Curvature and holonomy in 4-dimensional manifolds admitting a metric

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Abstract. This paper considers 4−dimensional manifolds which admit a metric of any signature and examines the relationships between the metric, its Levi-Civita connection, its curvature tensor and sectional curvature function and its Weyl conformal tensor. It is shown that, with some special cases excepted (some of which will be discussed), these various curvature concepts are very closely related. The relationship between them and the holonomy group associated with the connection is also explored. Some of these results, in the case of positive definite and Lorentz signature, have been given before and so this paper will concentrate mainly on the case of neutral signature $(+,+,-,-)$ and on the process of putting together simple arguments which cover all signatures simultaneously.

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1 Introduction and notation

The idea of this paper is to show the close relationships between the metric g, its Levi-Civita connection ∇, the curvature tensor $Riem$ with components $R^a{}_{bcd}$, the sectional curvature function, the holonomy group and the Weyl conformal tensor C with components $C^a{}_{bcd}$ in 4−dimensional manifolds admitting a metric. Some of these results for Lorentz and positive definite (and occasionally for neutral) signature have been discussed elsewhere and it is intended that this paper collects them together and adds new ones, mostly for neutral signature. To establish notation, M denotes a 4−dimensional, smooth, connected, paracompact, hausdorff manifold with smooth metric g of any signature, collectively labelled (M, g). The tangent space at $m \in M$ is denoted by T_mM and the vector space of 2−forms (usually referred to as bivectors) at m by Λ_mM. The symbol $u.v$ denotes the inner product at m, $g(m)(u, v)$, of $u, v \in T_mM$. To allow for all signatures, a non-zero member $u \in T_mM$ is called spacelike if $u.u > 0$, timelike if $u.u < 0$ and null if $u.u = 0$. The symbol $*$ denotes the usual Hodge duality (linear) operator on Λ_mM. For positive definite signature

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an orthogonal basis of unit vectors x, y, z, w is employed whilst for Lorentz signature an orthonormal basis x, y, z, t is sometimes used with $x.x = y.y = z.z = -t.t = 1$ together with its derived *real null* basis l, n, x, y where $\sqrt{2}l = z + t$ and $\sqrt{2}n = z - t$ so that l and n are null and $l.n = 1$ with all other such inner products zero. For neutral signature one may choose an orthonormal basis x, y, s, t at $m \in M$ with $x.x = y.y = -s.s = -t.t = 1$ and an associated *null* basis of (null) vectors l, n, L, N at m given by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l.n = L.N = 1$ with all other such inner products zero.

For all signatures a $2-$dimensional subspace ($2-$space) V of T_mM is called spacelike if each non-zero member of V is spacelike, or each non-zero member of V is timelike, timelike if V contains exactly two, null $1-$dimensional subspaces (*directions*), null if V contains exactly one null direction and totally null if each non-zero member of V is null. Thus a totally null $2-$space consists, apart from the zero vector, of null vectors any two of which are orthogonal and can only occur for neutral signature. A bivector E at m with components $E^{ab}(= -E^{ba})$ necessarily has even matrix rank. If this rank is 2, E is called simple and if 4, it is called non-simple. If E is simple it may be written $E^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_mM$ and the $2-$space spanned by u and v is uniquely determined by E and called the blade of E (and then, unless more precision is required, E or its blade is written $u \wedge v$). A simple bivector is called spacelike (respectively, timelike, null or totally null) if its blade is spacelike (respectively, timelike, null or totally null). All types may occur for neutral signature whereas for Lorentz signature $2-$spaces and simple bivector blade may only be spacelike, timelike or null. For positive definite metrics all tangent vectors, $2-$spaces and blades of simple bivectors are spacelike. In the positive definite and neutral signature cases any bivector E satisfies $\overset{**}{E} = E$ whilst in Lorentz signature $\overset{**}{E} = -E$.

For positive definite and neutral signatures define the subspaces $\overset{+}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = E\}$ and $\overset{-}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = -E\}$ and also the subset $\widetilde{S}_m \equiv \overset{+}{S}_m \cup \overset{-}{S}_m$, of $\Lambda_m M$. Then each member of $\Lambda_m M$ may be written uniquely as the sum of a member of $\overset{+}{S}_m$ and a member of $\overset{-}{S}_m$ and if $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, one has $[\overset{+}{E}, \overset{-}{E}] = 0$ where [ ] denotes matrix commutation. Thus one may write $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. Each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is a Lie algebra isomorphic to $o(3)$ for positive definite signature and to $o(1, 2)$ for neutral signature, each under [ ] and so $\Lambda_m M$ is the Lie algebra product $\overset{+}{S}_m \oplus \overset{-}{S}_m$ and which is isomorphic to $o(2, 2)$ or to $o(4)$. For Lorentz signature $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are trivial and play no further role here for this signature. The set $\widetilde{S}_m$ has no simple members in the positive definite case whilst in the neutral case its only simple members are totally null. More details on such matters may be found in [12, 19, 2, 3]. Finally, one may define an inner product P on $\Lambda_m M$ by $P(A, B) = A^{ab}B_{ab}$ for $A, B \in \Lambda_m M$ and then $A \in \Lambda_m M$ is simple $\Leftrightarrow P(A, \overset{*}{A}) = 0$. In the neutral case this inner product reduces to a Lorentz metric on each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$. If $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, $P(\overset{+}{E}, \overset{-}{E}) = 0$.

In what is to follow, those subalgebras of $o(4)$, $o(1, 3)$ and $o(2, 2)$ which can be holonomy algebras for the connection ∇ will be important. Thus, in Tables $1 - 3$, the first three columns list all possible holonomy algebras, their dimensions and a spanning set in the bivector representation which is convenient for present purposes. They are taken from [3, 17] for $o(1,3)$, from [12] for $o(4)$ and from [18, 19] for $o(2,2)$. Complete lists for $o(1,3)$ may be found in [3] and for $o(2,2)$ in [1, 18]. Also, in Tables 1 and 3, $\overset{+}{S}$ denotes the Lie algebra $\overset{+}{S}_m$ (and similarly for $\overset{-}{S}$), and for neutral signature $\overset{+}{B}$ denotes the 2−dimensional subalgebra of $\overset{+}{S}$ spanned in some null basis by $l \wedge N$ and $l \wedge n - L \wedge N$ and similarly $\overset{-}{B}$ the 2−dimensional subalgebra of $\overset{-}{S}$ spanned by $l \wedge L$ and $l \wedge n + L \wedge N$. In Table 1, $\alpha, \beta \in \mathbb{R}$, for subalgebra $2(j)$ $\alpha\beta \neq 0$ and for $2(h)$ and $3(d)$, $\alpha \neq \pm\beta$, whereas in Table 2, $0 \neq \omega \in \mathbb{R}$. As a general clause, and given the existence of the metric g on M, tangent and cotangent spaces (together with their tensor equivalents) will often be identified.

Table 1: Subalgebras for $(+,+,-,-)$

Type	Dimension	Basis	Curvature Type
$1(a)$	1	$l \wedge n$	O, D
$1(b)$	1	$x \wedge y$	O, D
$1(c)$	1	$l \wedge y$ or $l \wedge s$	O, D
$1(d)$	1	$l \wedge L$	O, D
$2(a)$	2	$l \wedge n - L \wedge N, l \wedge N (= \overset{+}{B})$	O, D, A
$2(b)$	2	$l \wedge n,\ L \wedge N$	O, D, B
$2(c)$	2	$l \wedge n - L \wedge N,\ l \wedge L + n \wedge N$	O, B
$2(d)$	2	$l \wedge n - L \wedge N,\ l \wedge L$	O, D, B
$2(e)$	2	$x \wedge y,\ s \wedge t$	O, D, B
$2(f)$	2	$l \wedge N + n \wedge L,\ l \wedge L$	O, D, B
$2(g)$	2	$l \wedge N,\ l \wedge L$	O, D, C
$2(h)$	2	$l \wedge N,\ \alpha(l \wedge n) + \beta(L \wedge N)$	O,D,C ($\alpha\beta = 0$), O,D,A ($\alpha\beta \neq 0$)
$2(j)$	2	$l \wedge N,\ \alpha(l \wedge n - L \wedge N) + \beta(l \wedge L)$	O, D, A
$2(k)$	2	$l \wedge y,\ l \wedge n$ or $l \wedge s,\ l \wedge n$	O, D, C
$3(a)$	3	$l \wedge n,\ l \wedge N,\ L \wedge N$	Any
$3(b)$	3	$l \wedge n - L \wedge N,\ l \wedge N,\ l \wedge L$	Any
$3(c)$	3	$x \wedge y,\ x \wedge t,\ y \wedge t$ or $x \wedge s,\ x \wedge t,\ s \wedge t$	O, D, C
$3(d)$	3	$l \wedge N,\ l \wedge L,\ \alpha(l \wedge n) + \beta(L \wedge N)$	O,D,C ($\alpha = 0$),O,D,C,A ($\alpha \neq 0$)
$4(a)$	4	$\overset{+}{S},\ l \wedge n + L \wedge N$	Any
$4(b)$	4	$\overset{+}{S},\ l \wedge L + n \wedge N$	O, D, B, A
$4(c)$	4	$\overset{+}{B},\ \overset{-}{B} =< l \wedge L,\ l \wedge N,\ l \wedge n,\ L \wedge N >$	Any
5	5	$\overset{+}{S},\ \overset{-}{B}$	Any
6	6	$o(2,2)$	Any

2 Sectional curvature and the Weyl conformal tensor

It has been shown that if the sectional curvature function σ_m at each $m \in M$, arising from g and *Riem*, is given then, under a weak restriction (for the positive definite case [14]) and under the same restriction (but with some very special cases excluded for

Table 2: Subalgebras for $(+,+,+,-)$

Type	Dimension	Basis	Curvature Types
R_2	1	$l \wedge n$	O,D
R_3	1	$l \wedge x$	O,D
R_4	1	$x \wedge y$	O,D
R_6	2	$l \wedge n,\ l \wedge x$	O,D,C
R_7	2	$l \wedge n,\ x \wedge y$	O,D,B
R_8	2	$l \wedge x,\ l \wedge y$	O,D,C
R_9	3	$l \wedge n,\ l \wedge x,\ l \wedge y$	O,D,C,A
R_{10}	3	$l \wedge n,\ l \wedge x,\ n \wedge x$	O,D,C
R_{11}	3	$l \wedge x,\ l \wedge y,\ x \wedge y$	O,D,C
R_{12}	3	$l \wedge x,\ l \wedge y,\ l \wedge n + \omega(x \wedge y)$	O,D,C,A
R_{13}	3	$x \wedge y,\ y \wedge z,\ x \wedge z$	O,D,C
R_{14}	4	$l \wedge n,\ l \wedge x,\ l \wedge y,\ x \wedge y$	Any
R_{15}	6	$o(1,3)$	Any

Table 3: Subalgebras for $(+,+,+,+)$

Type	Dimension	Basis	Curvature Types
S_1	1	$x \wedge y$	O,D
S_2	2	$x \wedge y,\ z \wedge w$	O,D,B
S_3	3	$x \wedge y,\ x \wedge z,\ y \wedge z$	O,D,C
$\overset{+}{S}_3$	3	$\overset{+}{S}$	O,A
$\overset{+}{S}_4$	4	$\overset{+}{S},\ G\ (G \in \overset{-}{S})$	O,D,B,A
S_6	6	$o(4)$	Any

Lorentz [4, 16, 3] and neutral signatures [5]), the metric g can be uniquely recovered from it. The exceptional cases can be described.

It is also true that the Weyl conformal tensor, with components $C^a{}_{bcd}$ arising from g, if nowhere zero, uniquely determines the conformal class to which g belongs in the positive definite case [6]. This result can be merged with the other signatures and the result (for all signatures) is [7] that if the (necessarily closed) subset U of points of M at which the equation $C^a{}_{bcd}k^d = 0$ has a non-trivial solution for $k \in T_mM$ has empty interior in the manifold topology on M, C uniquely determines the conformal class of g. In the positive definite case this condition on U is equivalent to $\{m : C(m) = 0\}$ having empty interior in M (and is slightly weaker than that in the first sentence above). Thus for all three signatures one has a close relationship between sectional curvature and metric and between the Weyl conformal tensor and the conformal class of the metric.

3 The curvature map

Now consider a similar problem this time imposed on the curvature tensor *Riem*. This latter tensor gives rise, for each signature, to a linear map $f : \Lambda_m M \to \Lambda_m M$

called the curvature map and given by

$$f : F^{ab} \to R^{a}{}_{bcd}F^{cd} \tag{3.1}$$

The rank of f at $m \in M$ will be referred to as the curvature rank at m and the range space of f is denoted $rgf(m)$. It is $rgf(m)$ which will be important in what is to follow and this section will be devoted to a classification of $rgf(m)$ for all signatures. To do this it is first noted that $rgf(m)$ is a subspace of the *infinitesimal holonomy algebra* $\phi'(m)$ *of* ∇ at m which, in turn, is a subalgebra of the holonomy algebra ϕ of ∇ [13] (but rgf may not be a subalgebra of ϕ). Next let $\widetilde{rg}f(m)$ be the *smallest* (not necessarily holonomy) subalgebra of the appropriate orthogonal algebra containing $rgf(m)$ (the intersection of all the subalgebras containing $rgf(m)$). The reason for this construction will be made clear later. Then $\widetilde{rg}f(m)$ is a subalgebra of the holonomy algebra ϕ which may differ from ϕ and may not itself be a holonomy algebra; hence the need for care in Tables $1-3$ where the first column is the *actual holonomy algebra of* (M,g). If dim $rgf(m)=1$, then $\widetilde{rg}f(m)=rgf(m)$ and one may, using the metric $g(m)$, write $R_{abcd}=g_{ae}R^{e}{}_{bcd}=aF_{ab}F_{cd}$ at m for $0\neq a\in\mathbb{R}$ where F spans $rgf(m)$. Then the identity $R_{a[bcd]}=0$ (where square brackets denote the usual skew-symmetrisation of the indices enclosed) gives $F_{a[b}F_{cd]}=0$ which implies that F is *simple* and so $rgf(m)$ is spanned by a *simple* bivector. So only those 1-dimensional subalgebras of $o(4)$, $o(1,3)$ and $o(2,2)$ spanned by *simple* bivectors are retained in Tables $1-3$.

It turns out convenient to classify the map f at m into one of five mutually disjoint and exhaustive classes A, B, C and D and O (with the latter being the trivial case when $Riem(m)=0$) which is determined by $rgf(m)$ and referred to as the curvature class (of $Riem$ or of the curvature map f) at m. This classification applies to all signatures although it was given in a slightly different, but equivalent form for the Lorentz case in [3] and positive definite case in [12].

Class D. This arises when dim $rgf(m)=1$ with $rgf(m)$ being spanned by a (necessarily simple) bivector. In this case $rgf(m)=\widetilde{rg}f(m)$

Class C. This arises when there exists a *unique* (up to a scaling) $0\neq k\in T_mM$ such that $F_{ab}k^b=0$ for each $F\in rgf(m)$ (and k will be said to *annihilate* F).

Class B. This arises when $rgf(m)=<F,G>$ (where $<>$ denotes a spanning set) for independent $F,G\in\Lambda_mM$ with $[F,G]=0$ and where F and G have no common annihilator. Thus $rgf(m)=\widetilde{rg}f(m)$. By writing, in the positive definite and neutral cases, $F=\overset{+}{F}+\overset{-}{F}$ for unique members $\overset{+}{F}\in\overset{+}{S}_m$ and $\overset{-}{F}\in\overset{-}{S}_m$, and similarly for G, it can easily be shown that class B can be equivalently described by the ability to choose $rgf(m)=<F,G>$ with $F\in\overset{+}{S}_m$ and $G\in\overset{-}{S}_m$. In the Lorentz case, the subalgebra type R_7 is the unique (up to isomorphism) $2-$dimensional abelian subalgebra without a common annihilator and similarly for S_2 in the positive definite case.

Class A. This arises when $rgf(m)$ is not of class B, C, D or O.

The main idea here is to first consider the holonomy group of (M,g) with holonomy algebra ϕ listed in Tables $1-3$. Since, for $m\in M$, $rgf(m)$ is a subset (and $\widetilde{rg}f(m)$ a subalgebra) of ϕ (and recalling that $\widetilde{rg}f(m)$ may not be a holonomy algebra) one can, after some calculation, complete the fourth column in the tables. This is mostly known for positive definite and Lorentz signatures and so only the neutral case need be discussed. As will be seen below, it is the subalgebra $\widetilde{rgf}(m)$ which will turn out to be important here.

Thus in the neutral case, for class B, $rgf(m)$ is a $2-$dimensional abelian subalgebra of $o(2,2)$ *and* for which there does not exist a non-trivial common annihilator k for the members of $rgf(m)$. For class C, it is clear that dim $rgf(m) \geq 2$ and it is noted that if k annihilates bivectors $F, G \in rgf(m)$ it annihilates their Lie bracket $[F,\, G]$ and so k annihilates each member of the $2-$ or $3-$ dimensional subalgebra $< F,\, G,\, [F,\, G] >$. (That it is a subalgebra follows since any subspace of $\Lambda_m M$ consisting entirely of simple bivectors has dimension at most 3 [19].) Thus for class C, dim $rgf(m)$ equals 2 or 3 as also does dim$\widetilde{rgf}(m)$. For class D, there are exactly two independent members $k \in T_m M$ such that, if $rgf(m) =< F >$, k annihilates F. For class A, dim $rgf(m) \geq 2$ and there does not exist k which annihilates *every* $F \in rgf(m)$. This last result follows since, otherwise, if dim $rgf(m)$ equals 2 or 3 one would get class C whilst if dim $rgf(m) \geq 4$ a contradiction follows since, then, each member of $rgf(m)$ would be simple. Said a little differently, $f(m)$ is of class D if and only if $rgf(m)$ $(=\widetilde{rgf}(m))$ is one of the four $1-$dimensional subalgebras $1(a) - 1(d)$ of $o(2,2)$ spanned by a simple bivector and $f(m)$ is of class B if and only if $rgf(m)$ $(=\widetilde{rgf}(m))$ is one of the $2-$dimensional abelian subalgebras $2(b), 2(c), 2(d), 2(e)$ or $2(f)$ of $o(2,2)$. Class C applies to $f(m)$ if and only if the subalgebras $\widetilde{rgf}(m)$ arising are $2(g), 2(h)$ (with $\alpha\beta = 0$), $2(k), 3(c)$ and $3(d)(\alpha = 0)$. If $rgf(m)$ is none of those above it is of class A and $\widetilde{rgf}(m)$ is one of the subalgebras $2(a)$, $2(h)(\alpha\beta \neq 0)$, $2(j)$, $3(a)$, $3(b)$ and $3(d)(\alpha \neq 0)$, (plus one other $2-$dimensional, non-holonomy subalgebra labelled $2(l)$ $(=< l\wedge N, \alpha(l\wedge n - L\wedge N) + \beta(l\wedge L + n\wedge N) >$, $\alpha\beta \neq 0)$ in [10] and two other $3-$dimensional, non-holonomy subalgebras labelled $3(e)$ $(=< \overset{+}{S} >)$ and $3(f)$ $(=< \overset{+}{B}, l\wedge L + n\wedge N >)$ in [10]) together with those subalgebras of dimension ≥ 4 (which necessarily have no common annihilator and which include one $4-$dimensional, non-holonomy subalgebra labelled $4(d)$ $(=< \overset{+}{S}, l\wedge L >)$ in [10]).

The possibilities for the curvature class for each *holonomy algebra* for ∇ and signature are mostly obvious except, maybe, for the following remarks in the neutral case; (i) if the holonomy algebra is $2(c)$ there are no simple members and so classes C and D cannot occur, (ii) holonomy algebras $2(h)$ $(\alpha\beta \neq 0)$ and $2(j)$ are not abelian and have no common annihilator and so classes C and B cannot occur, (iii) holonomy algebra $3(d)$ $(\alpha = 0)$ has a common annihilator l, (iv) holonomy type $4(a)$ contains the members $l\wedge n$, $L\wedge N$ and $l\wedge N$ and so classes B and C are possible, (v) holonomy type $4(b)$ contains the members $x\wedge y$ and $s\wedge t$ and so class B is possible and, in fact, all classes except C are possible since if the latter is a possibility one has a $2-$ or $3-$dimensional subalgebra E each of whose members is simple and with a common annihilator. Then E is not contained in $\overset{+}{S}$ since $\overset{+}{S}$ (and $\overset{-}{S}$) has no such subalgebras. Thus if $F' = F + \lambda(l\wedge L + n\wedge N)$ and $G' = G + \mu(l\wedge L + n\wedge N)$ are independent simple (non-zero) members of E with $F, G \in \overset{+}{S}$ and real numbers λ and μ with $\lambda \neq 0 \neq \mu$, then $F' + \alpha G'$ is also simple for each $\alpha \in \mathbb{R}$. It follows (Section 1) that $P(F', \overset{*}{F'}) = P(G', \overset{*}{G'}) = P((F' + \alpha G'), (\overset{*}{F'} + \alpha \overset{*}{G'})) = 0$. Since $l\wedge L + n\wedge N \in \overset{-}{S}$, this gives $P(F,F) = 4\lambda^2 > 0$, $P(G,G) = 4\mu^2 > 0$

and $P(F,G) = 4\mu\lambda$ (and hence $(P(F,G))^2 = P(F,F)P(G,G)$). Now if F and G are proportional one achieves the contradiction that some linear combination of the bivectors F' and G' is a multiple of the non-simple bivector $l \wedge L + n \wedge N$. Otherwise $P((F+\alpha G),(F+\alpha G)) = 4(\lambda+\alpha\mu)^2 \geq 0$. This inequality and those above contradict the fact that P is a Lorentz metric on $\overset{+}{S}$ of signature $(-1,-1,1)$. Similar remarks deal with the absence of class C in the positive definite $\overset{+}{S}_4$ case. The classes for the other signatures are straightforward to calculate (see e.g. [3, 12]).

Now, for any signature of g, consider the following equation for $Riem(m)$ and $0 \neq k \in T_m M$

$$R^a{}_{bcd}k^d = 0 \tag{3.2}$$

Since $Riem(m)$ may be written out as symmetrised products of the bivectors spanning $rgf(m)$, it follows that k satisfies (3.2) if and only if it annihilates each $F \in rgf(m)$. Thus if $f(m)$ is class D there are exactly two independent solutions of (3.2) for k, for class C there is exactly one and for classes A and B there are none.

It is remarked here that if one imposes the *Ricci flat* condition $Ricc = 0$ on (M,g), where $Ricc$ denotes the Ricci tensor with components $R_{ab} = R^c{}_{acb}$, serious restrictions arise on the range spaces available for $f(m)$ on M. Under such conditions the Weyl conformal tensor equals $Riem$ and further details may be found in [3, 8]. For example, the curvature class D can only arise for neutral signature and then rgf is of type $1(d)$ [and the Weyl tensor is of type $(\mathbf{N},\mathbf{O})$ in the classification given in [2]]. Similar restrictions apply if one imposes the proper Einstein space condition on (M,g).

Now, for any signature, let A also denote the subset of M consisting of precisely those points m where $rgf(m)$ is of class A and similarly for subsets B, C, D and O. Then one has the *disjoint* decomposition $M = A \cup B \cup C \cup D \cup O$. In order to do calculus on the individual regions of such a decomposition one requires a decomposition in terms of their (open) topological interiors in M.

Lemma 3.1. *For any signature the manifold M may be* disjointly *decomposed as*

$$M = intA \cup intB \cup intC \cup intD \cup intO \cup Z \tag{3.3}$$

where int denotes the interior operator in the manifold topology of M. In this decomposition A, $A \cup B$, $A \cup B \cup C$ and $A \cup B \cup C \cup D$ are open in M (and so $A = intA$) and Z is a closed subset of M satisfying $intZ = \varnothing$.

Proof. For the Lorentz case see [3], chapters 9 and 12. The following proof covers all cases. First note that the subset $A \cup B$ is precisely the subset of M on which (3.2) has no non-trivial solutions. Let $m \in A \cup B$, let U be an open coordinate neighbourhood of m and consider the continuous map $h : U \times S^3 \to \mathbb{R}^q$ (for q some positive integer and for some ordering of the tensor components which arise) given by $h : (m', X) \to \frac{R_{abcd}X^d}{(\gamma(X,X))^{\frac{1}{2}}}$ for $m' \in U$ where γ is some *positive definite* metric on M (which exists since M is paracompact and could be chosen as g in the positive definite case) and X will be used to denote both a non-zero member of $T_m M$ and the signed direction in S^3 which it uniquely determines (since the map h is indifferent to this choice). Now for $0 \neq X' \in T_m M$, $h(m, X') \neq (0, ..., 0)$ (q times) and so there exists an open neighbourhood of (m, X') of the form $W \times W'$ with $W \subset U$ and $W' \subset S^3$ both open and with h nowhere zero on $W \times W'$. Applying this to each $X' \in S^3$ yields an open covering with sets like W' of the compact space S^3. On taking a finite subcover of this covering the associated first factors in the pairs $W \times W'$ above give a finite collection of open subsets of U each containing m whose intersection is an open neighbourhood $W'' \subset U$ of m and with h non-vanishing on $W'' \times S^3$. It follows that $W'' \subset (A \cup B)$ and so $A \cup B$ is open. Next let $m \in A$ and, since $A \cup B$ is open, choose an open neighbourhood U of m with $U \subset A \cup B$ (and hence $U \cap C = U \cap D = U \cap O = \varnothing$). If dim $rgf(m) \geq 3$ the rank theorem shows that $U \cap B = \varnothing$ and so $U \subset A$. If dim $rgf(m) = 2$ let $rgf(m) =< F, G >$ with $[F, G] \neq 0$ (since m is not in B). Then there exists an open neighbourhood U' of m on which smooth extensions F' and G' of F and G, respectively, exist, which are in rgf on U' and which are independent with $[F', G'] \neq 0$ at each point of U'. Thus $U' \subset A$ and it follows that A is open. The openness of $A \cup B \cup C$ and $A \cup B \cup C \cup D$ follow from a consideration of rank. Finally, let $U \subset Z$ be open. Then by the previous results and the disjointness of the decomposition $U \cap A = \varnothing$ and if $U \cap B (= U \cap (A \cup B)) \neq \varnothing$ it is open by the previous result and contradicts $A \cap \text{int} B = \varnothing$. Thus $U \cap B = \varnothing$ and, by definition, $U \cap \text{int} C = \varnothing$. Suppose $U \cap C \neq \varnothing$ and let $m \in U \cap C$. Then dim $rgf(m) \geq 2$ and so there exists an open neighbourhood W of m with $W \subset U$ ($\Rightarrow W \cap A = W \cap B = \varnothing$) with dim $rgf(m) \geq 2$ on W. So $W \cap D = W \cap O = \varnothing$. This implies that $W \subset C$ and hence that $W \cap \text{int} C \neq \varnothing$ and gives the contradiction that $U \cap \text{int} C \neq \varnothing$ (by disjointness since $U \subset Z$). So $U \cap C = \varnothing$. Similarly one shows that $U \cap D = \varnothing$ and so $U \subset O$ which gives the contradiction that $U \cap \text{int} O \neq \varnothing$. Thus $U = \varnothing$ and $\text{int} Z = \varnothing$ and this completes the proof. □

4 The determination of the metric

Retaining the notation above and with g of Lorentz signature, the equivalents of Lemma 4.1 and Theorem 4.2 below can be found in [3] whereas if g is of positive definite signature, these equivalents can be found in [12]. Now let g be of neutral signature and let g' be another smooth metric on M with the *same* curvature tensor *Riem* as g. Then $g'_{ae} R^e{}_{bcd} + g'_{be} R^e{}_{acd} = 0$ on M and so each member F of rgf at each $m \in M$ satisfies (and it is remarked that any index movement is done using the original metric g)

$$g'_{ae} F^e{}_b + g'_{be} F^e{}_a = 0 \tag{4.1}$$

This is just the statement that the bivectors F in $rgf(m)$ are also in the orthogonal algebra of g'. Now if $F, G \in rgf(m)$, so that they each satisfy (4.1), then it is easily checked that $[F, G]$ also satisfies (4.1) even if it is not in $rgf(m)$). Thus (4.1) holds for each member of $\widetilde{rgf}(m)$ (and it is recalled from Section 3 that if $k \in T_m M$ annihilates F and G, it annihilates $[F, G]$). In this sense, only subalgebras of $o(2, 2)$ need to be considered for examining the curvature type in what is to follow and explains the introduction of $\widetilde{rgf}(m)$, as promised earlier. The idea is then to consider (4.1) for $\widetilde{rgf}(m)$ at each $m \in M$ using the decomposition of Lemma 3.1, and for this the following Lemma is useful.

Lemma 4.1. (a) *If (4.1) holds at $m \in M$ and $F(m)$ is simple the blade of F is an eigenspace of the linear map associated with g' with respect to g at m (that is, $k^a \to g'^a{}_b k^b$ $(g'^a{}_b \equiv g^{ac} g'_{cb})$ for $k \in T_m M$).*

(b) *If (4.1) holds at $m \in M$ and V is the α-eigenspace ($\alpha \in \mathbb{C}$) of the linear map associated with F with respect to g then V is an invariant subspace of g' and, in particular, if k is a (real or complex) non-degenerate eigenvector of F (that is, $< k >$ is a $1-$dimensional eigenspace of F) at m then k is an eigenvector of g' with respect to g at m. This result also follows if F and g' are interchanged.*

(c) *If (4.1) holds at m for $F = l \wedge n - L \wedge N$, then $l \wedge N$ and $n \wedge L$ are invariant subspaces for g' with respect to g at m.*

Proof. The proof for (a) is the same as in [3] (chapter 9) even though this latter proof is for Lorentz signature. For (b) the proof consists of assuming k is in the α-eigenspace of F, that is, $F^a{}_b k^b = \alpha k^a$ at m and then contracting (4.1) with k^a. If one defines a 1-form p by $p_a \equiv g'_{ab} k^b$, then one can see that the vector P given by $P^a \equiv g^{ab} p_b$ satisfies $F^a{}_b P^b = \alpha P^a$ and is hence in the α-eigenspace of F. Thus $g'^a{}_b k^b (= g^{ac} g'_{cb} k^b) = P^a$ and the result follows. The result (c) now follows from (b) since $l \wedge N$ and $n \wedge L$ are eigenspaces of F (cf. [19]). □

For $m \in A$ and for the subalgebras in Table 1, it follows from a similar proof in [9] (which included only subalgebras of $o(2,2)$ giving holonomy algebras), that the only solution of (4.1) is $g' = cg$ $(0 \neq c \in \mathbb{R})$. However, this result also follows for the subalgebras $3(e)$, $3(f)$ and $4(d)$ (not given in Table 1—see section 3) and all subalgebras of dimension ≥ 4 (including type $4(d)$), since they each contain a subalgebra isomorphic to $2(a)$. It also follows for the subalgebra $2(l)$ (also not given in Table 1—see section 3). To see this note that Lemma 4.1(a) shows that $l \wedge N$ is an eigenspace of g' with respect to g. Then with $F = \alpha(l \wedge n - L \wedge N) + \beta(l \wedge L + n \wedge N)$, $l \pm iN$ and $n \pm iL$ are non-degenerate eigenvectors of F and hence from Lemma 4.1(b), eigenvectors of g' with respect to g. It then easily follows that l, n, L, N are each eigenvectors of g' with the same eigenvalue and so g' is a multiple of g.

If $m \in M \setminus (A \cup O)$ one can similarly find an algebraic expression for g' in terms of g and the geometry of $rgf(m)$, but they are more complicated. For example, for $m \in C$ the members of $rgf(m)$ are all simple and the algebraic determination of g' is straightforward (Lemma 4.1(a)). In fact one gets $g'_{ab} = \alpha g_{ab} + \beta k_a k_b$ at m, where $k \in T_m M$ represents the common annihilator and which may be spacelike, timelike or null, and $\alpha, \beta \in \mathbb{R}$. For the open subset A of M, one has the following result.

Theorem 4.2. *Suppose $dim M = 4$ and g and g' are two metrics on M of arbitrary signature and which have the same tensor Riem. Then $g' = cg\,(0 \neq c \in \mathbb{R})$ on each component of the open subset A of M (c being, possibly, component dependent). In particular, if the subset A is open and dense in M, $g' = cg$ $(0 \neq c \in \mathbb{R})$ on M and g and g' have the same Levi-Civita connection on M.*

Proof. This mostly follows from the above work. On the open region A, g and g' are smooth and conformally related and so $g' = \phi g$, for some smooth, nowhere-zero, real-valued function on each component of A (ϕ being, possibly, component dependent). Now we use the Bianchi identities derived from the respective Levi-Civita connections ∇ and ∇' of g and g'. With a semi-colon and a vertical stroke denoting, respectively, a covariant derivative with respect to ∇ and ∇' on any coordinate neighbourhood in A, they are

$$R^a{}_{bcd;a} + R_{bc;d} - R_{bd;c} = 0 \qquad R^a{}_{bcd|a} + R_{bc|d} - R_{bd|c} = 0, \tag{4.2}$$

where $R_{ab} \equiv R^c{}_{acb}$ are the components of the *common* Ricci tensors of g and g'. The relation between the Christoffel symbols Γ^a_{bc} of ∇ and Γ'^a_{bc} of ∇' is easily computed and is

$$\Gamma'^a_{bc} - \Gamma^a_{bc} = \frac{1}{2\phi}(\phi_{,c}\delta^a_b + \phi_{,b}\delta^a_c - \phi^a g_{bc}), \tag{4.3}$$

where a comma denotes a partial derivative and $\phi^a = g^{ab}\phi_{,b}$. The remainder of the proof consists of subtracting the equations in (4.2) (to remove the partial derivatives) and performing some judicious contractions (which can be found in [3]) to achieve the result $R^a{}_{bcd}\phi^d = 0$. By recalling the results following (3.2), it follows that $\phi_{,a} = 0$ in any coordinate neighbourhood in A, and the first result follows. If A is (open and) dense in M, g' and g are conformally related on A, and hence on M with a smooth conformal factor with vanishing derivative on A and hence on M. Since M is connected, this conformal factor is constant on M. It follows that $\nabla = \nabla'$ on M (and so $\nabla g' = 0$). □

It is remarked that for the region $M \setminus A$, the comments before the statement of Theorem 4.2 show the more complicated relations between g and g' there and hence that g and g' may have different signatures (for example, if $M = D$ or $M = C$).

A comparison of the above results with those arising from the study of recurrence given for each of the three signatures in [3, 9] is worthwhile. In particular, the above results are quite different from these obtained in recurrence theory, since in the latter study a fixed metric g and its Levi-Civita connection ∇ were assumed and solutions to (amongst others) the equation $\nabla h = 0$ sought for a second order, symmetric tensor h. Amongst these results it is shown that, starting from the original g and ∇, if the holonomy algebra arising from ∇ has dimension ≥ 4, then the only solutions for h to $\nabla h = 0$ for a second order, non-degenerate, symmetric tensor h is when h is a (non-zero) constant multiple of g. Thus in these cases the connection uniquely determines the metric up to a *constant* conformal factor. If one of the other holonomy types occurs, the solutions for h can still be found and the (non-degenerate) solutions amongst them generate all the alternative metrics compatible with ∇. In Theorem 4.2, however, the Levi-Civita connection ∇' of g' is *not* assumed equal to the Levi-Civita connection ∇ of g, but is proved equal to it if $M = A$ and if g and g' have the same tensor *Riem*.

5 Remarks and examples

There are a number of issues arising between the holonomy group, the space rgf, the curvature tensor, the holonomy algebra ϕ and the infinitesimal holonomy algebra ϕ'. In fact, independently of signature, given $m \in M$, no two of $rgf(m)$, $\phi'(m)$ and ϕ need be equal. In addition one has the powerful Ambrose-Singer theorem which supplies the holonomy algebra if *Riem* and the parallel transport map are known on M. Of course, examples of the non-equality of $rgf(m)$ for m in some subset $U \subset M$, and ϕ can be constructed by a judicious (smooth) joining of U with the rest of M. But non-trivial, simpler examples also exist for metrics of any of the three signatures (and are easily generalised, if $\dim M > 4$, by taking suitable products) where $\dim rgf(m) = 1$ for all $m \in M$ (so that $M = D$) but where ϕ is three dimensional. They were studied in connection with projective relatedness and can be found in [11, 3, 19, 12].

The connection ∇ of the original metric g does not necessarily determine the metric (up to a constant conformal factor) from which it came and, in fact, may not even determine the signature of such a metric. From the viewpoint of the present paper one may have metrics g and g' on M with the same tensor *Riem* but with distinct Levi-Civita connections (but not in some open subset of A as Theorem 1 shows). As an example of this latter feature consider the metric g on $M = \mathbb{R}^4$ with a (connected) global coordinate domain u, v, y, s and given by (cf [3])

$$H(u, y, s)du^2 + 2dudv + dy^2 - ds^2 \tag{5.1}$$

for some nowhere-zero function H. This metric has neutral signature and on the open subset of M where $\partial^2 H/\partial y^2 . \partial^2 H/\partial s^2 - (\partial^2 H/\partial y \partial s)^2 \neq 0$ it has curvature rank 2, curvature class C and at each m, $rgf(m)$ is of type $2(g)$. The vector field $l^a = g^{ab}u_{,b}$ spans the unique common annihilator of the members of rgf and is null and parallel. Now let $g'_{ab} = g_{ab} + \lambda(u)u_{,a}u_{,b}$ for some nowhere-zero function λ so that g' is non-degenerate and also has neutral signature. Then g and g' can be checked to have the same tensor *Riem* but $\nabla' \neq \nabla$ provided λ is not a constant function. [If one assumes, in addition, that $\partial^2 H/\partial y^2 = \partial^2 H/\partial s^2$ in (5.1), $Ricc \equiv 0$ and g and g' have the same Weyl conformal tensor but are not conformally related. One can then calculate in this case that the Weyl tensor satisfies $C^a{}_{bcd}k^d = 0$: see Section 2.] A particular example of this type is the neutral signature analogue of the plane wave metric in general relativity and arises, for example, when $H = a(u)y^2 + b(u)s^2 + c(u)ys$ for functions a, b and c on the open subset where $ab - \frac{c^2}{4} \neq 0$ (and its conformally flat special case when $a = -b$ and $c = 0$). For this example the Weyl tensor types, in the notation of [2], are $(\mathbf{N}, \mathbf{N})$ (for regions where $(a+b)^2 \neq c^2$), $(\mathbf{N}, \mathbf{O})$ (for regions where $(a+b)^2 = c^2 \neq 0$) and $(\mathbf{O}, \mathbf{O})$ (the conformally flat case for regions where $a = -b$ and $c = 0$). These examples may be converted to Lorentz signature by changing the sign in the last term in (5.1) (see [3]) where similar results are obtained. [In fact, quite generally, for any Lorentz or neutral signature metric g which admits a parallel null vector field l the metric $g'_{ab} = g_{ab} + \lambda(u)u_{,a}u_{,b}$ has the same signature as g and the same tensor *Riem* but whose connection differs from that of g if λ is not a constant function. In addition, if one of g and g' is Ricci flat the other is and the Weyl tensors of g and g' agree.]

As another example consider the metric

$$\epsilon_1 dt^2 + dx^2 + x^2 dy^2 + \epsilon_2 (x + t)^2 ds^2 \tag{5.2}$$

where M is that open submanifold of $\mathbb{R}^4$ given in the global coordinate system $-\infty < y, s < \infty$ and $0 < x, t < \infty$. This metric is positive definite if $\epsilon_1 = \epsilon_2 = 1$, Lorentz if $-\epsilon_1 = \epsilon_2 = 1$ and neutral if $\epsilon_1 = \epsilon_2 = -1$. In the neutral case one may compute with Maple that the curvature class is D and that $R_{abcd} = AH_{ab}H_{cd}$ for some function $A : M \to \mathbb{R}$ where $H = L \wedge N$ in some global null basis l, n, L, N and in the language of

Section 1. Thus, in this notation, the global null vector fields l and n satisfy $R_{abcd}l^d = R_{abcd}n^d = 0$ and hence $R_{ab}l^b = R_{ab}n^b = 0$. Further, the Ricci tensor has Jordan-Segre type $\{(11)(11)\}$ with L and N also spanning a Ricci eigenspace. From the expression for the Weyl conformal tensor with components C_{abcd} one then finds $C_{abcd}l^bl^d = -\frac{R}{6}l_al_c$ and $C_{abcd}n^bn^d = -\frac{R}{6}n_an_c$ where R is the Ricci scalar, $R = R_{ab}g^{ab}$. A similar calculation then shows that $C_{abcd}L^bL^d = AL_aL_c$ and $C_{abcd}N^bN^d = AN_aN_c$ for some function A. Thus l, n, L, N are *repeated principal null directions* (repeated pnds) for C in the classification of this latter tensor [2]. The Weyl tensor, considered as a 6×6 matrix in the usual way, has (maximum) rank 6 everywhere. Now C may admit at most four repeated pnds and this, only when the algebraic type is $(\mathbf{D_1}, \mathbf{D_1})$, in the notation of [2], and so this latter is the algebraic type of C on M.

To examine the metric more closely consider the possibility of parallel or properly recurrent vector fields on M (see, for example, [9] for definitions, etc). Any parallel vector field k satisfies $R_{abcd}k^d = 0$ from the Ricci identity and any recurrent *non-null* vector field may be scaled so that it is parallel. Any parallel vector field must therefore lie everywhere in the span of $\partial/\partial x$ and $\partial/\partial t$ and it is then easily computed that there are no parallel or recurrent vector fields in this case. Thus any *properly recurrent* vector field k is *null* and satisfies $k_{a;b} = k_ap_b$ for some $1-$form p. A differentiation of this last equation and use again of the Ricci identity shows that $R_{abcd}k^bk^d = Bk_ak_c$ and $R_{ab}k^b = -Bk_a$ for some function B. It follows, as before, that k is a repeated pnd of C and hence is proportional to either L or N. But it can then be computed that neither L nor N are properly recurrent for (5.2). Thus no recurrent vector fields are admitted by the metric (5.2). The holonomy algebra ϕ may now be calculated from Table 1 by first noting that no recurrent vector fields are allowed and second that rg$f =< L \wedge N >$ must be a subalgebra of ϕ and hence that ϕ must have a subalgebra of type $1a$ (and this latter condition rules out $2(c)$, $2(e)$, $2(f)$ and $4(b)$). Thus only types $4(a)$, the $5-$dimensional subalgebra and $o(2,2)$ remain.

Finally, rgf fixes the null vectors L and N at each point and one may supplement these with null vectors l and n to give a null tetrad, l, n, L, N on some connected, open neighbourhood of any $m \in M$. Now suppose that (5.2) admits a (local) *totally null recurrent* bivector field F on (a possible reduced version of) U, so that $F(m) \in \overset{+}{S}_m$ or $F(m) \in \overset{-}{S}_m$ for each $m \in U$. Thus F is simple and it is easily checked that if $F = P \wedge Q$ for orthogonal, null vector fields P and Q on U then F is recurrent on U if and only if $P_{a;b} = P_ar_b + Q_ar'_b$ for $1-$forms r and r' and with a similar obvious expression for Q, on U. Now since F is recurrent on U the Ricci identity for F and the above expression for *Riem* shows, after an obvious contraction to expose H from rgf, that F and the bivector H spanning rgf satisfy $[F, H]$ is a multiple of F on U. Since $H = L \wedge N$, this can only happen if (for $F \in \overset{+}{S}_m$) $F = l \wedge N$ or $F = n \wedge L$, or ($F \in \overset{-}{S}_m$) if $F = l \wedge L$ or $F = n \wedge N$ on U, in the above basis. This follows by writing $2L \wedge N = (l \wedge n + L \wedge N) - (l \wedge n - L \wedge N)$ and using the fact that $[F, H]$ is a multiple of F. However, it is easily checked that neither of these bivectors is recurrent for (5.2). Thus (5.2) admits no totally null recurrent bivectors and hence cannot have holonomy type $4a$ or be $5-$dimensional since each of these admit a totally null, recurrent bivector. (In fact, $4a$ admits a pair of recurrent totally null bivectors whilst the $5-$dimensional case is characterised by admitting only one such bivector.) It follows that the holonomy algebra for (5.2) is $o(2,2)$. This metric thus has dim $rgf(m) = 1$ at each $m \in M$ yet its holonomy algebra is the most general possible. Essentially identical results apply to the Lorentz metric option in (5.2). The calculation is rather similar, in fact slightly easier, since now at most two repeated principal null directions for C are possible at any point (since C is nowhere zero) and the number of holonomy types is less. The curvature rank is 1 with curvature class D and the holonomy type is again the most general one (labelled R_{15} in Table 2; see [17, 3]). In the well-known Petrov classification of the Weyl conformal tensor for this signature [15], C is of type **D**. Again, with the positive definite option in (5.2) almost identical results can be obtained, and are much easier to achieve. In this case [12] is helpful.

One final theorem may be given and which involves certain types of symmetries on (M, g). A curvature collineation on M is a global, *smooth* vector field X satisfying $\mathcal{L}_X Riem = 0$ and a Weyl collineation on M is a global, *smooth* vector field X satisfying $\mathcal{L}_X C = 0$. The collection of such vector fields is, in each case, a Lie algebra which, in general, is infinite-dimensional as is easily shown by modifying the Lorentz examples in [3]. A consideration of the local flows associated with such vector fields then easily leads to the following theorem, the first of which is a consequence of Theorem 4.2, and the second of the work in Section 2.

Theorem 5.1. (i) *Let M be a $4-$dimensional, smooth, connected, paracompact, Hausdorf manifold admitting a smooth metric g of any signature and suppose $M = A$. Then the Lie algebra of curvature collineations on M equals the Lie algebra of homothetic vector fields on M and is hence finite-dimensional.*

(ii) *Let M be a $4-$dimensional, smooth, connected, paracompact, Hausdorf manifold admitting a smooth metric g of any signature and suppose that at no $m \in M$ is there a non-trivial solution for $k \in T_mM$ to $C^a{}_{bcd}k^d = 0$. Then the Lie algebra of Weyl collineations on M equals the Lie algebra of conformal vector fields on M and is hence finite-dimensional.*

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