Some inequalities for submanifolds in Bochner-Kaehler manifold

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Abstract. In 1999, De Smet, Dillen, Verstraelen and Vrancken conjectured the generalized Wintgen inequality for submanifolds in real space form. This conjecture is also known as DDVV conjecture. It has been proved by G. Jianquan and T. Zizhou (2008). Recently, Ion Mihai (2014) established such inequality for Lagrangian submanifold in complex space form. In this paper, we obtain the generalized Wintgen inequality for bi-slant submanifold in Bochner-Kaehler manifold. Further, we discuss the inequality for semi-slant submanifold, hemi-slant submanifold, CR-submanifold, invariant submanifold, and anti-invariant submanifold in the same ambient space. We also obtain B.Y. Chen inequality for totally real submanifold in Bochner-Kaehler manifold.

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1 Introduction

In 1948, S. Bochner [1] introduced Bochner curvature tensor on a Kaehler manifold as the complex version of the Weyl conformal curvature tensor on a Riemannian manifold. If the Bochner curvature tensor of a Kaehler metric vanishes, then it is called Bochner-Kaehler metric. A complex manifold with Bochner-Kaehler metric is called Bochner-Kaehler manifold. Many geometers obtained various results for different submanifolds in Bochner-Kaehler manifold [9, 11].

On the other hand, the Wintgen inequality is a sharp geometric inequality for surface in 4-dimensional Euclidean space involving Gauss curvature (intrinsic invariant), normal curvature, and square mean curvature (extrinsic invariant).

P. Wintgen [13], proved that the gauss curvature $\mathcal{K}$, the normal curvature $\mathcal{K}^{\perp}$ and the squared mean curvature $\|\mathcal{H}\|^2$ for any surface $\mathcal{N}^2$ in E^4 satisfies the following inequality:

$$\|\mathcal{H}\|^2 \geq \mathcal{K} + |\mathcal{K}^{\perp}|,$$

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and the equality holds if and only if the ellipse of curvature $\mathcal{N}^2$ in E^4 is a circle.

Later, it was extended by I. V. Gaudalupe et al. [4] for arbitrary codimension m in real space forms $\overline{\mathcal{N}}^{m+2}(c)$ as

$$\|\mathcal{H}\|^2 + c \geq \mathcal{K} + |\mathcal{K}^{\perp}|.$$

They also discussed the equality case of the inequality.

Recently, I. Mihai [8] obtained the DDVV inequality for Lagrangian submanifolds in complex space forms and investigated some of its applications.

In present article, we obtain the generalized Wintgen inequality for bi-slant submanifold in Bochner-Kaehler manifold. We also investigate such inequality for different slant cases. Further, we discuss the B. Y. Chen inequality for totally real submanifold in Bochner-Kaehler manifold.

2 Submanifolds in Bochner-Kaehler manifold

Let $\mathcal{N}$ be a real p-dimensional submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$ of complex dimension m. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on $\mathcal{N}$ and $\overline{\mathcal{N}}$ respectively. Let J be the complex structure on $\overline{\mathcal{N}}$. Then the Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$

$$\overline{\nabla}_X N = -S_N X + \nabla^{\perp}_X Y, \tag{2.2}$$

for all X, Y tangent to $\mathcal{N}$ and vector field N normal to $\mathcal{N}$. Where σ, $\nabla^{\perp}_X$, S_N denotes the second fundamental form, normal connection and the shape operator respectively. The shape operator and the second fundamental form are related by

$$g(\sigma(X,Y),N) = g(S_N X, Y). \tag{2.3}$$

Let R be the curvature tensor of $\mathcal{N}$ and let $\overline{R}$ be the curvature tensor of $\overline{\mathcal{N}}$, then the Gauss equation is given by [3]

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &\quad -g(\sigma(X,W),\sigma(Y,Z)),\end{aligned} \tag{2.4}$$

for any vector fields X, Y, Z, W tangent to $\mathcal{N}$.

The curvature tensor $\overline{R}$ of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$ is defined as [10]

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= \mathcal{L}(Y,Z)g(X,W) - \mathcal{L}(X,Z)g(Y,W) \\ &\quad +\mathcal{L}(X,W)g(Y,Z) - \mathcal{L}(Y,W)g(X,Z) \\ &\quad +\mathcal{M}(Y,Z)g(JX,W) - \mathcal{M}(X,Z)g(JY,W) \\ &\quad +\mathcal{M}(X,W)g(JY,Z) - \mathcal{M}(Y,W)g(JX,Z) \\ &\quad -2\mathcal{M}(X,Y)g(JZ,W) - 2\mathcal{M}(Z,W)g(JX,Y),\end{aligned} \tag{2.5}$$

where

$$\mathcal{L}(Y,Z) = \frac{1}{2p+4}\overline{Ric}(Y,Z) - \frac{\overline{\rho}}{2(2p+2)(2p+4)}g(Y,Z), \tag{2.6}$$

$$\mathcal{M}(Y,Z) = -\mathcal{L}(Y,JZ), \tag{2.7}$$

$$\mathcal{L}(Y,Z) = \mathcal{L}(Z,Y), \quad \mathcal{L}(Y,Z) = \mathcal{L}(JY,JZ), \quad \mathcal{L}(Y,JZ) = -\mathcal{L}(JY,Z), \tag{2.8}$$

where $\overline{Ric}$ and $\overline{\rho}$ are the Ricci tensor and scalar curvature of $\overline{\mathcal{N}}$.

Let $\{e_1,\dots,e_p\}$ and $\{e_{p+1},\dots,e_{2m}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on $\mathcal{N}$. The mean curvature vector field is given by

$$\mathcal{H} = \frac{1}{p}\sum_{i=1}^{p}\sigma(e_i,e_i). \tag{2.9}$$

The norm of the squared mean curvature of the submanifold is defined by

$$\|\mathcal{H}\|^2 = \frac{1}{p^2}\sum_{\gamma=p+1}^{m}\left(\sum_{i=1}^{p}\sigma_{ii}^{\gamma}\right)^2.$$

Further, we set

$$\|\sigma\|^2 = \sum_{i,j=1}^{p} g(\sigma(e_i,e_j),\sigma(e_i,e_j)). \tag{2.10}$$

For any $x\in\mathcal{N}$ and $X\in T_x\mathcal{N}$, we put $JX = TX + FX$, where TX and FX are the tangential and normal components of JX, respectively.

We denote by

$$\|P\|^2 = \sum_{i,j=1}^{p} g^2(Je_i,e_j). \tag{2.11}$$

Definition 2.1 ([2])**.** A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a slant submanifold if for any $p\in\mathcal{N}$ and a non zero vector $X\in T_p\mathcal{N}$, the angle between JX and $T_p\mathcal{N}$ is constant, i.e., the angle does not depend on the choice of $p\in\mathcal{N}$ and $X\in T_p\mathcal{N}$. The angle $\theta\in[0,\frac{\pi}{2}]$ is called the slant angle of $\mathcal{N}$ in $\overline{\mathcal{N}}$.

Definition 2.2 ([2])**.** A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a bi-slant submanifold, if there exist two orthogonal distributions D_1 and D_2, such that (i) $T\mathcal{N}$ admits the orthogonal direct decomposition i.e $T\mathcal{N} = D_1 + D_2$. (ii) For i=1,2, D_i is the slant distribution with slant angle θ_i.

In fact, semi-slant submanifold, hemi-slant submanifold, CR-submanifold, slant submanifold can be obtained from bi-slant submanifold in particular. We can see the case in the following table:

Table 1: Definition

S.N.	$\overline{\mathcal{N}}$	$\mathcal{N}$	D_1	D_2	θ_1	θ_2
(1)	$\overline{\mathcal{N}}$	bi-slant	slant	slant	slant angle	slant angle
(2)	$\overline{\mathcal{N}}$	semi-slant	invariant	slant	0	slant angle
(3)	$\overline{\mathcal{N}}$	hemi-slant	slant	anti-invariant	slant angle	$\frac{\pi}{2}$
(4)	$\overline{\mathcal{N}}$	CR	invariant	anti-invariant	0	$\frac{\pi}{2}$
(5)	$\overline{\mathcal{N}}$	slant	either $D_1 = 0$ or $D_2 = 0$		either $\theta_1 = \theta_2 = \theta$ or $\theta_1 = \theta_2 \neq \theta$	

Invariant and anti-invariant submanifold (or totally real submanifold) are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively and when $0 < \theta < \frac{\pi}{2}$, then slant submanifold is called proper slant submanifold.

We may also state the totally real submanifolds as:

Definition 2.3 ([14](pp. 199))**.** A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a totally real submanifold, if $JT_x\mathcal{N} \subset T_x\mathcal{N}^{\perp}$ for each point $x \in \mathcal{N}$.

Also, we may state Einstein manifold as:

Definition 2.4 ([15](pp. 5))**.** An almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be Einstein manifold if the Ricci tensor $\overline{Ric}$ is proportional to the metric tensor g, i.e. $\overline{Ric}(X,Y) = \lambda g(X,Y)$, for some constant λ.

If $\mathcal{N}$ is a bi-slant submanifold in Bochner Kaehler manifold $\overline{\mathcal{N}}$, then one can easily see that

$$\|P\|^2 = 2(d_1 cos^2\theta_1 + d_2 cos^2\theta_2), \tag{2.12}$$

where $dimD_1 = 2d_1$ and $dimD_2 = 2d_2$.

3 Generalized Wintgen inequality

We denote by $\mathcal{K}$ and $R^{\perp}$ the sectional curvature function and the normal curvature tensor on $\mathcal{N}$, respectively. Then the normalized scalar curvature ρ is given by [8]

$$\rho = \frac{2\tau}{p(p-1)} = \frac{2}{p(p-1)} \sum_{1 \le i < j \le p} \mathcal{K}(e_i \wedge e_j), \tag{3.1}$$

where τ is scalar curvature, and the normalized normal scalar curvature by [8]

$$\rho^{\perp} = \frac{2\tau^{\perp}}{p(p-1)} = \frac{2}{p(p-1)} \sqrt{\sum_{1 \le i < j \le p} \sum_{1 \le r < s \le 2m} (R^{\perp}(e_i, e_j, \xi_r, \xi_s))^2}.$$

Following [14] we put

$$\mathcal{K}_N = \frac{1}{4}\sum_{r,s=1}^{2m-p} Trace[S_r, S_s]^2 \tag{3.2}$$

and called it the scalar normal curvature of $\mathcal{N}$. The normalized scalar normal curvature is given by [8] $\rho_N = \frac{2}{p(p-1)}\sqrt{\mathcal{K}_N}$.
Obviously

$$\begin{aligned}\mathcal{K}_N &= \frac{1}{2}\sum_{1\le r<s\le 2m-p} Trace[S_r, S_s]^2\\ &= \sum_{1\le r<s\le 2m-p}\sum_{1\le i<j\le p} g([S_r, S_s]e_i.e_j)^2,\end{aligned} \tag{3.3}$$

for $i, j \in \{1, \ldots, p\}$ and $r, s \in \{1, \ldots, 2m-p\}$.
In term of the components of the second fundamental form, we can express $\mathcal{K}_N$ by the formula [8]

$$\mathcal{K}_N = \sum_{1\le r<s\le 2m-p}\sum_{1\le i<j\le p}\big(\sum_{k=1}^{p}\sigma^r_{jk}\sigma^s_{ik} - \sigma^r_{jk}\sigma^s_{ik}\big)^2. \tag{3.4}$$

Now, we shall state and proof the generalized Wintgen inequality for bi-slant submanifold in Bochner-Kaehler manifold.

Theorem 3.1. *Let $\mathcal{N}$ be a bi-slant submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \le\ & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6(d_1cos^2\theta_1 + d_2cos^2\theta_2)}{4p(p^2-1)(p+2)}\overline{\rho}\\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\le i<j\le p}\overline{Ric}(e_i, e_j)g(e_i, e_j)\\ & +\frac{3}{2p(p-1)(p-2)}\sum_{1\le i<j\le p}\overline{Ric}(e_i, Je_j)g(e_i, Je_j).\end{aligned} \tag{3.5}$$

Proof. Let $\mathcal{N}$ be a submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}(c)$. We choose $\{e_1, \ldots, e_p\}$ and $\{e_{p+1}, \ldots, e_{2m}\}$ as orthonormal frame and orthonormal normal frame on $\mathcal{N}$ respectively. Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$ from (2.5), we have

$$\begin{aligned}\overline{R}(e_i, e_j, e_j, e_i) =\ & \mathcal{L}(e_j, e_j)g(e_i, e_i) - \mathcal{L}(e_i, e_j)g(e_j, e_i)\\ & +\mathcal{L}(e_i, e_i)g(e_j, e_j) - \mathcal{L}(e_j, e_i)g(e_i, e_j)\\ & +\mathcal{M}(e_j, e_j)g(Je_i, e_i) - \mathcal{M}(e_i, e_j)g(Je_j, e_i)\\ & +\mathcal{M}(e_i, e_i)g(Je_j, e_j) - \mathcal{M}(e_j, e_i)g(Je_i, e_j)\\ & -2\mathcal{M}(e_i, e_j)(Je_j, e_i) - 2\mathcal{M}(e_j, e_i)g(Je_i, e_j).\end{aligned} \tag{3.6}$$

From Gauss equation and (3.6), we get

$$\begin{aligned} R(e_i,e_j,e_j,e_i) &= \mathcal{L}(e_j,e_j)g(e_i,e_i)-\mathcal{L}(e_i,e_j)g(e_j,e_i) \\ &+\mathcal{L}(e_i,e_i)g(e_j,e_j)-\mathcal{L}(e_j,e_i)g(e_i,e_j) \\ &+\mathcal{M}(e_j,e_j)g(Je_i,e_i)-\mathcal{M}(e_i,e_j)g(Je_j,e_i) \\ &+\mathcal{M}(e_i,e_i)g(Je_j,e_j)-\mathcal{M}(e_j,e_i)g(Je_i,e_j) \\ &-2\mathcal{M}(e_i,e_j)(Je_j,e_i)-2\mathcal{M}(e_j,e_i)g(Je_i,e_j) \\ &+g(\sigma(e_j,e_j),\sigma(e_i,e_i))-g(\sigma(e_i,e_j),\sigma(e_j,e_i)). \end{aligned} \tag{3.7}$$

By taking summation $1\leq i<j\leq p$ and using (3.6) in (3.7), we derive

$$\begin{aligned} \sum_{1\leq i<j\leq p} R(e_i,e_j,e_j,e_i) &= \frac{3p^2+5p-3\|P\|^2}{8(p+1)(p+2)}\overline{\rho} \\ &-\frac{1}{2(p+1)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ &+\frac{3}{2(p+2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \\ &+\sum_{r=p+1}^{2m-p}\sum_{1\leq i<j\leq p}[\sigma_{ii}^r\sigma_{jj}^r-(\sigma_{ij}^r)^2]. \end{aligned} \tag{3.8}$$

Also, we know that

$$\tau=\sum_{1\leq i<j\leq p} R(e_i,e_j,e_j,e_i). \tag{3.9}$$

Now, using equation (2.12) and (3.9) in (3.8), gives

$$\begin{aligned} \tau = &\frac{3p^2+5p-6(d_1cos^2\theta_1+d_2cos^2\theta_2)}{8(p+1)(p+2)}\overline{\rho}-\frac{1}{2(p+1)}\textstyle\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ &+\frac{3}{2(p+2)}\textstyle\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)+\sum_{r=p+1}^{2m-p}\sum_{1\leq i<j\leq p}[\sigma_{ii}^r\sigma_{jj}^r-(\sigma_{ij}^r)^2]. \end{aligned} \tag{3.10}$$

On the other hand, we have

$$\begin{aligned} p^2\|\mathcal{H}\|^2 &= \sum_{r=p+1}^{2m-p}\Big(\sum_{i=1}^{p}\sigma_{ii}^r\Big)^2 \\ &= \frac{1}{p-1}\sum_{r=p+1}^{2m-p}\sum_{1\leq i<j\leq p}(\sigma_{ii}^r-\sigma_{jj}^r)^2+\frac{2p}{p-1}\sum_{r=p+1}^{2m-p}\sum_{1\leq i<j\leq p}\sigma_{ii}^r\sigma_{jj}^r. \end{aligned} \tag{3.11}$$

Further, from [7]

$$\begin{aligned} &\sum_{r=p+1}^{2m-p}\sum_{1\leq i<j\leq p}(\sigma_{ii}^r-\sigma_{jj}^r)^2+2p\sum_{r=p+1}^{2m-p}\sum_{1\leq i<j\leq p}(\sigma_{ij}^r)^2 \\ &\geq 2p\Big[\sum_{p+1\leq r<s\leq 2m-p}\sum_{1\leq i<j\leq p}\Big(\sum_{k=1}^{p}(\sigma_{jk}^r\sigma_{ik}^s-\sigma_{ik}^r\sigma_{jk}^s)\Big)^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{3.12}$$

Combining (3.4), (3.11) and (3.12), we find

$$p^2\|\mathcal{H}\|^2 - p^2\rho_N \geq \frac{2p}{p-1}\sum_{r=p+1}^{2m-p}\sum_{1\leq i<j\leq p}[\sigma^r_{ii}\sigma^r_{jj})^2 - (\sigma^r_{ij})^2]. \tag{3.13}$$

Taking into account (3.1), (3.10) and (3.13), we obtain

$$\begin{aligned}\rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6(d_1cos^2\theta_1 + d_2cos^2\theta_2)}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ & +\frac{3}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j).\end{aligned} \tag{3.14}$$

□

4 Immediate applications

An immediate consequence of the Theorem 3.1 yields the following.

Corollary 4.1. *Let $\mathcal{N}$ be a semi-slant submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6(d_1 + d_2cos^2\theta_2)}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ & +\frac{3}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j).\end{aligned} \tag{4.1}$$

Corollary 4.2. *Let $\mathcal{N}$ be a hemi-slant submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6d_1cos^2\theta_1}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ & +\frac{3}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j).\end{aligned} \tag{4.2}$$

Corollary 4.3. *Let $\mathcal{N}$ be a CR-submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6d_1}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j).\end{aligned} \tag{4.3}$$

Corollary 4.4. *Let $\mathcal{N}$ be a slant submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \leq\ & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-3pcos^2\theta}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ & +\frac{3}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j). \end{aligned} \tag{4.4}$$

Corollary 4.5. *Let $\mathcal{N}$ be a invariant submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \leq\ & \|\mathcal{H}\|^2 - \rho - \frac{3p+2}{4(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ & +\frac{3}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j). \end{aligned} \tag{4.5}$$

Corollary 4.6. *Let $\mathcal{N}$ be a anti-invariant submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \leq\ & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{1}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ & +\frac{3}{2p(p-1)(p-2)}\sum_{1\leq i<j\leq p}\overline{Ric}(e_i,Je_j)g(e_i,Je_j). \end{aligned} \tag{4.6}$$

Corollary 4.7. *Let $\mathcal{N}$ be a bi-slant submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \leq\ & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6(d_1+d_2cos^2\theta_2)}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{\lambda}{(p-1)(p-2)}+\frac{3\lambda\|P\|^2}{p(p-1)(p-2)}. \end{aligned} \tag{4.7}$$

Corollary 4.8. *Let $\mathcal{N}$ be a semi-slant submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N \leq\ & \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6(d_1+d_2cos^2\theta_2)}{4p(p^2-1)(p+2)}\overline{\rho} \\ & -\frac{\lambda}{(p-1)(p-2)}+\frac{3\lambda\|P\|^2}{p(p-1)(p-2)}. \end{aligned} \tag{4.8}$$

Corollary 4.9. *Let $\mathcal{N}$ be a hemi-slant submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N &\leq \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6d_1cos^2\theta_1}{4p(p^2-1)(p+2)}\overline{\rho} \\ &\quad - \frac{\lambda}{(p-1)(p-2)} + \frac{3\lambda\|P\|^2}{p(p-1)(p-2)}.\end{aligned} \tag{4.9}$$

Corollary 4.10. *Let $\mathcal{N}$ be a CR-submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\rho_N \leq \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-6d_1}{4p(p^2-1)(p+2)}\overline{\rho} - \frac{\lambda}{(p-1)(p-2)}. \tag{4.10}$$

Corollary 4.11. *Let $\mathcal{N}$ be a slant submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N &\leq \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p-3pcos^2\theta}{4p(p^2-1)(p+2)}\overline{\rho} \\ &\quad - \frac{\lambda}{(p-1)(p-2)} + \frac{3\lambda\|P\|^2}{p(p-1)(p-2)}.\end{aligned} \tag{4.11}$$

Corollary 4.12. *Let $\mathcal{N}$ be a invariant submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N &\leq \|\mathcal{H}\|^2 - \rho - \frac{3p+2}{4(p^2-1)(p+2)}\overline{\rho} \\ &\quad - \frac{\lambda}{(p-1)(p-2)} + \frac{3\lambda\|P\|^2}{p(p-1)(p-2)}.\end{aligned} \tag{4.12}$$

Corollary 4.13. *Let $\mathcal{N}$ be a anti-invariant submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then*

$$\begin{aligned}\rho_N &\leq \|\mathcal{H}\|^2 - \rho - \frac{3p^2+5p}{4p(p^2-1)(p+2)}\overline{\rho} \\ &\quad - \frac{\lambda}{(p-1)(p-2)} + \frac{3\lambda\|P\|^2}{p(p-1)(p-2)}.\end{aligned} \tag{4.13}$$

Remark 4.1. The proof of the Corollary 4.1 - Corollary 4.6 is similar to the Theorem 3.1. We obtain the proof of the Corollary 4.1 - Corollary 4.4 with the help of Table 1 and Theorem 3.1. The Corollary 4.5 and Corollary 4.6 is obtain by putting $\theta = 0$ and $\theta = \frac{\pi}{2}$ in Corollary 4.4 respectively. Proof of Corollary 4.7 to Corollary 4.13 is similar to the proof of Corollary 4.1 - Corollary 4.6 and using Definition 2.4.

5 B. Y. Chen inequality for totally real submanifolds in Bochner-Kaehler manifold

In 1993, B. Y. Chen [3] has obtained a sharp inequality for the sectional curvature of a submanifold in a real space forms in term of the scalar curvature and squared mean curvature. Afterward, several geometers obtained similar inequality for different submanifolds in different ambient spaces [12].

In this section, we derive B. Y. Chen inequality for totally real submanifold in Bochner-Kaehler manifold. Before that, we recall the following lemma.

Lemma 5.1 ([3]). *Let $b_1, b_2, \dots, b_p, l$ be $p+1$ real numbers for $p \geq 2$ such that*

$$\Big(\sum_{i=1}^{p} b_i\Big)^2 = (p-1)\Big(\sum_{i=1}^{p} b_i^2 + l\Big).$$

Then, $2b_1b_2 \geq l$ and the equality holds if and only if $b_1 + b_2 = b_3 = \cdots = b_p$.

Now, we state and proof the following.

Theorem 5.2. *Let $\mathcal{N}$ be a totally real submanifold in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then for each point $x \in \mathcal{N}$ and each plane section $\pi \in T_x\mathcal{N}$, we have*

$$\begin{aligned} \tau - K(\pi) \quad \leq \quad & \frac{3p^2 - 13p - 6}{8(p+1)(p+2)}\overline{\rho} - \frac{1}{2(p+2)}\overline{Ric}(e_i, e_j)g(e_i, e_j) \\ & + \frac{p^2(p-2)}{2(p-1)}\|\mathcal{H}\|^2. \end{aligned} \tag{5.1}$$

Equality holds if and only if there exists an orthonormal basis $\{e_1, \dots, e_p\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{p+1}, \dots, e_{2m}\}$ of $T^{\perp}N$ such that the shape operators takes the following forms

$$S_{p+1} = \begin{pmatrix} \varsigma & 0 & 0 & \dots & 0 \\ 0 & \upsilon & 0 & \dots & 0 \\ 0 & 0 & \xi & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \xi \end{pmatrix}, \qquad \varsigma + \upsilon = \xi. \tag{5.2}$$

$$S_r = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & \dots & 0 \\ \sigma_{21}^r & -\sigma_{22}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad r = p+2, \dots, 2m. \tag{5.3}$$

Proof. Combining equations (2.4) and (2.5), we get

$$\begin{aligned} R(e_i, e_j, e_j, e_i) \quad = \quad & \mathcal{L}(e_j, e_j)g(e_i, e_i) - \mathcal{L}(e_i, e_j)g(e_j, e_i) \\ & + \mathcal{L}(e_i, e_i)g(e_j, e_j) - \mathcal{L}(e_j, e_i)g(e_i, e_j) \\ & + g(\sigma(e_j, e_j), \sigma(e_i, e_i)) - g(\sigma(e_i, e_j), \sigma(e_j, e_i)). \end{aligned} \tag{5.4}$$

Now, taking summation over $1 \leq i, j \leq p, i \neq j$ and using (2.6) and (5.4), we find

$$2\tau = \frac{3p^2+5p}{4(p+1)(p+2)}\overline{\rho} - \frac{1}{p+2}\overline{Ric}(e_i,e_j)g(e_i,e_j) + p^2\|\mathcal{H}\|^2 - \|\sigma\|^2. \tag{5.5}$$

Further, we put

$$\begin{aligned} \delta &= 2\tau - \frac{3p^2+5p}{4(p+1)(p+2)}\overline{\rho} + \frac{1}{p+2}\overline{Ric}(e_i,e_j)g(e_i,e_j) \\ &\quad -\frac{p^2(p-2)}{(p-1)}\|\mathcal{H}\|^2. \end{aligned} \tag{5.6}$$

Using equation (5.5) and (5.6), we obtain

$$p^2\|\mathcal{H}\|^2 = (p-1)(\delta + \|\sigma\|^2). \tag{5.7}$$

In the view of chosen orthomormal basis above equation can be written as

$$\left(\sum_{i=1}^{p}\sigma_{ii}^{p+1}\right)^2 = (p-1)\left[\sum_{i=1}^{p}(\sigma_{ii}^{p+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{p+1})^2 + \sum_{r=p+1}^{2m}\sum_{i,j=1}^{p}(\sigma_{ij}^{r})^2 + \delta\right]. \tag{5.8}$$

Now, with the help of Lemma 5.1 and (5.8), we derive

$$2\sigma_{11}^{p+1}\sigma_{22}^{p+1} \geq \sum_{i\neq j}(\sigma_{ij}^{p+1})^2 + \sum_{r=p+1}^{2m}\sum_{i,j=1}^{p}(\sigma_{ij}^{r})^2 + \delta. \tag{5.9}$$

On the other hand, we have

$$K(\pi) = R(e_1,e_2,e_2,e_1), \tag{5.10}$$

which implies

$$\begin{aligned} K(\pi) &= \overline{R}(e_1,e_2,e_2,e_1) + g\big(\sigma(e_2,e_2),\sigma(e_1,\sigma_1)\big) - g\big(\sigma(e_1,e_2),\sigma(e_2,\sigma_1)\big) \\ &= \frac{4p+3}{(2p+2)(2p+4)}\overline{\rho} + \sigma_{11}^{p+1}\sigma_{22}^{p+1} + \sum_{r=p+2}^{2m}\sigma_{11}^{r}\sigma_{22}^{r} - \sum_{r=p+1}^{2m}(\sigma_{12}^{r})^2. \end{aligned} \tag{5.11}$$

Taking into account (5.9) and (5.11), we obtain

$$\begin{aligned} K(\pi) &\geq \frac{1}{2}\sum_{i\neq j}(\sigma_{ij}^{p+1})^2 + \frac{1}{2}\sum_{r=p+1}^{2m}\sum_{i,j=1}^{p}(\sigma_{ij}^{r})^2 \\ &\quad +\frac{1}{2}\delta + \frac{4p+3}{(2p+2)(2p+4)}\overline{\rho} + \sum_{r=p+2}^{2m}\sigma_{11}^{r}\sigma_{22}^{r} - \sum_{r=p+1}^{2m}(\sigma_{12}^{r})^2 \\ &\geq \frac{1}{2}\delta + \frac{4p+3}{4(p+1)(p+2)}\overline{\rho}. \end{aligned} \tag{5.12}$$

Equations (5.7) and (5.12) gives

$$\tau - K(\pi) \leq \frac{3p^2-13p-6}{8(p+1)(p+2)}\overline{\rho} - \frac{1}{2(p+2)}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{p^2(p-2)}{2(p-1)}\|\mathcal{H}\|^2 \tag{5.13}$$

and the equality holds if the shape operator takes the above stated forms. Further, if the equality holds in the inequality (5.1), then we have

$$\begin{cases} \sigma^r_{ii} = 0,\ \forall i \neq j, i,j = 3,\dots,p,\ r = p+1,\dots,2m, \\ \sigma^{p+1}_{1j} = \sigma^{p+1}_{2j} = \sigma^{p+1}_{ij} = 0,\ i \neq j > 2, \\ \sigma^r_{11} + \sigma^r_{22} = 0,\ \forall r = p+2,\dots,2m, \\ \sigma^{p+1}_{11} + \sigma^{p+1}_{22} = \cdots = \sigma^{p+1}_{nn} = 0. \end{cases} \tag{5.14}$$

Now, if we take e_1, e_2 such that $\sigma^{p+1}_{12} = 0$ and putting $\varsigma = \sigma^r_{11}, \upsilon = \sigma^r_{22}, \xi = \sigma^{p+1}_{33} = \cdots = \sigma^{p+1}_{nn}$. Then, it follows that the shape operator have the desired forms. □

Corollary 5.3. *Let $\mathcal{N}$ be a totally real submanifold in Einstein Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then for each point $x \in \mathcal{N}$ and each plane section $\pi \in T_x\mathcal{N}$, we have*

$$\tau - K(\pi) \leq \frac{3p^2-13p-6}{8(p+1)(p+2)}\overline{\rho} - \frac{\lambda p}{2(p+2)} + \frac{p^2(p-2)}{2(p-1)}\|\mathcal{H}\|^2. \tag{5.15}$$

Equality holds if and only if there exists an orthonormal basis $\{e_1,\dots,e_p\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{p+1},\dots,e_{2m}\}$ of $T^\perp N$ such that the shape operators take forms (5.2) and (5.3).

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