(the main extrinsic invariant) of a submanifold in a real space form. The same author obtained the inequalities for submanifolds between the k-Ricci curvature, the squared mean curvature and the shape operator in the real space form with arbitrary codimension [5]. Since then different geometers proved similar inequalities for different submanifolds and ambient spaces [3, 4, 11, 15, 16].

The Casorati curvature(extrinsic invariant) of a submanifold of a Riemannian manifold introduced by Casorati defined as the normalized square length of the second fundamental form [2]. The concept of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold [10]. The geometrical meaning and the importance of the Casorati curvature discussed by some distinguished geometers [12, 20, 21, 7, 8]. Therefore it attracts the attention of geometers to obtain the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [13, 14, 17].

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The purpose of the present paper is to establish two sharp inequalities for the generalized normalized δ-Casorati curvature for slant submanifolds of generalized complex space form.

2 Preliminaries

Let $\overline{M}$ be an almost Hermitian manifold with an almost complex structure J and a Riemannian metric g. An almost Hermitian manifold is said to be a nearly Kaehler manifold if $(\overline{\nabla}_X J)X = 0$ and becomes a Kaehler manifold if $\overline{\nabla} J = 0$ for all $X \in T\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of the Riemannian metric g.

Gray [9] introduced the concept of constant type for the nearly Kaehler manifold, which led to the definition of RK-manifolds. An almost Hermitian manifold for which the Riemannian curvature tensor $\overline{R}$ is J-invariant, that is

$$\overline{R}(JX, JY, JZ, JW) = \overline{R}(X, Y, Z, W), \qquad \forall\, X, Y, Z, W \in T\overline{M}$$

is called an RK-manifold. All nearly Kaehler manifolds belong to the class of RK-manifolds. An almost Hermitian manifold $\overline{M}$ is said to have pointwise constant type if for each $p \in \overline{M}$ and for vector field $X, Y, Z \in T_p\overline{M}$ such that

$$g(X, Y) = g(X, Z) = g(X, JY) = g(X, JZ) = 0 \quad \textit{and} \quad g(Y, Y) = g(Z, Z) = 1,$$

we have

$$\overline{R}(X, Y, X, Y) - \overline{R}(X, Y, JX, JY) = \overline{R}(X, Z, X, Z) - \overline{R}(X, Z, JX, JZ).$$

An RK-manifold $\overline{M}$ has pointwise constant type if and only if there is a differentiable function α on $\overline{M}$ satisfying [19]

$$\begin{aligned} \overline{R}(X, Y, X, Y) - \overline{R}(X, Y, JX, JY) = \alpha\{&g(X, X)g(Y, Y) \\ &-g^2(X, Y) - g^2(X, JY)\}, \end{aligned} \tag{2.1}$$

for all $X, Y \in T\overline{M}$. $\overline{M}$ has global constant type if and only if (2.1) holds with a constant function α.

An RK-manifold of constant holomorphic sectional curvature c and constant type α is denoted by $\overline{M}(c, \alpha)$. For $\overline{M}(c, \alpha)$, we have [19]

$$\begin{aligned} 4\overline{R}(X, Y)Z = (c + 3\alpha)\{g(Y, Z)X - g(X, Z)Y\} + (c - \alpha)\{&g(X, JZ)JY \\ &-g(Y, JZ)JX + 2g(X, JY)JZ\}, \end{aligned}$$

for all $X, Y, Z \in T\overline{M}$. If $c = \alpha$ then $\overline{M}(c, \alpha)$ is a space of constant curvature. A complex space form $\overline{M}(c)$ belongs to the class of almost Hermitian manifolds $\overline{M}(c, \alpha)$ with constant type zero.

Tricerri and Vanhecke [18] introduced the concept of generalized complex space form as a generalization of the complex space form. An almost Hermitian manifold $\overline{M}$ is called the generalized complex space form, denoted by $\overline{M}(f_1, f_2)$, if the Riemannian curvature tensor $\overline{R}$ satisfies

$$\begin{aligned} \overline{R}(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{&g(X, JZ)JY \\ &-g(Y, JZ)JX + 2g(X, JY)JZ\}, \end{aligned} \tag{2.2}$$

for all $X, Y, Z \in T\overline{M}$. Where f_1 and f_2 are smooth functions on $\overline{M}(f_1, f_2)$.

Let M be an n-dimensional submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$ of dimension m. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on M and $\overline{M}$ respectively. The Gauss and Weingarten formulae are defined as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\overline{\nabla}_X \xi = -S_\xi X + \nabla^\perp_X Y,$$

for the vector fields $X, Y \in TM$ and $\xi \in T^\perp M$, where h, S and $\nabla^\perp$ is the second fundamental form, the shape operator and the normal connection respectively. The second fundamental form and the shape operator are related by the following equation

$$g(h(X, Y), \xi) = g(S_\xi X, Y),$$

for vector fields $X, Y \in TM$ and $\xi \in T^\perp M$.

The equation of Gauss is given by

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \tag{2.3}$$

for $X, Y, Z, W \in TM$, where $\overline{R}$ and R represent the curvature tensor of $\overline{M}(f_1, f_2)$ and M respectively.

Let M be an n-dimensional submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$ of dimension m. For any tangent vector field $X \in TM$, we can write $JX = PX + QX$, where P and Q are the tangential and normal components of JX respectively. If $P = 0$, the submanifold is said to be an anti-invariant submanifold and if $Q = 0$, the submanifold is said to be an invariant submanifold. The squared norm of P at $p \in M$ is defined as

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j), \tag{2.4}$$

where $\{e_1, \dots, e_n\}$ is any orthonormal basis of the tangent space T_pM.

A submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be a slant submanifold if for any $p \in M$ and a non zero vector $X \in T_pM$, the angle between JX and T_pM is constant, i.e., the angle does not depend on the choice of $p \in M$ and $X \in T_pM$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called *the slant angle of M in $\overline{M}$*.

The slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively are invariant and anti-invariant submanifolds, and when $0 < \theta < \frac{\pi}{2}$, then the slant submanifold is called *proper slant submanifold*.

If M is a θ-slant submanifold in a generalized complex space form $\overline{M}(f_1, f_2)$, then

$$\|P\|^2 = \sum_{i,j}^{n} g^2(Pe_i, e_j) = n \cos^2 \theta.$$

Let M be a Riemannian manifold and $K(\pi)$ denotes the sectional curvature of M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_m\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ at any $p \in M$, then the scalar curvature τ at that point is given by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

The mean curvature vector denoted by H is defined as

$$H = \frac{1}{n} \sum_{i,j=1}^{n} h(e_i, e_i)$$

and also we put

$$h_{ij}^{\gamma} = g(h(e_i, e_j), e_\gamma), \ i, j \in 1, 2, .., n, \ \gamma \in \{n+1, n+2, ..., m\}.$$

The norm of the squared mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\gamma=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\gamma} \right)^2$$

and the squared norm of second fundamental form h is denoted by $\mathcal{C}$ defined as

$$\mathcal{C} = \frac{1}{n} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\gamma}\right)^2,$$

known as *Casorati curvature* of the submanifold.

If we suppose that L is an r-dimensional subspace of TM, $r \ge 2$, and $\{e_1, e_2, \dots, e_r\}$ is an orthonormal basis of L, then the scalar curvature of the r-plane section L is given as

$$\tau(L) = \sum_{1 \le \gamma < \beta \le r} K(e_\gamma \wedge e_\beta)$$

and the Casorati curvature $\mathcal{C}$ of the subspace L is as follows

$$\mathcal{C}(L) = \frac{1}{r} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\gamma}\right)^2.$$

A point $p \in M$ is said to be an *invariantly quasi-umbilical point* if there exist $m-n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \dots, \xi_m$ such that the shape operators with respect to all directions ξ_γ have an eigenvalue of multiplicity $n-1$ and that for each ξ_γ the distinguished eigendirection is the same. The submanifold is said to be an invariantly quasi-umbilical submanifold if each of its points is an invariantly quasi-umbilical point [1].

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n} inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\} \tag{2.5}$$

and

$$[\widehat{\delta}_c(n-1)]_p = 2\mathcal{C}_p + \frac{2n-1}{2n} sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}. \tag{2.6}$$

Some authors use the coefficient $\frac{n+1}{2n(n-1)}$ instead of $\frac{2n-1}{2n}$ in the equation(2.6). It was pointed out that the coefficient $\frac{n+1}{2n(n-1)}$ is not suitable and therefore modified by the coefficient $\frac{2n-1}{2n}$. For a positive real number $t \neq n(n-1)$, put

$$a(t) = \frac{1}{nt}(n-1)(n+t)(n^2-nt), \tag{2.7}$$

then the generalized normalized δ-Casorati curvatures $\delta_c(t; n-1)$ and $\widehat{\delta}_c(t; n-1)$ are given as

$$[\delta_c(t; n-1)]_p = t\mathcal{C}_p + a(t) inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\},$$

if $0 < t < n^2 - n$, and

$$[\widehat{\delta}_c(t; n-1)]_p = r\mathcal{C}_p + a(t) sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\},$$

if $t > n^2 - n$.

3 The main Theorem

Theorem 3.1. *Let M be a n-dimensional θ-slant submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$ of dimension m. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t; n-1)}{n(n-1)} + f_1 + \frac{3f_2}{(n-1)} \cos^2\theta \tag{3.1}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t; n-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta}_c(t; n-1)}{n(n-1)} + f_1 + \frac{3f_2}{(n-1)} \cos^2\theta \tag{3.2}$$

for any real number $t > n(n-1)$. Moreover, the equality holds in (3.1) and (3.2) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \ldots, e_m\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}a \end{pmatrix}, \quad S_{n+2} = \cdots = S_m = 0. \tag{3.3}$$

Proof. Let $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_m\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ respectively at any point $p \in M$. Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$ from (2.2), we have

$$\overline{R}(e_i, e_j, e_j, e_i) = f_1\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} \\ +f_2\{g(e_i, Je_j)g(Je_j, e_i) - g(e_j, Je_j)g(Je_i, e_i) + 2g(e_i, Je_j)g(Je_j, e_i)\}. \tag{3.4}$$

From Gauss equation and (3.4), we infer

$$f_1\{g(e_j, e_j)g(e_i, e_i) + 3f_2\{g(e_i, Je_j)g(Je_j, e_i)\} = R(e_i, e_j, e_j, e_i) \\ +g(h(e_i, e_j), h(e_j, e_i)) - g(h(e_i, e_i), h(e_j, e_j)). \tag{3.5}$$

By taking summation $1 \leq i, j \leq n$ and using (2.4) and (3.5), we get

$$2\tau = n^2\|H\|^2 - n\mathcal{C} + n(n-1)f_1 + 3ncos^2\theta f_2. \tag{3.6}$$

Define the following function, denoted by $\mathcal{Q}$, a quadratic polynomial in the components of the second fundamental form

$$\mathcal{Q} = t\mathcal{C} + a(t)\mathcal{C}(L) - 2\tau + n(n-1)f_1 + 3ncos^2\theta f_2, \tag{3.7}$$

where L is the hyperplane of T_pM. Without loss of generality, we suppose that L is spanned by $e_1, \dots, e_{n-1}$, it follows from (3.7) that

$$\mathcal{Q} = \frac{n+t}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^{\gamma})^2 + \frac{a(t)}{n-1}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n-1}(h_{ij}^{\gamma})^2 - \sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n}h_{ii}^{\gamma}\right)^2,$$

which can be easily written as

$$\mathcal{Q} = \sum_{\gamma=n+1}^{m}\sum_{i=1}^{n-1}\left[\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right)(h_{ii}^{\gamma})^2 + \frac{2(n+t)}{n}(h_{in}^{\gamma})^2\right] \\ +\sum_{n+1}^{m}\left[2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right)\sum_{(i<j)=1}^{n}(h_{ij}^{\gamma})^2 - 2\sum_{(i<j)=1}^{n}h_{ii}^{\gamma}h_{jj}^{\gamma} + \frac{t}{n}(h_{nn}^{\gamma})^2\right]. \tag{3.8}$$

From(3.8), we can see that the critical points

$$h^c = (h_{11}^{n+1}, h_{12}^{n+1}, \dots, h_{nn}^{n+1}, \dots, h_{11}^{m}, \dots, h_{nn}^{m})$$

of $\mathcal{Q}$ are the solutions of the following system of homogenous equations:

$$\begin{cases} \frac{\partial\mathcal{Q}}{\partial h_{ii}^{\gamma}} = 2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right)(h_{ii}^{\gamma}) - 2\sum_{k=1}^{n}h_{kk}^{\gamma} = 0 \\ \frac{\partial\mathcal{Q}}{\partial h_{nn}^{\gamma}} = \frac{2t}{n}h_{nn}^{\gamma} - 2\sum_{k=1}^{n-1}h_{kk}^{\gamma} = 0 \\ \frac{\partial\mathcal{Q}}{\partial h_{ij}^{\gamma}} = 4\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right)(h_{ij}^{\gamma}) = 0 \\ \frac{\partial\mathcal{Q}}{\partial h_{in}^{\gamma}} = 4(\frac{n+t}{n}(h_{in}^{\gamma}) = 0, \end{cases} \tag{3.9}$$

where $i, j = \{1, 2, \ldots, n-1\}, i \neq j$, and $\gamma \in \{n+1, n+2, \ldots, m\}$.

Hence, every solution h^c has $h^\gamma_{ij} = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathcal{Q}$ is of the following form

$$\mathcal{H}(\mathcal{Q}) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)-2 & -2 & \ldots & -2 & -2 \\ -2 & 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)-2 & \ldots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \ldots & 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)-2 & -2 \\ -2 & -2 & \ldots & -2 & \frac{2t}{n} \end{pmatrix},$$

H_2 and H_3 are the diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = diag\left(4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right), 4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right), \ldots, 4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)\right),$$

and

$$H_3 = diag\left(\frac{4(n+t)}{n}, \frac{4(n+t)}{n}, \ldots, \frac{4(n+t)}{n}\right).$$

Hence, we find that $\mathcal{H}(\mathcal{Q})$ has the following eigenvalues

$$\lambda_{11} = 0,\ \lambda_{22} = 2\left(\frac{2t}{n}+\frac{a(t)}{n-1}\right),\ \lambda_{33} = \cdots = \lambda_{nn} = 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right),$$

$$\lambda_{ij} = 4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right), \lambda_{in} = \frac{4(n+t)}{n},\ \forall\, i, j \in \{1, 2, \ldots, n-1\},\ i \neq j.$$

Thus, $\mathcal{Q}$ is parabolic and reaches a minimum $\mathcal{Q}(h^c) = 0$ for the solution h^c of the system (3.9). Hence $\mathcal{Q} \geq 0$ and

$$2\tau \leq t\mathcal{C} + a(t)\mathcal{C}(L) + n(n-1)f_1 + 3n\cos^2\theta f_2,$$

whereby we obtain

$$\rho \leq \frac{t}{n(n-1)}\mathcal{C} + \frac{a(t)}{n(n-1)}\mathcal{C}(L) + f_1 + \frac{3f_2}{(n-1)}cos^2\theta$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$h_{ij}^{\gamma} = 0,\ \forall\ i,j \in \{1,\dots,n\},\ i \neq j \text{ and } \gamma \in \{n+1,\dots,m\} \tag{3.10}$$

and

$$h_{nn}^{\gamma} = \frac{n(n-1)}{t}h_{11}^{\gamma} = \cdots = \frac{n(n-1)}{t}h_{n-1n-1}^{\gamma}, \forall\gamma \in \{n+1,\dots,m\}. \tag{3.11}$$

From (3.10) and (3.11), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\overline{M}$, such that the shape operator takes the form (3.3) with respect to the orthonormal tangent and orthonormal normal frames. In the same way, we can prove (ii). □

Corollary 3.2. *Let M be a n-dimensional θ-slant submanifold of a generalized complex space form $\overline{M}$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + f_1 + \frac{3f_2}{(n-1)}cos^2\theta.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1,\dots,e_n\}$ and normal orthonormal frame $\{e_{n+1},\dots,e_m\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1,\dots,m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & 2a \end{pmatrix},\ S_{n+2} = \cdots = S_m = 0.$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + f_1 + \frac{3f_2}{(n-1)}cos^2\theta.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1,\dots,e_n\}$ a and normal orthonormal frame $\{e_{n+1},\dots,e_m\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1,\dots,m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \dots & 0 & 0 \\ 0 & 2a & 0 & \dots & 0 & 0 \\ 0 & 0 & 2a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2a & 0 \\ 0 & 0 & 0 & \dots & 0 & a \end{pmatrix}, S_{n+2} = \cdots = S_m = 0.$$

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