A note on Yamabe solitons

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Abstract. We find sufficient conditions on the soliton vector field under which the metric of a Yamabe soliton is a Yamabe metric, that is, a metric of constant scalar curvature.

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R. Hamilton introduced the notion of Yamabe flow (cf. [8]), in which the metric on a Riemannian manifold is deformed by evolving according to

$$\frac{\partial}{\partial t} g(t) = -R(t)g(t), \tag{1}$$

where $R(t)$ is the scalar curvature of the metric $g(t)$. Yamabe solitons correspond to self-similar solutions of the Yamabe flow.

In dimension $n = 2$ the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t} g(t) = -2Ric(t)$, where Ric denotes the Ricci tensor). However, in dimension $n > 2$ the Yamabe and Ricci flows do not agree, since the first one preserves the conformal class of the metric but the Ricci flow does not in general.

A connected Riemannian n-manifold (M, g) with $n \geq 2$ is called a *Yamabe soliton* if it admits a vector field ξ such that

$$\frac{1}{2}\pounds_\xi g = (R - \lambda)g, \tag{2}$$

where $\pounds_\xi$ denotes the Lie derivative in the direction of the vector field ξ and λ is a real number. Moreover, the vector field ξ in the definition is called a soliton vector field for (M, g). In the following, we denote the Yamabe soliton satisfying (2) by (M, g, ξ, λ). A Yamabe soliton is said to be gradient Yamabe soliton if the soliton vector field ξ is gradient of a smooth function. Recall that on a Riemannian manifold (M, g), the metric g is said to be a *Yamabe metric* if the scalar curvature R is a constant.

Yamabe flows and Yamabe solitons have been studied quite extensively (cf. [1]-[3], [5], [6], [9]). In [6], it is shown that the metric of a compact gradient Yamabe soliton is a Yamabe metric and the same result is achieved in [9] giving a shorter proof. One of the interesting questions is to find conditions on soliton vector field of a Yamabe soliton so that the metric is of a constant scalar curvature.

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In this article, a vector field ξ on a Riemannian manifold (M,g) is called *closed* if its dual 1-form is closed. A smooth vector field ξ on (M,g) is said to be a *geodesic vector field* if it satisfies

$$\nabla_\xi \xi = 0.$$

It follows that integral curves of a geodesic vector field are geodesics. Natural examples of geodesic vector fields are Reeb vector fields on Sasakian manifolds, Killing vector fields of constant length on a Riemannian manifold. Also, all parallel vector fields on the Euclidean space $\mathbb{R}^n$ in particular constant vector fields on $\mathbb{R}^n$ are geodesic vector fields. Next, we define generalized geodesic vector fields on a Riemannian manifold. A smooth vector field ξ on a Riemannian manifold (M,g) is said to be a *generalized geodesic vector field* if it satisfies

$$\nabla_\xi \xi = f\xi,$$

where f is a smooth function on M called the potential function. It is clear that a geodesic vector fields are generalized geodesic vector field but the converse is not true. Physically, a generalized geodesic vector fields are those whose integral curves have acceleration that is multiple of the velocity. Natural examples of generalized geodesic vector field are, positions vector field on the Euclidean space $\mathbb{R}^n$ (with potential function constant 1), a closed conformal vector field ξ on a Riemannian manifold (M,g). Let Z be a nonzero constant vector field on the Euclidean space $\mathbb{R}^{n+1}$ and Z^T be the tangential projection of Z on the unit sphere $\mathbb{S}^n$. Then with respect to the Riemannian connection ∇ on $\mathbb{S}^n$, we have

$$\nabla_{Z^T} Z^Y = \rho Z^T,$$

where $\rho = \langle Z, N\rangle$, N is the unit normal vector field to the sphere $\mathbb{S}^n$ and $\langle\ ,\ \rangle$ is the Euclidean metric on $\mathbb{R}^{n+1}$. Thus Z^T is generalized geodesic vector field on $\mathbb{S}^n$. The following Lemma gives a sufficient condition for a generalized geodesic vector field to be geodesic vector field.

Lemma. *A nonzero generalized geodesic vector field ξ of constant length on a Riemannian manifold (M,g) is a geodesic vector field.*

Proof. Let ξ be a generalized geodesic vector field with potential function f. Then for $\varphi = \frac{1}{2}\|\xi\|^2$, using definition of generalized geodesic vector field, we get

$$\xi(\varphi) = f\varphi,$$

thus on open subset where $\varphi \neq 0$, we have

$$f = \xi(\ln\varphi).$$

If φ is a constant, the above equation gives $f = 0$ and consequently, ξ is a geodesic vector field. □

As a direct consequence of above Lemma, we have the following result for a Yamabe soliton.

Corollary. *If the soliton field of Yamabe soliton (M, g, ξ, λ) is a generalized geodesic vector field with potential function $R - \lambda$ and length of soliton field is a constant, then g is a Yamabe metric.*

In this short note, we use generalized geodesic vector field to prove the following results for the cases when the soliton field is non-closed and other for when it is closed:

Theorem 1. *A connected Yamabe soliton (M, g, ξ, λ) with soliton vector field is a generalized geodesic vector field with potential function $R - \lambda$ that is not closed has constant scalar curvature.*

Proof. Let η be smooth 1-form dual to soliton vector field ξ, that is, $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being Lie algebra of smooth vector fields on M. Define a skew symmetric $(1,1)$ tensor field φ on M by

$$g(\varphi X, Y) = \frac{1}{2} d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{3}$$

Then using Koszul's formula (cf. [4]) and equations (2) and (3), the covariant derivative of the soliton field ξ is given by

$$\nabla_X \xi = (R - \lambda)X + \varphi X, \quad X \in \mathfrak{X}(M). \tag{4}$$

Using above equation, it is easy to see that the curvature tensor of the Yamabe soliton (M, g, ξ, λ) satisfies

$$R(X, Y)\xi = X(R)Y - Y(R)X + (\nabla\varphi)(X, Y) - (\nabla\varphi)(Y, X), \tag{5}$$

where the covariant derivative $\nabla\varphi$ is defined by

$$(\nabla\varphi)(X, Y) = \nabla_X (\varphi Y) - \varphi(\nabla_X Y), \quad X, Y \in \mathfrak{X}(M).$$

Note the smooth 2-form $\Omega = d\eta$ is closed and therefore using equations (3) and (5) and symmetries of curvature tensor, we conclude that

$$(\nabla\varphi)(X, Y) = R(X, \xi)Y + Y(R)X - g(X, Y)\nabla R, \quad X, Y \in \mathfrak{X}(M), \tag{6}$$

where ∇R is the gradient of scalar curvature R. Note that the condition on the soliton filed and equation (4) gives $\varphi(\xi) = 0$. Thus the equation (6) with $X = Y = \xi$ gives

$$\xi(R)\xi = \|\xi\|^2 \nabla R,$$

that is, vector fields ξ and ∇R are parallel and consequently there exists a smooth function α on M such that

$$\nabla R = \alpha\xi. \tag{7}$$

Let A_R be the Hessian operator of the scalar curvature R, then the above equation and the equation (4) gives

$$A_R X = X(\alpha)\xi + \alpha(R - \lambda)X + \alpha\varphi X, \quad X \in \mathfrak{X}(M). \tag{8}$$

Using the symmetry of the operator A_R and skew-symmetry of φ in above equation leads to

$$2\alpha\varphi X = g(X,\xi)\nabla\alpha - X(\alpha)\xi, \quad X \in \mathfrak{X}(M). \tag{9}$$

Taking $X = \xi$ in above equation gives

$$\|\xi\|^2 \nabla\alpha = \xi(\alpha)\xi, \tag{10}$$

which on taking inner product with $\nabla\alpha$, gives

$$\|\xi\|^2 \|\nabla\alpha\|^2 = \xi(\alpha)^2. \tag{11}$$

Now, with a local orthonormal frame $\{e_1, e_2, ..e_n\}$ the equation (9) gives

$$4\alpha^2 \|\varphi\|^2 = 4\alpha^2 \sum \|\varphi e_i\|^2 = 2\|\xi\|^2 \|\nabla\alpha\|^2 - 2\xi(\alpha)^2,$$

which in view of equation (11), gives

$$\alpha^2 \|\varphi\|^2 = 0.$$

If $\alpha = 0$, then equation (7) implies R is a constant. If $\varphi = 0$, then the equation (4) takes the form

$$\nabla_X \xi = (R - \lambda)X, \tag{12}$$

which gives ξ is closed and this is not allowed by the hypothesis that ξ is not closed. □

Next, we consider the Yamabe soliton (M, g, ξ, λ) with soliton field ξ a generalized geodesic vector field with potential function $(R - \lambda)$ and it is closed. Recall that the Hessian of the scalar curvature H_R is given by

$$H_R(X,Y) = g\left(A_R X, Y\right), \quad X, Y \in \mathfrak{X}(M),$$

and if $H_R(u,u) = 0$, for some vector field u, Hessian is said to be degenerate in the direction of u.

Theorem 2. *Let the soliton field of a connected Yamabe soliton (M, g, ξ, λ) be closed generalized geodesic vector field with potential function $R - \lambda$. If the function*

$$\frac{\xi(R)}{\|\xi\|^2}$$

(defined on the open subset where $\xi \neq 0$) is a constant along the integral curves of ξ and the Hessian H_R is degenerate in the direction of ξ, then Yamabe soliton has constant scalar curvature.

Proof. As the soliton field is closed, we have $\varphi = 0$ in equation (4) and it takes the form of equation (12), and equation (6), gives

$$R(X,\xi)Y = g(X,Y)\nabla R - Y(R)X, \quad X,Y \in \mathfrak{X}(M), \tag{13}$$

which on taking $X = Y = \xi$, implies

$$\|\xi\|^2 \nabla R = \xi(R)\xi. \tag{14}$$

Thus, on the open subset where $\xi \neq 0$, we have

$$\nabla R = \left(\frac{\xi(R)}{\|\xi\|^2}\right)\xi,$$

which on taking divergence both sides and noting that the coefficient function of ξ in above equation is a constant along integral curves of ξ, we get

$$\Delta R = n(R-\lambda)\left(\frac{\xi(R)}{\|\xi\|^2}\right), \tag{15}$$

where we used equation (12) to conclude $\operatorname{div}\xi = n(R-\lambda)$, $n = \dim M$. Now, taking an orthonormal frame $\{e_1, .., e_n\}$ and substituting $X = Y = e_i$ in equation (13) and summing the resulting equation, leads to

$$Q(\xi) = -(n-1)\nabla R, \tag{16}$$

where Q is the Ricci operator defined by $Ric(X,Y) = g(QX,Y)$. Using equation (12) in taking divergence on both sides of equation (16), we get

$$-(n-1)\Delta R = R(R-\lambda) + \frac{1}{2}\xi(R), \tag{17}$$

where we used the following equation (cf. [4], [7])

$$\sum (\nabla Q)(e_i, e_i) = \frac{1}{2}\nabla R.$$

Eliminating ΔR in equations (15) and (17), we get

$$n(n-1)(R-\lambda)\left(\frac{\xi(R)}{\|\xi\|^2}\right) + R(R-\lambda) + \frac{1}{2}\xi(R) = 0,$$

which on differentiating along ξ, yields

$$n(n-1)\left(\frac{\xi(R)^2}{\|\xi\|^2}\right) + (R-\lambda)\xi(R) + R\xi(R) + \frac{1}{2}\xi\xi(R) = 0.$$

Now, using the expression for the Hessian $H_R(\xi,\xi) = \xi\xi(R) - \nabla_\xi\xi(R) = \xi\xi(R) - (R-\lambda)\xi(R)$ and the condition in the statement that the Hessian H_R is degenerate in the direction of ξ, in above equation leads to

$$n(n-1)\left(\frac{\xi(R)^2}{\|\xi\|^2}\right) + (R-\lambda)\xi(R) + R\xi(R) + \frac{1}{2}(R-\lambda)\xi(R) = 0,$$

that is,

$$\xi(R)\left(n(n-1)\left(\frac{\xi(R)}{\|\xi\|^2}\right)+R+\frac{3}{2}(R-\lambda)\right)=0.$$

If $\xi(R)=0$, then equations (2) and (14) imply that R is a constant. Therefore suppose

$$n(n-1)\left(\frac{\xi(R)}{\|\xi\|^2}\right)+R+\frac{3}{2}(R-\lambda)=0,$$

which on differentiating along ξ, gives

$$\frac{5}{2}\xi(R)=0,$$

and it proves that R is a constant. □

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References

[1] S. Brendle, Convergence of the Yamabe flow in dimension 6 and higher, Invent. Math. 170 (3) (2007) 541–576.

[2] S. Brendle, Convergence of the Yamabe flow for arbitrary initial energy, J. Differential Geom. 69 (2) (2005) 217–278.

[3] A. Burchard, R. J. Mccan, A. Smith, Explicit Yamabe flow of an asymmetric cigar, Methods Appl. Anal. 15 (1) (2008) 65–80.

[4] B.-Y. Chen, Total mean curvature and submanifolds of finite type, 2nd edition. World Scientific, Hackensack, NJ (2015).

[5] B. Chow, The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature, Comm. Pure Appl. Math. 45 (1992) 1003–1014.

[6] P. Daskalopoulos, N. Sesum, The classification of locally conformally flat Yamabe solitons, Advances in Math. 240 (2013) 346–369.

[7] S. Deshmukh, A. Al-Eid, Curvature bounds for the spectrum of a compact Riemannian manifold of constant scalar curvature. J. Geom. Anal. 15(4) (2005) 589–606.

[8] R. S. Hamilton, The Ricci flow on surfaces. Math. Gen. Relativ. (Santa Cruz, CA, 1986), 237-262, Contemp. Math. 71 (1998).

[9] S. Y. Hsu, A note on compact gradient Yamabe solitons, J. Math. Anal. Appl. 388 (2012) 725-726.

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