The new Minkowski norm and integral formulas for a manifold endowed with a set of one-forms

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Abstract. Integral formulas are the power tool for obtaining global results in Analysis and Geometry. We explore the problem: *Find integral formulas for a closed manifold endowed with a set of linearly independent 1-forms (or vector fields).* In our recent works in common with P. Walczak, the problem was examined for a manifold endowed with a codimension-one foliation and a 1-form β, using approach of Randers norm. Continuing this study, we introduce new Minkowski norm, determined by Euclidean norm α, linearly independent 1-forms $\beta_i,\ (1 \le i \le p)$ and a function ϕ of p variables; this produces a new class of "computable" Finsler metrics generalizing Matsumoto's (α, β)-metric. The geometrical meaning of our Minkowski norm is that its indicatrix is a rotation hypersurface with the axis $\bigcap_{i=1}^{p} \ker \beta_i$ passing through the origin. We explore a Riemannian structure, naturally arising from this norm and a codimension-one distribution $\ker \omega$ of 1-form $\omega \ne 0$, and find the second fundamental form of $\ker \omega$ through invariants of α, ω, β_i and ϕ. Then we apply the above to prove new integral formulas for a closed Riemannian manifold endowed with a codimension-one distribution and linearly independent 1-forms $\beta_i,\ (1 \le i \le p)$, which generalize the Reeb's integral formula and its counterpart for the second mean curvature of the distribution.

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Integral formulas are the power tool for obtaining global results in Analysis and Geometry (e.g. generalized Gauss-Bonnet theorem and Minkowski-type formulas for submanifolds). Such formulas are usually proved applying the Divergence theorem to appropriate vector field. The first known integral formula by G. Reeb [10], for a closed Riemannian manifold (M, a) endowed with a 1-form $\omega \ne 0$ tells us that the total mean curvature H of the distribution $\ker \omega$ vanishes:

$$\int_M H \ \mathrm{d}\,\mathrm{vol}_a = 0; \tag{0.1}$$

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thus, either $H \equiv 0$ or $H(x)H(x') < 0$ for some points $x \neq x'$. Its counterpart (6.1) for the second mean curvature of a codimension one foliation (see [9]) has been used to estimate the energy of a vector field [3] and to prove that codimension-one foliations with negative Ricci curvature are far from being totally umbilical [6]. Recently, these were extended into infinite series of integral formulas including the higher order mean curvatures of the leaves and curvature tensor, see [1, 7, 11]. The integral formulas for foliations can be used for prescribing the mean curvatures of the leaves, e.g. characterizing totally geodesic, totally umbilical and Riemannian foliations.

We explore the **problem**: *Find integral formulas for a closed Riemannian manifold endowed with a set of linearly independent 1-forms (or vector fields).* The "maximal number of pointwise linearly independent vector fields on a closed manifold" is an important topological invariant; such vector fields on a sphere S^l are built using orthogonal multiplications on $\mathbb{R}^{l+1}$.

In [12, 13], the problem was examined for (M, a) endowed with 1-forms $\omega \neq 0$ and β, using approach of Randers norm, that is a Euclidean norm α shifted by a small vector. In the paper we extend this approach for (M, a) with the codimension-one distribution $\ker\omega$ and p linearly independent 1-forms $\beta_1, \ldots, \beta_p$, by introducing new Minkowski norm, generalizing (α, β)-norm of M. Matsumoto, see [8]. Remark that navigation (α, β)-norms appear when $p = 2$. The (α, β)-metrics form a rich class of computable Finsler metrics and play an important role in geometry, see [2, 8, 14, 17], thus we expect that our so called $(\alpha, \vec{\boldsymbol{\beta}})$-metrics will also find many applications.

The paper contains an introduction and six sections. In Section 1 we introduce and explore the $(\alpha, \vec{\boldsymbol{\beta}})$-norm, determined by Euclidean norm α, linearly independent 1-forms $\beta_1, \ldots, \beta_p$ and a function ϕ of p variables; the indicatrix is a rotational hypersurface with p-dimensional rotation axis. The norm produces a class of "computable" Finsler metrics generalizing Matsumoto's (α, β)-metric. In Sections 2–4 we study a new Riemannian structure, naturally arising on M endowed with $(\alpha, \vec{\boldsymbol{\beta}})$-metric with $\vec{\boldsymbol{\beta}} = (\beta_1, \ldots, \beta_p)$ and 1-form $\omega \neq 0$, and calculate the second fundamental form of the distribution $\ker\omega$ through invariants of α, ω, β_i and ϕ. Sections 5–6 contain applications to proving new integral formulas for a closed M endowed with a codimension-one distribution $\ker\omega$ and a set of linearly independent 1-forms, which generalize the Reeb's formula (0.1) and its counterpart for the second mean curvature of the distribution. Using our norm and assuming for simplicity $p = 1$, we get new estimates of the "non-umbilicity" of a codimension-one distribution and the energy of a vector field.

1 The $(\alpha, \vec{\boldsymbol{\beta}})$-norm

In this section, we define a new Minkowski norm, generalizing the (α, β)-norm of M. Matsumoto.

A *Minkowski norm* on a vector space V^{m+1} $(m \geq 1)$ is a function $F : V \to [0, \infty)$ with the properties of regularity, positive 1-homogeneity and strong convexity [14]:

$\mathrm{M}_1 : F \in C^\infty(V \setminus \{0\})$, $\quad \mathrm{M}_2 : F(\lambda\, y) = \lambda F(y)$ for $\lambda > 0$ and $y \in V$,

M_3 : For any $y \in V \setminus \{0\}$, the following symmetric bilinear form is *positive definite*:

$$g_y(u, v) = \frac{1}{2}\,\frac{\partial^2}{\partial s\,\partial t}\left[F^2(y + su + tv)\right]_{|\,s=t=0}. \tag{1.1}$$

By M_2–M_3, $g_{\lambda y} = g_y$ ($\lambda > 0$) and $g_y(y,y) = F^2(y)$. As a result of M_3, the indicatrix $S := \{y \in V : F(y) = 1\}$ is a closed, convex smooth hypersurface that surrounds the origin.

The following symmetric trilinear form is called the *Cartan torsion* for F:

$$C_y(u,v,w) = \frac{1}{4}\,\frac{\partial^3}{\partial r\,\partial s\,\partial t}\left[F^2(y + ru + sv + tw)\right]_{|\,r=s=t=0}, \tag{1.2}$$

where $y,u,v,w \in V$ and $y \neq 0$. Note that $C_y(u,v,y) = 0$ and $C_{\lambda y} = \lambda^{-1} C_y$ for $\lambda > 0$. Vanishing of a 1-form $I_y(u) = \mathrm{Tr}_{g_y} C_y(u,\cdot\,,\cdot)$, called the *mean Cartan torsion*, characterizes Euclidean norms among all Minkowski norms, see e.g. [14].

Definition 1.1. Given $p \in \mathbb{N}$ and $\delta_i > 0$ $(1 \le i \le p)$, let $\phi : \Pi \to (0,\infty)$ be a smooth function on $\Pi = \prod_{i=1}^{p}[-\delta_i, \delta_i]$, and $a(\cdot,\cdot) = \langle\cdot\,,\cdot\rangle$ a scalar product with the Euclidean norm $\alpha(y) = \langle y, y\rangle^{1/2}$ on a $(m+1)$-dimensional vector space V. Given linearly independent 1-forms β_i $(1 \le i \le p)$ on V of the norm $\alpha(\beta_i) < \delta_i$, the $(\alpha, \vec{\boldsymbol\beta})$-*norm* (see below Lemma 1.3 on regularity) with $\vec{\boldsymbol\beta} = (\beta_1, \ldots, \beta_p)$ is defined on $V \setminus \{0\}$ by

$$F(y) = \alpha(y)\,\phi(s), \quad s = (s_1, \ldots, s_p), \quad s_i = \beta_i(y)/\alpha(y). \tag{1.3}$$

Usually, we assume $\phi(0,\ldots,0) = 1$. We call α the *associated norm* (or metric).

The geometrical meaning of (1.3) is that the indicatrix of F is a rotation hypersurface in V with the axis $\bigcap_{i=1}^{p} \ker \beta_i$ passing through the origin, see below Proposition 1.1. For $p = 1$, (1.3) defines the (α,β)-norm. By shifting the indicatrix of an (α,β)-norm, we obtain new Minkowski norms, called *navigation* (α,β)-*norms*, [17]. The indicatrix of this norm is still a rotation hypersurface, but the rotation axis does not pass the origin in general. Meanwhile, this is a case of $(\alpha, \vec{\boldsymbol\beta})$-norm with $p = 2$, whose indicatrix has a two-dimensional rotation axis passing through the origin.

The "musical isomorphisms" $\sharp$ and $\flat$ will be used for rank one and symmetric rank 2 tensors. For example, $\langle \beta_i^\sharp, u\rangle = \beta_i(u) = u^\flat(\beta_i^\sharp)$. We will use Einstein summation convention. Set

$$b_{ij} = \langle \beta_i, \beta_j\rangle = \langle \beta_i^\sharp, \beta_j^\sharp\rangle.$$

A Minkowski norm on V^{m+1} is Euclidean if and only if it is preserved under the action of $O(m+1)$. Next, we will clarify the geometric property about the indicatrices of $(\alpha, \vec{\boldsymbol\beta})$-metrics.

Definition 1.2 (The symmetry of a Minkowski norm, see [17])**.** Let F be a Minkowski norm on V^{m+1} and G a subgroup of $GL(m+1,\mathbb{R})$. Then F is called G-*invariant* if the following holds for some affine coordinates $(y^1, \ldots, y^{m+1})$ of V:

$$F(y^1, \ldots, y^{m+1}) = F((y^1, \ldots, y^{m+1})f), \quad y \in V,\ f \in G. \tag{1.4}$$

The next proposition for $p = 1$ belongs to [17].

Proposition 1.1. *Let F be a Minkowski norm and β_i $(1 \le i \le p)$ linearly independent 1-forms on a vector space V^{m+1}. Then F is an $(\alpha, \vec{\boldsymbol\beta})$-norm with $\vec{\boldsymbol\beta} = (\beta_1, \ldots, \beta_p)$ if and only if F is G-invariant, where $G = \{x \in GL(m+1,\mathbb{R}) : x = \begin{pmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathrm{id}_p \end{pmatrix},\ C \in GL(m-p+1,\mathbb{R})\,\}$.*

Proof. Let $F=\alpha\,\phi(\frac{\beta_1}{\alpha},\ldots,\frac{\beta_p}{\alpha})$ be the $(\alpha,\vec{\boldsymbol\beta})$-norm. Let $\{e_1,\ldots,e_{m+1}\}$ be an $\langle\cdot\,,\cdot\rangle$-orthonormal basis such that $\bigcap_{i=1}^p\ker\beta_i=\operatorname{span}\{e_1,\ldots,e_{m-p+1}\}$. Then $\beta_i(y)=\sum_{j=m-p+2}^{m+1}\beta_i(e_j)y^j$ where

$$F(y)=\sqrt{(y^1)^2+\ldots+(y^{m+1})^2}\,\phi\Big(\frac{\sum_{j=m-p+2}^{m+1}\beta_1(e_j)y^j}{\sqrt{(y^1)^2+\ldots+(y^{m+1})^2}},\ldots,\frac{\sum_{j=m-p+2}^{m+1}\beta_p(e_j)y^j}{\sqrt{(y^1)^2+\ldots+(y^{m+1})^2}}\Big)$$

and $y=y^ie_i$. Hence, F is G-invariant.

Conversely, let F obey (1.4) for G and affine coordinates $y=(y^1,\ldots,y^{m+1})$. If $p=m+1$ then for $G=\{\mathrm{id}_{m+1}\}$ one may take $\beta_i=e_i^\flat$ and use axiom M_2. Let $p\le m$. By restricting F on the $(m-p+1)$-dimensional linear subspace U given by p equations $y^{m-p+2}=\ldots=y^{m+1}=0$, one obtains an $O(m-p+1)$-invariant Minkowski norm, which must be Euclidean. Thus, there exists $B>0$, such that the norm $\alpha(y)=B\sqrt{(y^1)^2+\ldots+(y^{m+1})^2}$ on V obeys $\alpha|_U=F|_U$. Set

$$\tilde\phi(y)=F(y)/\alpha(y)\quad(y\ne0).$$

Then $\tilde\phi$ is G-invariant, hence $\tilde\phi$ depends on p variables $y^{m-p+2},\ldots,y^{m+1}$ only. Since $\tilde\phi$ is 0-homogeneous, we have $\tilde\phi(y)=\tilde\phi(By^{m-p+2}/\alpha(y),\ldots,By^{m+1}/\alpha(y))$, that is $\beta_i=Be^\flat_{m-p+1+i}$. □

Define real functions ρ, ρ_0^{ij},ρ_1^i $(1\le i,j\le p)$ of variables $s=(s_1,\ldots,s_p)$, see also (1.3):

$$\rho=\phi\big(\phi-\sum\nolimits_i s_i\dot\phi_i\big),\quad \rho_0^{ij}=\phi\,\ddot\phi_{ij}+\dot\phi_i\dot\phi_j,\quad \rho_1^i=\phi\,\dot\phi_i-\sum\nolimits_j s_j\big(\phi\,\ddot\phi_{ij}+\dot\phi_i\,\dot\phi_j\big),$$

where $\dot\phi_i=\frac{\partial\phi}{\partial s_i}$, $\ddot\phi_{ij}=\frac{\partial^2\phi}{\partial s_i\partial s_j}$, etc. Assume in the paper that $\rho>0$, thus

$$\phi-\sum\nolimits_i s_i\dot\phi_i>0.$$

The following relations hold:

$$\dot\rho_i=\rho_1^i,\quad \ddot\rho_{ij}=(\rho_1^i)'_j=-s_k(\rho_0^{ik})'_j.$$

Proposition 1.2. *For $(\alpha,\vec{\boldsymbol\beta})$-norm, the bilinear form g_y $(y\ne0)$ in* (1.1) *is given by*

$$\begin{aligned} g_y(u,v)\ &=\ \rho\langle u,v\rangle+\rho_0^{ij}\beta_i(u)\beta_j(v)\\ &+\ \rho_1^i(\beta_i(u)\langle y,v\rangle+\beta_i(v)\langle y,u\rangle)/\alpha(y)-\beta_i(y)\rho_1^i\langle y,u\rangle\langle y,v\rangle/\alpha^3(y). \end{aligned}\tag{1.5}$$

The Cartan tensor of $(\alpha,\vec{\boldsymbol\beta})$-norm is expressed by

$$\begin{aligned} 2\,C_y(u,v,w)\ &=\ \alpha^{-1}(y)\sum\nolimits_i\rho_1^i\big(K_y(u,v)p_{yi}(w)+K_y(v,w)p_{yi}(u)+K_y(w,u)p_{yi}(v)\big)\\ &+\ \alpha^{-1}(y)\sum\nolimits_{i,j,k}(\dot\phi_i\ddot\phi_{jk}+\dot\phi_j\ddot\phi_{ik}+\dot\phi_k\ddot\phi_{ij}+\phi\,\dddot\phi_{ijk})\,p_{yi}(u)p_{yj}(v)p_{yk}(w), \end{aligned}\tag{1.6}$$

where $p_{yi}=\beta_i-s_iy^\flat/\alpha(y)$ $(1\le i\le p)$ are 1-forms and $K_y(u,v)=\langle u,v\rangle-\langle y,u\rangle\langle y,v\rangle/\alpha^2(y)$ is the angular metric *of the associated metric $a=\langle\cdot,\cdot\rangle$.*

Proof. From (1.1) and (1.3) we find

$$g_y(u,v) \;=\; [F^2/2]_\alpha K_y(u,v)/\alpha(y) + [F^2/2]_{\alpha\alpha}\langle y,u\rangle\langle y,v\rangle/\alpha^2(y)$$
$$+\sum\nolimits_i([F^2/2]_{\alpha\beta_i}/\alpha(y))\big(\langle y,u\rangle\beta_i(v)+\langle y,v\rangle\beta_i(u)\big)+\sum\nolimits_{i,j}[F^2/2]_{\beta_i\beta_j}\beta_i(u)\beta_j(v). \tag{1.7}$$

Calculating derivatives of $\frac12 F^2=\frac12\,\alpha^2\phi^2(\beta_1/\alpha,\dots,\beta_p/\alpha)$,

$$[F^2/2]_\alpha=\alpha\rho,\quad [F^2/2]_{\beta_i}=\alpha\phi\dot\phi_i,\quad [F^2/2]_{\alpha\beta_i}=\rho^i_1,\quad [F^2/2]_{\beta_i\beta_j}=\rho_0^{ij},$$
$$[F^2/2]_{\alpha\alpha}=\rho+(\sum\nolimits_i s_i\,\dot\phi_i)^2+\phi\sum\nolimits_{i,j}s_is_j\,\ddot\phi_{ij} \tag{1.8}$$

and comparing (1.5) and (1.7), completes the proof of (1.5).

We calculate the Cartan tensor of $(\alpha,\vec{\boldsymbol\beta})$-norm using (1.2) as

$$2\,C_y(u,v,w) \;=\; \alpha^{-1}(y)\sum\nolimits_i[F^2/2]_{\alpha\beta_i}\big(K_y(u,v)p_{yi}(w)+K_y(v,w)p_{yi}(u)+K_y(w,u)p_{yi}(v)\big)$$
$$+\;\sum\nolimits_{i,j,k}[F^2/2]_{\beta_i\beta_j\beta_k}\,p_{yi}(u)p_{yj}(v)p_{yk}(w). \tag{1.9}$$

Then using equalities (1.8) and

$$[F^2/2]_{\beta_i\beta_j\beta_k}=\alpha^{-1}(y)(\dot\phi_i\,\ddot\phi_{jk}+\dot\phi_j\,\ddot\phi_{ik}+\dot\phi_k\,\ddot\phi_{ij}+\phi\,\dddot\phi_{ijk}),$$

and comparing (1.9) and (1.6) completes the proof of (1.6). □

Note that if $s_i=0$ $(1\le i\le p)$ then $\rho=1$. By Proposition 1.2, g_y (for small s_i and $\rho>0$) of $(\alpha,\vec{\boldsymbol\beta})$-norm can be viewed as a perturbed scalar product $\langle\cdot\,,\cdot\rangle$.

Define nonnegative quantities: $R_1=\max_{s\in\Pi}\|\boldsymbol\rho_1(s)\|$ – the maximal norm of the vector $\boldsymbol\rho_1=(\rho^1_i)$, $R_0=\max_{s\in\Pi}\|\boldsymbol\rho_0(s)\|$ – the maximal norm of the symmetric matrix $\boldsymbol\rho_0=(\rho_0^{ij})$, and $R=\min_{s\in\Pi}\rho(s)$, where $\Pi=\prod_{i=1}^p[-\delta_i,\delta_i]$ and $\delta_i>0$.

Lemma 1.3 (Regularity)**.** *Let* $\delta_0:=(\delta_1^2+\ldots+\delta_p^2)^{\frac12}$ *obeys the following inequality:*

$$\delta_0<\frac{2R}{3R_1+\sqrt{9R_1^2+4RR_0}}\,. \tag{1.10}$$

Then F *in* (1.3) *is a Minkowski norm on* V*.*

Proof. Since $\alpha(\beta_i)\le\delta_i$ $(1\le i\le p)$, the terms in (1.5) obey the inequalities when $y\ne0$:

$$|\rho_0^{ij}\beta_i\otimes\beta_j|\le|\rho_0^{ij}\delta_i\delta_j|\le R_0\delta_0^2,$$
$$\alpha^{-1}(y)|\rho_1^i(\beta_i\otimes y^\flat+y^\flat\otimes\beta_i)|\le 2|\rho_1^i\delta_i|\le 2R_1\delta_0,$$
$$\alpha^{-3}(y)|(\beta_i(y)\rho_1^i)y^\flat\otimes y^\flat|\le|\rho_1^i\delta_i|\le R_1\delta_0.$$

Thus, $g_y\ge R-3R_1\delta_0-R_0\delta_0^2$. The RHS of the last inequality (quadratic polynomial in $\delta_0\ge0$) is positive if and only if $\delta_0<\frac{\sqrt{9R_1^2+4RR_0}-3R_1}{2R_0}$, that is (1.10) holds. □

We restrict ourselves to regular $(\alpha,\vec{\boldsymbol\beta})$-norms alone, that is $\det g_y\ne0$ $(y\ne0)$.

Let $\{e_1,\ldots,e_{m+1}\}$ be a basis of V. A scalar product (metric) a on V and similarly, the metric g_y for any $y\ne 0$, define volume forms by

$$\mathrm{d}\,\mathrm{vol}_a(e_1,\ldots,e_{m+1})=\sqrt{\det b_{ij}},\quad \mathrm{d}\,\mathrm{vol}_{g_y}(e_1,\ldots,e_{m+1})=\sqrt{\det g_y(e_i,e_j)}.$$

Then

$$\mathrm{d}\,\mathrm{vol}_{g_y}=\mu_{g_y}(y)\,\mathrm{d}\,\mathrm{vol}_a$$

for some function $\mu_{g_y}(y)>0$. Let $q_k=(q_k^1,\ldots,q_k^p)\in\mathbb{R}^p$ be unit eigenvectors with eigenvalues λ^k of the matrix $\{\rho_0^{ij}+\varepsilon^{-1}\rho_1^i\rho_1^j\}$. Define vectors $\tilde\beta_k=q_k^i\beta_i$ $(1\le k\le p)$. Then (1.5) takes the form

$$g_y(u,v)=\rho\langle u,v\rangle+\sum\nolimits_i\lambda^i\,\tilde\beta_i(u)\,\tilde\beta_i(v)-\varepsilon\,\tilde Y(u)\tilde Y(v), \tag{1.11}$$

which can be used to find $\mu_{g_y}(y)$.

Let M^{m+1} $(m\ge 2)$ be a connected smooth manifold with Riemannian metric $a=\langle\cdot\,,\cdot\rangle$ and the Levi-Civita connection $\bar\nabla$. We will generalize definition in [17] for $p=1$.

Definition 1.3. A *general* $(\alpha,\vec{\boldsymbol\beta})$*-metric* F on M is a family of $(\alpha,\vec{\boldsymbol\beta})$-norms F_x in tangent spaces T_xM depending smoothly on a point $x\in M$.

The study of a sphere S^{m+1} endowed with a general $(\alpha,\vec{\boldsymbol\beta})$-metric (e.g., the bounds of curvature, and totally geodesic submanifolds) seem to be interesting and is delegated to further work.

2 The $(\alpha,\vec{\boldsymbol\beta})$-modification of a scalar product

Let $\omega\ne 0$ be a 1-form and $\beta_1,\ldots,\beta_p$ linear independent 1-forms on a vector space V^{m+1} endowed with Euclidean scalar product $\langle\cdot\,,\cdot\rangle$. Let N be a unit normal to a hyperplane $W=\ker\omega$ in V,

$$\langle N,v\rangle=0\quad(v\in W),\qquad\langle N,N\rangle=1.$$

If $W\ne\ker\beta_i$ $(1\le i\le p)$ then $\beta_i^{\sharp\top}\ne 0$ (the projection of $\beta_i^\sharp$ onto W) and $|\beta_i(N)|<b_i$. For any Minkowski norm on V, there are two normal directions to W, opposite when this norm is reversible, see [15]. Hence, there is a unique α-unit vector $n\in V$, which is g_n-orthogonal to W and lies in the same half-space as N:

$$g_n(n,v)=0\quad(v\in W),\quad \alpha(n)=1,\quad\langle n,N\rangle>0.$$

Remark that $\nu=F(n)^{-1}n$ is a g_n-unit normal to W, where $F(n)=\alpha\,\phi(s)$, and we get $g_n(n,n)=\phi^2(s)$, where $s=(s_1,\ldots,s_p)$ and

$$s_i=\beta_i(n),\quad 1\le i\le p. \tag{2.1}$$

In what follows, in all expressions with s_i, ϕ and ρ's we assume (2.1). Put $g:=g_n$, thus

$$g(u,v)=\rho\langle u,v\rangle+\rho_0^{ij}\beta_i(u)\beta_j(v)+\rho_1^i(\beta_i(u)\langle n,v\rangle+\beta_i(v)\langle n,u\rangle)-(\rho_1^is_i)\langle n,u\rangle\langle n,v\rangle, \tag{2.2}$$

see (1.5) with $y = n$. Define the quantities (needed for two lemmas in what follows),

$$\begin{aligned} \gamma_1^i &= (\rho_1^i + \rho_0^{ij} s_j)/\rho = \dot\phi_i\,/(\phi - \sum\nolimits_j \dot\phi_j s_j) \quad (1 \le i \le p),\\ \gamma_2^{ij} &= \rho_0^{ij} - \gamma_1^i\rho_1^j - \gamma_1^j\rho_1^i - \gamma_1^i\gamma_1^j\rho_1^k s_k \quad (1 \le i, j \le p),\\ c_1 &= \gamma_1^i\beta_i(N) + (1 - \gamma_1^i\gamma_1^j b_{ij}^\top)^{1/2}, \end{aligned} \tag{2.3}$$

where $b_{ij}^\top := b_{ij} - \beta_i(N)\beta_j(N)$. Assume that

$$b_{ij}^\top\,\gamma_1^i\gamma_1^j \le 1. \tag{2.4}$$

By (2.4), discriminant in the formula (2.3) for c_1 is nonnegative, hence c_1 is real. In the following lemma we express g-normal n to W through the a-normal N and the auxiliary functions (2.3).

Lemma 2.1. *Let* (2.4) *holds, then the value of* c_1 *is real and*

$$n = c_1 N - \gamma_1^i\beta_i^\sharp, \tag{2.5}$$

$$g(u, v) = \rho\,\langle u, v\rangle + \gamma_2^{ij}\beta_i(u)\,\beta_j(v) \quad (u, v \in W)\,. \tag{2.6}$$

Moreover, the values $s_i = \beta_i(n)$ *can be found from the system*

$$s_i = c_1\beta_i(N) - \gamma_1^j b_{ij} \quad (1 \le j \le p). \tag{2.7}$$

Proof. From (2.2) with $u = n$ and $v \in W$ and $g(n, v) = 0$ we find

$$\langle \rho\, n + \gamma_1^i\beta_i^\sharp,\ v\rangle = 0 \quad (v \in W). \tag{2.8}$$

From (2.8) and $\rho > 0$ we conclude that $\rho\, n + \gamma_1^i\beta_i^{\sharp\top} = c_1 N$ for some real c_1. Using

$$1 = \langle n, n\rangle = c_1^2 - 2\,c_1\gamma_1^i\beta_i^\sharp + \gamma_1^i\gamma_1^j\langle\beta_i^\top, \beta_j^\top\rangle$$

and $\langle\beta_i^\top, \beta_j^\top\rangle = b_{ij} - \beta_i(N)\beta_j(N)$, we get two real solutions

$$(c_1)_{1,2} = \gamma_1^i\beta_i(N) \pm (1 - \gamma_1^i\gamma_1^j b_{ij}^\top)^{1/2}.$$

The greater value (with +) provides inequality $\langle n, N\rangle > 0$, that proves (2.5). Thus, we get (2.7):

$$s_i = \beta_i(n) = \beta_i(c_1 N - \gamma_1^j\beta_j^\sharp) = c_1\beta_i(N) - \gamma_1^j b_{ij} \quad (1 \le i \le p).$$

Finally, (2.6) follows from (2.2), (2.5) and $\langle n, u\rangle = -\gamma_1^i\beta_i(u)$ $(u \in W)$. □

Remark 2.1 (Case $\beta_i^\sharp \in W$)**.** An interesting particular case appears when all vectors $\beta_i^\sharp$ belong to W, that is $\beta_i(N) = 0$. Then, rather complicated system (2.7) reads

$$\sum\nolimits_i \dot\phi_i/\phi\,(b_{ij} - s_i s_j) = -s_j \quad (1 \le j \le p), \tag{2.9}$$

from which all $\dot\phi_i$ at $s_i = \beta_i(n)$ can be expressed through ϕ and $\{s_i\}$.

Define a matrix P with elements

$$P^j_k = \gamma_2^{ij} b^\top_{ik}.$$

$Q = \rho\,\mathrm{id} + P$ is non-singular, if γ_2^{ij} are "small" relative to $\rho > 0$, i.e.,

$$\det[\,\rho\,\delta^j_k + \gamma_2^{ij} b^\top_{ik}\,] \neq 0. \tag{2.10}$$

Using the inverse matrix Q^{-1}, define the quantities (needed for the following lemma),

$$\gamma_3^{ij} = -\gamma_2^{kj}(Q^{-1})^i_k \quad (1 \le i, j \le p).$$

In the following lemma, we find relation between $u \in W$ and $U \in W$ such that

$$g(u, v) = \langle U, v\rangle, \quad \forall\, v \in W. \tag{2.11}$$

Lemma 2.2. *Let* (2.4) *and* (2.10) *hold. If the vectors* u, U *belong to* W *and obey* (2.11) *then*

$$\rho\, u = U + \gamma_3^{ij}\, \beta_i(U)\, \beta_j^{\sharp\top}. \tag{2.12}$$

Proof. By (2.6), $g(u, v) = \langle \rho\, u + \gamma_2^{ij}\beta_i(u)\beta_j^\sharp,\, v\rangle$ for $u, v \in W$. By conditions, and since $U, \beta_j^{\sharp\top} \in W$, we find $\rho\, u + \gamma_2^{ij}\beta_i(u)\beta_j^{\sharp\top} = U$. Applying β_k and using $\beta_k(\beta_j^{\sharp\top}) = b^\top_{jk}$ yields

$$(\rho\,\delta^j_k + P^j_k)\beta_j(u) = \beta_k(U) \quad (1 \le k \le p),$$

and then (2.12). □

3 Examples

The following lemma is used to compute the volume forms of $(\alpha, \vec{\boldsymbol\beta})$-norm for $p = 1, 2$. This extends the Silvester's determinant identity, see [14],

$$\det(\mathrm{id}_m + C_1 P_1^t) = 1 + C_1^t P_1,$$

where C_1, P_1 are m-vectors (columns), and id_m is the identity m-matrix.

Lemma 3.1. *Let* C_i, P_i $(1 \le i \le j \le m)$ *be* m*-vectors. Then* $\mathrm{Tr}(C_i P_j^t) = C_i^t P_j = P_j^t C_i$ *and*

$$\det(\mathrm{id}_m + C_1 P_1^t + C_2 P_2^t) = 1 + C_1^t P_1 + C_2^t P_2 + C_1^t P_1 \cdot C_2^t P_2 - C_1^t P_2 \cdot C_2^t P_1\,, \tag{3.1}$$

$$\begin{aligned}\det(\mathrm{id}_m + C_1 P_1^t + C_2 P_2^t + C_3 P_3^t) &= 1 + C_1^t P_1 + C_2^t P_2 + C_3^t P_3 + C_1^t P_1 \cdot C_2^t P_2\\ &+ C_2^t P_2 \cdot C_3^t P_3 + C_1^t P_1 \cdot C_3^t P_3 - C_1^t P_2 \cdot C_2^t P_1 - C_1^t P_3 \cdot C_3^t P_1 - C_2^t P_3 \cdot C_3^t P_2\\ &+ C_1^t P_1 \cdot C_2^t P_2 \cdot C_3^t P_3 + C_1^t P_2 \cdot C_2^t P_3 \cdot C_3^t P_1 + C_1^t P_3 \cdot C_2^t P_1 \cdot C_3^t P_2\\ &- C_1^t P_1 \cdot C_2^t P_3 \cdot C_3^t P_2 - C_1^t P_2 \cdot C_2^t P_1 \cdot C_3^t P_3 - C_1^t P_3 \cdot C_2^t P_2 \cdot C_3^t P_1\,,\end{aligned} \tag{3.2}$$

and so on.

For $p=1$, (1.3) defines (α,β)-norm $F=\alpha\phi(s)$ for $s=\beta/\alpha$. This function F is a Minkowski norm on V for any α and β with $\alpha(\beta)<\delta_0$ if and only if $\phi(s)$ satisfies

$$\phi - s\,\dot\phi + (b^2-s^2)\,\ddot\phi > 0, \tag{3.3}$$

where real s,b obey $|s|<b$, see [14]. Taking $s\to b$ in (3.3), we get $\phi - s\,\dot\phi>0$. By (1.5),

$$\begin{aligned} g_y(u,v) \,&=\, \rho\langle u,v\rangle + \rho_0\beta(u)\beta(v) + \rho_1(\beta(u)\langle y,v\rangle + \beta(v)\langle y,u\rangle)/\alpha(y)\\ &-\, \rho_1\beta(y)\langle y,u\rangle\langle y,v\rangle/\alpha^3(y). \end{aligned}\tag{3.4}$$

Here $\rho>0$ and ρ_0,ρ_1 are the following functions of s:

$$\rho=\phi(\phi-s\,\dot\phi),\quad \rho_0=\phi\,\ddot\phi+\dot\phi^2,\quad \rho_1=\phi\,\dot\phi - s(\phi\,\ddot\phi+\dot\phi^2).$$

The following relations hold: $\dot\rho=\rho_1$, $\ddot\rho=\dot\rho_1=-s\,\dot\rho_0$. Set $\tilde Y=s^{-1}\beta - y^\flat/\alpha(y)$ and $\varepsilon=s\rho_1$. Then (3.4) takes the form

$$g_y(u,v)=\rho\langle u,v\rangle + (\rho_0+\rho_1^2/\varepsilon)\,\beta(u)\,\beta(v) - \varepsilon\,\tilde Y(u)\tilde Y(v), \tag{3.5}$$

From (3.5) and (3.1) with $C_1=(\rho_0+\rho_1^2/\varepsilon)\rho^{-1}\beta^\sharp$, $P_1=\beta^\sharp$, $C_2=-\varepsilon\rho^{-1}\tilde Y^\sharp$, $P_2=\tilde Y^\sharp$, for the volume form $\mathrm{d\,vol}_{g_y}=\mu_{g_y}(y)\,\mathrm{d\,vol}_a$ we obtain, see also [14],

$$\begin{aligned} \mu_{g_y}(y) \,&=\, \rho^{m-1}(\rho^2+\rho_0\rho_1 s^3+\rho_1^2 s^2+(\rho-\rho_0 b^2)\rho_1 s + (\rho\rho_0-\rho_1^2)b^2)\\ &=\, \phi^{m+2}(\phi-s\,\dot\phi)^{m-1}[\,\phi - s\,\dot\phi+(b^2-s^2)\,\ddot\phi\,]. \end{aligned}\tag{3.6}$$

Set $p_y=\beta^\sharp - sy/\alpha(y)$. The Cartan tensor of (α,β)-norm has an interesting special form [8]:

$$\begin{aligned} 2\,C_y(u,v,w) \,&=\, \rho_1\alpha^{-1}(y)(K_y(u,v)\langle p_y,w\rangle + K_y(v,w)\langle p_y,u\rangle + K_y(w,u)\langle p_y,v\rangle)\\ &+\, (3\dot\phi\,\ddot\phi+\phi\,\dddot\phi)\alpha^{-1}(y)\,\langle p_y,u\rangle\langle p_y,v\rangle\langle p_y,w\rangle, \end{aligned}$$

see (1.6) for $p=1$. For a hyperplane $W\subset V$ we have $s=\beta(n)$ and

$$\begin{aligned} &c_1=\gamma_1\beta(N)+(1-\gamma_1^2(b^2-\beta(N)^2))^{1/2},\\ &\gamma_1=(\rho_1+\rho_0\beta(n))/\rho=\dot\phi/(\phi-s\,\dot\phi),\\ &\gamma_2=\rho_0-\gamma_1\rho_1(\beta(n)\gamma_1+2)=\phi\,(\phi^2\ddot\phi-\phi\,\dot\phi^2+s\,\dot\phi^3)/(\phi-s\,\dot\phi)^2,\\ &\gamma_3=-\frac{\gamma_2}{\rho+(b^2-\beta(N)^2)\,\gamma_2}\,. \end{aligned}$$

Then (2.7) reads

$$\frac{\dot\phi}{\phi}=-\frac{s\sqrt{b^2-s^2}+\beta(N)\sqrt{b^2-\beta(N)^2}}{(b^2-s^2-\beta(N)^2)\sqrt{b^2-s^2}},$$

which for $\beta^\sharp\in W$ reads $\frac{\dot\phi}{\phi}=-\frac{s}{b^2-s^2}$, see also (2.9) for $p=1$.

Example 3.1 ($p=1$)**.** Some progress was achieved for particular cases of (α,β)-norms. Below we consult some of (α,β)-norms to illustrate the above metric g on V.

(i) For $\phi(s) = 1+s,\ |s| < b < \delta_0 = 1$, we have the norm $F = \alpha+\beta$, introduced by a physicist G. Randers to consider the unified field theory. We have $\rho = 1+s,\ \rho_0 = 1$ and $\rho_1 = 1$. For a hyperplane $W \subset V$ and $g = g_n$, we get $n = c_1 N - \beta^\sharp,\ s = \beta(n) = c\,c_1 - 1,\ \phi(s) = c\,c_1$, where $c_1 = c + \beta(N)$ and $c = \sqrt{1 - b^2 + \beta(N)^2} \in (0, 1]$, see also [13]. Then

$$\gamma_1 = 1, \quad \gamma_2 = -c\,c_1, \quad \gamma_3 = c^{-2}.$$

Conditions (2.4) and (2.10) become trivial: $c > 0$. Next, $\mu_g(n) = (c\,c_1)^{\,m+2}$ and

$$g(u, v) = (1 + s)\langle u, v\rangle - s\langle n, u\rangle\langle n, v\rangle + \beta(u)\langle n, v\rangle + \beta(v)\langle n, u\rangle + \beta(u)\beta(v).$$

(ii) The (α, β)-norms $F = \alpha^{l+1}/\beta^l$ $(l > 0)$, i.e., $\phi(s) = 1/s^l$ $(0 < s < b)$, are called *generalized Kropina metrics*, see [8], and have applications in general dynamical systems. The *Kropina metric*, i.e., $l = 1$, first introduced by L. Berwald in connection with a Finsler plane with rectilinear extremal, and investigated by V.K. Kropina in 1961. We have $\rho = 2/s^2,\ \rho_0 = 3/s^4$ and $\rho_1 = -4/s^3$. For a hyperplane $W \ne \ker\beta$ in V and $g = g_n$ we get

$$c_1 = (b - 2\beta(N))/\sqrt{2\,b(b - \beta(N))}, \qquad \beta(n) = s = \sqrt{b(b - \beta(N))/2},$$
$$\gamma_1 = -1/(2s) = -1/\sqrt{2\,b(b - \beta(N))}, \qquad \gamma_2 = \gamma_3 = 0,$$

and $\mu_g(n) = \frac{4^{m+1}}{b^m(b-\beta(N))^{m+2}}$. Note that conditions (2.4) and (2.10) become trivial.

(iii) The (α, β)-norm $F = \frac{\alpha^2}{\alpha-\beta}$, i.e., $\phi(s) = \frac{1}{1-s}$ with $|s| < b < \delta_0 = \frac12$, (called *slope-metric*) was introduced by M. Matsumoto to study the time it takes to negotiate any given path on a hillside. We have $\rho = \frac{1-2s}{(1-s)^3}$, $\rho_0 = \frac{3}{(1-s)^4}$ and $\rho_1 = \frac{1-4s}{(1-s)^4}$. For a hyperplane $W \ne \ker\beta$ and $g = g_n$, from (2.7) we find that $s = \beta(n)$ obeys 4th-order equation

$$4s^4 - 4s^3 + (1 - 4b^2)s^2 + 2(b^2 + \beta(N)^2)s + b^4 - (b^2 + 1)\beta(N)^2 = 0,$$

and $s = \frac14\,(1 - \sqrt{1 + 8b^2})$ if $\beta^\sharp \in W$, see (2.9). We find $\mu_g(n) = \frac{(1-2s)^{m-1}}{(1-s)^{3m+3}}\,(2b^2 - 3s + 1)$ and

$$c_1 = \frac{\beta(N) + \sqrt{(1 - 2s)^2 - b^2 + \beta(N)^2}}{1 - 2s},$$
$$\gamma_1 = \frac{1}{1 - 2s}, \quad \gamma_2 = \frac{1}{(1 - 2s)^2(1 - s)^3}, \quad \gamma_3 = \frac{1}{(1 - 2s)^3 + b^2 - \beta(N)^2}.$$

Thus, (2.10) becomes trivial and (2.4) reads as $(1 - 2s)^2 \ge b^2 - \beta(N)^2$.

(iv) A Finsler metric is a *polynomial* (α, β)*-norm* if $\phi(s) = \sum_{i=0}^k C_i s^i$, $C_0 = 1$, $C_k \ne 0$. The *quadratic metric* $F = (\alpha + \beta)^2/\alpha$, i.e., $\phi(s) = (1 + s)^2$ with $|s| < b < \delta_0 = 1$, appears in many geometrical problems, [14]. We have $\rho = (1-s)(1+s)^3$, $\rho_0 = 6(1 + s)^2$ and $\rho_1 = 2(1 - 2s)(1 + s)^2$. For a hyperplane $W \ne \ker\beta$ in V and $g = g_n$, from (2.7) we find that s obeys 4th-order equation

$$s^4 - 2s^3 + (1 - 4b^2 + 3\beta(N)^2)s^2 + 2(2b^2 - \beta(N)^2)s + 4b^4 - (4b^2 + 1)\beta(N)^2 = 0,$$

and $s=(1-\sqrt{1+8b^2})/2$ if $\beta^\sharp\in W$, see (2.9). Then we obtain

$$c_1=\frac{2\beta(N)+\sqrt{(1-s)^2-4(b^2-\beta(N)^2)}}{1-s},$$
$$\gamma_1=\frac{2}{1-s},\quad \gamma_2=\frac{2(3s-1)(1+s)^3}{(1-s)^2},\quad \gamma_3=\frac{2(3s-1)}{(1-s)^3-2(1-3s)^2(b^2-\beta(N)^2)}$$

and $\mu_g(n)=(1+s)^{3m+3}(1-s)^{m-1}(2b^2-3s^2+1)$. Conditions (2.4) and (2.10) read

$$(1-s)^2\ge 4(b^2-\beta(N)^2),\quad (1-s)^3\ne 2(1-3s)(b^2-\beta(N)^2).$$

(v) Define by $\phi(s)=e^{s/k}$, $|s|<b<\delta_0:=|k|$, the *exponential metric* $F=\alpha\,e^{\beta/(k\alpha)}$. Condition (3.3) reads as a quadratic inequality $s^2+ks-(b^2+k^2)<0$. Taking $s=b$ in (3.3) yields $k(s-k)<0$ when $|s|<|k|$. Thus, (3.3) is satisfied for arbitrary numbers s and b with $|s|\le b<|k|$. We have $\rho=e^{2s/k}(k-s)/k>0$, $\rho_0=2\,e^{2s/k}/k^2$ and $\rho_1=e^{2s/k}(k-2s)/k^2$. For a hyperplane $W\ne\ker\beta$ in V and $g=g_n$, by (2.7), $s=\beta(n)$ obeys 4th-order equation

$$s^4-2ks^3+(k^2-2b^2+\beta(N)^2)s^2+2b^2ks+b^4-(b^2+k^2)\beta(N)^2=0,$$

and $s=(k-\sqrt{k^2+4b^2})/2$ if $\beta^\sharp$ is tangent to the foliation, see (2.9). Then we get

$$c_1=\frac{\beta(N)+((k-s)^2-b^2+\beta(N)^2)^{1/2}}{k-s},$$
$$\gamma_1=\frac{1}{k-s},\quad \gamma_2=\frac{s\,e^{2s/k}}{k(k-s)^2},\quad \gamma_3=\frac{s}{(k-s)^3+s(b^2-\beta(N)^2)}.$$

and $\mu_g(n)=\frac{(k-s)^{m-1}}{k^{m+1}}\,(b^2+k^2-ks-s^2)\,e^{(2m+2)s/k}$. Conditions (2.4) and (2.10) read, respectively,

$$(k-s)^2\ge b^2-\beta(N)^2,\qquad (k-s)^3\ne -s(b^2-\beta(N)^2).$$

Fig. 3.1 shows the dependence of s on $\beta(N)\in[-b,b]$, see (2.7), for four of above metrics. For $\beta(N)=0$ we obtain the values of s: a) 0.64, b) -0.13, c) -0.26, d) -0.53.

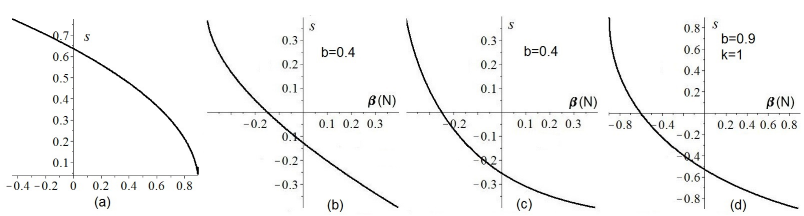

Figure 1: Dependence of s on $\beta(N)$ for metrics: a) Kropina, b) Matsumoto, c) quadratic, d) exponential.

For $p=2$, we can use (1.11) to find $\mu_g(y)$. By (1.5) we get

$$g_y(u,v)\ =\ \rho\langle u,v\rangle+(\rho_0^{ij}+\varepsilon^{-1}\rho_1^i\rho_1^j)\beta_i(u)\beta_j(v)-\varepsilon\,\tilde Y(u)\tilde Y(v),$$
$$\tilde Y\ =\ \varepsilon^{-1}\rho_1^i\beta_i-y^\flat/\alpha(y),\quad \varepsilon=s_i\rho_1^i.$$

From (3.2) with

$$C_1=\lambda^1\rho^{-1}\tilde\beta_1^\sharp,\quad P_1=\tilde\beta_1^\sharp,\quad C_2=\lambda^2\rho^{-1}\tilde\beta_2^\sharp,\quad P_2=\tilde\beta_2^\sharp,\quad C_3=-\varepsilon\rho^{-1}\tilde Y^\sharp,\quad P_3=\tilde Y^\sharp,$$

using $\tilde Y$ from (3.7), $\tilde b_{ij}=\langle\tilde\beta_i,\tilde\beta_j\rangle$, $\tilde\beta_i=q_i^1\beta_1+q_i^2\beta_2$ and $\varepsilon=\rho_1^1 s_1+\rho_1^2 s_2$, we obtain

$$\begin{aligned}\mu_{g_y}(y) \;=\; &\rho^{m-1}\big(\rho^2+\rho(\lambda^1\tilde b_{11}+\lambda^2\tilde b_{22})-\rho\,\varepsilon\langle\tilde Y,\tilde Y\rangle+\lambda^1\lambda^2(\tilde b_{11}\tilde b_{22}-\tilde b_{12}^2)\\ &-\,\varepsilon\langle\tilde Y,\tilde Y\rangle(\lambda^1\tilde b_{11}+\lambda^2\tilde b_{22})+\lambda^1\varepsilon\langle\tilde\beta_1,\tilde Y\rangle+\lambda^2\varepsilon\langle\tilde\beta_2,\tilde Y\rangle+\lambda^1\lambda^2\,\varepsilon/\rho\big[\tilde b_{11}\langle\tilde\beta_2,\tilde Y\rangle^2\\ &+\,\tilde b_{22}\langle\tilde\beta_1,\tilde Y\rangle^2+\tilde b_{12}\langle\tilde Y,\tilde Y\rangle^2-\tilde b_{11}\tilde b_{22}\langle\tilde Y,\tilde Y\rangle-2\tilde b_{12}\langle\tilde\beta_1,\tilde Y\rangle\langle\tilde\beta_2,\tilde Y\rangle\big]\big).\end{aligned}$$

Example 3.2 ($p=2$)**.** A navigation (α,β)-norm is the $(\alpha,\vec{\boldsymbol\beta})$-norm with $p=2$.

(a) For shifted Kropina norm $\phi=1+\frac{1}{s_1}+s_2$ for $s_1>0$, hence $F=\alpha(1+\frac{\alpha}{\beta_1}+\frac{\beta_2}{\alpha})$, we have

$$\begin{aligned}&\rho=(2+s_1)(1+s_1+s_1s_2)/s_1^2,\quad \rho_1^1=-(4+3s_1+2s_1s_2)/s_1^3,\quad \rho_1^2=(2+s_1)/s_1,\\ &\rho_0^{11}=(3+2s_1+2s_1s_2)/s_1^4,\qquad \rho_0^{12}=\rho_0^{21}=-1/s_1^2,\qquad \rho_0^{22}=1.\end{aligned}$$

For a hyperplane $W\ne\ker\beta_i$ $(i=1,2)$ in V and the metric $g=g_n$ we get

$$\begin{aligned}&c_1=\tfrac{s_1^2\beta_2(N)-\beta_1(N)}{s_1(2+s_1)}+\Big(1-\tfrac{b_{11}-\beta_1(N)^2}{s_1^2(2+s_1)^2}+\tfrac{2(b_{12}-\beta_1(N)\beta_2(N))}{(2+s_1)^2}-\tfrac{s_1^2(b_{22}-\beta_2(N)^2)}{(2+s_1)^2}\Big)^{1/2},\\ &\gamma_1^1=-\tfrac{1}{s_1(2+s_1)},\quad \gamma_1^2=\tfrac{s_1}{2+s_1},\quad \gamma_2^{11}=-\tfrac{2-s_1-10s_1^2-10s_1^3-3s_1^4-s_1^2s_2(2-2s_1^2-s_1^3)}{s_1^4(2+s_1)},\\ &\gamma_2^{12}=\tfrac{12+13s_1+3s_1^2+s_1s_2(2-2s_1-s_1^2)}{s_1^2(2+s_1)},\quad \gamma_2^{22}=\tfrac{4+3s_1-s_1^2(1+s_2)}{s_1^2}.\end{aligned}$$

If $\beta_i^\sharp\in W$ then s_1,s_2 obey the system

$$(1+2s_2)s_1^3-b_{12}s_1^2+b_{11}=0,\quad (1+2s_1)s_1s_2^2-b_{22}s_1^2+b_{12}=0.$$

Thus $s_2=\frac12\,[(b_{11}-s_1^2b_{12})/s_1^3-1]$, where s_1 is a positive root of the 6th-order polynomial:

$$2\,b_{22}s_1^6+b_{12}s_1^5-(b_{12}^2+2b_{12})s_1^4-b_{11}s_1^3+2b_{11}b_{12}s_1^2-b_{11}^2=0;$$

for example, if $b_{12}=0$ then $s_1=(\frac{b_{11}}{4\,b_{22}}\,(1+\sqrt{1+8\,b_{22}}))^{1/3}$ and $s_2=\frac12\,(b_{11}/s_1^3-1)$.

(b) For shifted Matsumoto norm $\phi=\frac{1}{1-s_1}+s_2$ with $\delta_i<1$, hence $F=\alpha(\frac{\alpha}{\alpha-\beta_1}+\frac{\beta_2}{\alpha})$, we have

$$\begin{aligned}&\rho=\frac{(1-2s_1)(1+s_2-s_1s_2)}{(1-s_1)^3},\quad \rho_1^1=\frac{1+2s_1(s_1s_2-s_2-2)}{(1-s_1)^4},\quad \rho_1^2=\frac{1-2s_1}{(1-s_1)^2},\\ &\rho_0^{11}=(3-2s_1s_2+2s_2)/(1-s_1)^4,\quad \rho_0^{12}=\rho_0^{21}=1/(1-s_1)^2,\quad \rho_0^{22}=1.\end{aligned}$$

For a hyperplane $W \neq \ker\beta_i$ $(1 \le i \le p)$ in V and the metric $g = g_n$ we get

$$
\begin{aligned}
c_1 &= \frac{(1-s_1)^2\beta_2(N)+\beta_1(N)}{1-2\,s_1} + \Big(1 - \frac{(1-s_1)^4(b_{22}-\beta_2(N)^2)}{(1-2\,s_1)^2} \\
&- \frac{2(1-s_1)^2(b_{12}-\beta_1(N)\beta_2(N))}{(1-2\,s_1)^2} - \frac{b_{11}-\beta_1(N)^2}{(1-2\,s_1)^2}\Big)^{1/2},\\
\gamma^1_1 &= \frac{1}{1-2s_1},\quad \gamma^2_1 = \frac{(1-s_1)^2}{1-2s_1},\quad \gamma^{11}_2 = \frac{1+2s_1+8s_1^2+s_2(1+5s_1-6s_1^2)}{(1-s_1)^3(1-2s_1)},\\
\gamma^{22}_2 &= -\frac{1-3s_1+2s_1^2-4s_1^3+s_1^4+s_2(1-4s_1+3s_1^2)}{(1-s_1)^4},\\
\gamma^{12}_2 &= -\frac{1-5s_1+3s_1^2+4s_1^3+s_2(1-8s_1+17s_1^2-12s_1^3+2s_1^4)}{(1-2s_1)(1-s_1)^4}.
\end{aligned}
$$

If $\beta_i^\sharp \in W$ then s_1 and s_2 obey the system

$$b_{11}+(1-s_1)^2(b_{12}-2s_1s_2)=s_1,\quad b_{12}+(1-s_1)^2(b_{22}-2s_2^2)=s_2.$$

Then $s_1=(2b_{11}s_2^2-b_{12}s_2-b_{11}b_{22}+b_{12}^2)/(2b_{12}s_2-b_{22})$, where s_2 is a root of a 6th-order polynomial.

Similarly to graphs on Fig. 3.1, one may calculate and graph pairs of surfaces in $\mathbb{R}^3$, showing dependence of s_1 and s_2 on variables $(\beta_1(N),\beta_2(N))$ for the above navigation (α,β)-metrics. For $\beta_i(N)=0$ we obtain the values: a) $s_1\approx-0.79$ and $s_2=-1.5$ for Kropina norm; b) $s_1\approx-0.42$ and $s_2=s_1^3-2s_1^2+s_1\approx-0.84$ for Matsumoto norm.

4 The shape operator and the curvature of normal curves

Let $(M^{m+1}, a=\langle\cdot\,,\cdot\rangle)$ $(m\ge 2)$ be a connected Riemannian manifold with the Levi-Civita connection $\bar\nabla$. Let N be a unit normal field to a codimension-one distribution $\mathcal{D}:=\ker\omega$ on (M,α). Due to Section 2, there exists a g_n-normal (to $\mathcal{D}$) vector field n such that $\langle n,N\rangle>0$ and $\langle n,n\rangle=1$. Define a new Riemannian metric $g:=g_n$ on M, see (2.2), with the Levi-Civita connection ∇. Let $\ker\beta_i\neq\mathcal{D}$ everywhere for all i, hence $|\beta_i(N)|<\sqrt{b_{ii}}$. By (2.7), $s_i=\beta_i(n)$ are smooth functions on M, and $\nu=n/\phi(s)$ is a g-unit normal to the leaves.

The shape operators $\bar A$ and A^g of $\mathcal{D}$ and the curvature vectors of ν- and N- curves for both metrics $\langle\cdot,\cdot\rangle$ and g belong to *Extrinsic Geometry* and are defined by

$$\bar A(u)=-\bar\nabla_u N,\quad A^g(u)=-\nabla_u\,\nu\quad(u\in\mathcal{D}),\tag{4.1}$$

$$Z=\nabla_\nu\,\nu,\qquad \bar Z=\bar\nabla_N N.\tag{4.2}$$

Let $\bar T^\sharp:\mathcal{D}\to\mathcal{D}$ be a linear operator adjoint to the integrability tensor $\bar T$ of $\mathcal{D}$ with respect to a,

$$2\,\bar T(u,v)=\langle[u,v],N\rangle\quad(u,v\in\mathcal{D}).$$

Note that $\bar T^\sharp = \frac12(\bar A - \bar A^*)$, where $\bar A^*$ is a linear operator adjoint to $\bar A$. The deformation tensor,

$$\overline{\rm Def}_u = (\bar\nabla u + (\bar\nabla u)^t)/2,$$

measures the degree to which the flow of a vector field u distorts $\langle\cdot,\cdot\rangle$. Here, $\bar\nabla u$ and $(\bar\nabla u)^t$ are

$$(\bar\nabla u)\,(v) = \bar\nabla_v\, u, \quad \langle(\bar\nabla\, u)^t(v),\, w\rangle = \langle v,\, (\bar\nabla\, u)(w)\rangle \quad (v, w \in TM).$$

In the next proposition, we express A^g through $\bar A$ and invariants of $\mathcal D$ with respect to a.

Proposition 4.1 (The shape operator)**.** *Let (M^{m+1}, a) be a Riemannian manifold with a form $\omega\ne 0$ and linear independent 1-forms $\beta_1,\dots\beta_p$ obeying conditions* (2.4) *and* (2.10)*. Let g be a Riemannian metric* (2.2) *determined by a distribution $\mathcal D = \ker\omega$, $\vec{\boldsymbol\beta} = (\beta_1,\dots,\beta_p)$ and a smooth function $\phi(x,s)$ on $M\times\mathbb R^p$. Then*

$$\rho\,\phi A^g = -\mathcal A - \gamma_3^{ij}\,(\beta_i\circ\mathcal A)\otimes\beta_j^{\sharp\top}, \tag{4.3}$$

where the linear operator $\mathcal A:\mathcal D\to\mathcal D$ is given by

$$\mathcal A = -\rho\, c_1\bar A - \rho\gamma_1^i(\overline{\rm Def}_{\beta_i^\sharp})^\top + \frac12\, n(\rho)\,{\rm id}^\top + {\rm Sym}(U^j\otimes\beta_j^\top), \tag{4.4}$$

and the vector fields U^j are given by

$$\begin{aligned} U^j \;=\;& \frac12\big(n(\gamma_2^{ij})\beta_i^{\sharp\top} + \gamma_2^{ij}\bar\nabla_n^\top\,\beta_i^{\sharp\top}\big) - \rho\,\bar\nabla^\top\gamma_1^j\\ &+\,(\rho_0^{ij} - \gamma_1^j\rho_1^i)\,\big(\beta_i(N)\bar\nabla^\top c_1 - (\gamma_1^k/2)\bar\nabla^\top b_{ik} - b_{ik}\bar\nabla^\top\gamma_1^k\big)\\ &+\,(c_1 - \beta_k(N)\gamma_1^k)\big((\rho_0^{ij} - \gamma_1^j\rho_1^i)\beta_i(N) + c_1\rho_1^j(1 + s_k\gamma_1^k)\big)\bar Z\\ &+\,\big(c_1\rho_1^i(1 + s_k\gamma_1^k)\gamma_1^j - (\rho_0^{ij} - \gamma_1^j\rho_1^i)(c_1 - \beta_k(N)\gamma_1^k)\big)\bar A^*(\beta_i^{\sharp\top}). \end{aligned} \tag{4.5}$$

Proof. By known formula for the Levi-Civita connection ∇ of g,

$$2\,g(\nabla_u v, w) = u(g(v,w)) + v(g(u,w)) - w(g(u,v)) + g([u,v],w) - g([u,w],v) - g([v,w],u), \tag{4.6}$$

where $u, v, w\in C^\infty(TM)$, we have

$$2\,g(\nabla_u\, n, v) = n(g(u,v)) + g([u,n],v) + g([v,n],u) - g([u,v],n) \quad (u, v\in\mathcal D). \tag{4.7}$$

Assume $\bar\nabla^\top_X\, u = \bar\nabla^\top_X\, v = 0$ for $X\in T_xM$ at a given point $x\in M$. Using (2.2) and (2.6), we get

$$\begin{aligned} & n(g(u,v)) = n(\rho\langle u,v\rangle) + n(\gamma_2^{ij}\,\beta_i(u)\,\beta_j(v))\\ &\quad = n(\rho)\langle u,v\rangle + \big[n(\gamma_2^{ij})\beta_i(u)\beta_j(v) + \gamma_2^{ij}\big(\beta_i(u)(\bar\nabla_n(\beta_j^\top))(v) + \beta_i(v)(\bar\nabla_n(\beta_j^\top))(u)\big)\big],\\ & g([u,v],n) = 2\,\rho\, c_1\bar T(u,v),\\ & g([u,n],v) = \rho\langle\bar\nabla_u n, v\rangle + \rho_0^{ij}\beta_i([u,n])\beta_j(v) + \rho_1^i(\beta_i([u,n])\langle n,v\rangle + \beta_i(v)\langle n,[u,n]\rangle)\\ &\quad -\rho_1^i s_i\langle n,[u,n]\rangle\langle n,v\rangle, \end{aligned}$$

where $u,v\in\mathcal{D}$. Using equalities

$$
\begin{aligned}
&\langle\bar\nabla_u n, v\rangle = -\langle c_1\bar A(u),v\rangle - \gamma^i_1\langle\bar\nabla_u\,\beta^\sharp_i, v\rangle - \beta_i(v)\langle\bar\nabla\gamma^i_1, u\rangle = \langle U_3, v\rangle,\\
&\beta_i([u,n]) = -\gamma^j_1\langle\bar\nabla_u\,\beta^\sharp_j,\beta^\sharp_i\rangle + \big\langle\beta_i(N)\bar\nabla c_1 - b_{ij}\bar\nabla\gamma^j_1\\
&\qquad + \beta_i(N)\big[(c_1-\gamma^j_1\beta_j(N))\bar Z + \gamma^j_1\bar A^*(\beta^{\sharp\top}_j)\big], u\big\rangle = \langle U_{2i}, u\rangle - \gamma^j_1\langle\bar\nabla_u\,\beta^\sharp_j,\beta^\sharp_i\rangle,\\
&\langle n,[u,n]\rangle = \langle(c_1-\gamma^j_1\beta_j(N))\bar\nabla c_1 + (\gamma^i_1 b_{ji} - c_1\beta_j(N))\bar\nabla\gamma^j_1\\
&\qquad - c_1\gamma^j_1\bar\nabla(\beta_j(N)) - c_1\gamma^j_1\beta_j(N)\bar Z, u\rangle = \langle U_1, u\rangle,\\
&\langle n, v\rangle = -\gamma^i_1\beta_i(v),
\end{aligned}
$$

we then obtain

$$
\begin{aligned}
&g([u,n],v) = -\rho\, c_1\langle\bar A(u),v\rangle - \rho(\gamma^i_1\langle\bar\nabla_u\,\beta^\sharp_i, v\rangle + \beta_i(v)\langle\bar\nabla\gamma^i_1,u\rangle)\\
&+\rho^{ij}_0\beta_j(v)\big[\big\langle\beta_i(N)\bar\nabla c_1 - b_{ik}\bar\nabla\gamma^k_1 + \beta_i(N)\big[(c_1-\gamma^k_1\beta_k(N))\bar Z + \gamma^k_1\bar A^*(\beta^{\sharp\top}_k)\big], u\big\rangle\\
&-\gamma^k_1\langle\bar\nabla_u\,\beta^\sharp_k,\beta^\sharp_i\rangle\big] - \gamma^j_1\beta_j(v)\rho^i_1\big[\big\langle\beta_i(N)\bar\nabla c_1 - b_{ik}\bar\nabla\gamma^k_1\\
&+\beta_i(N)\big[(c_1-\gamma^k_1\beta_k(N))\bar Z + \gamma^k_1\bar A^*(\beta^{\sharp\top}_k)\big], u\big\rangle - \gamma^k_1\langle\bar\nabla_u\,\beta^\sharp_k,\beta^\sharp_i\rangle\big]\\
&+\rho^i_1\beta_i(v)\big\langle(c_1-\gamma^j_1\beta_j(N))\bar\nabla c_1 + (\gamma^k_1 b_{jk} - c_1\beta_j(N))\bar\nabla\gamma^j_1\\
&-c_1\gamma^k_1\bar\nabla(\beta_k(N)) - c_1(\gamma^k_1\beta_k(N))\bar Z, u\big\rangle + \rho^i_1 s_i\gamma^j_1\beta_j(v)\big\langle(c_1-\gamma^k_1\beta_k(N))\bar\nabla c_1\\
&+(\gamma^k_1 b_{jk} - c_1\beta_j(N))\bar\nabla\gamma^j_1 - c_1\gamma^k_1\bar\nabla(\beta_k(N)) - c_1\gamma^k_1\beta_k(N)\bar Z, u\big\rangle\\
&= -\rho\, c_1\langle\bar A(u),v\rangle - \rho(\gamma^i_1\langle\bar\nabla_u\,\beta^\sharp_i, v\rangle + \beta_i(v)\langle\bar\nabla\gamma^i_1,u\rangle)\\
&+(\rho^{ij}_0 - \rho^i_1\gamma^j_1)\big\langle\beta_i(N)\bar\nabla c_1 - (\frac12\gamma^k_1\bar\nabla b_{ki} - b_{ik}\bar\nabla\gamma^k_1)\\
&+(c_1 - \beta_k(N)\gamma^k_1)(\beta_i(N)\bar Z - \bar A^*(\beta^{\sharp\top}_i)), u\big\rangle\beta_j(v)\\
&+c_1\rho^j_1(1+s_k\gamma^k_1)\langle(c_1-\beta_k(N)\gamma^k_1)\bar Z + \gamma^k_1\bar A^*(\beta^{\sharp\top}_k), u\rangle\beta_j(v),
\end{aligned}
$$

where $u,v\in\mathcal{D}$. Formula for $g([v,n],u)$ is obtained from $g([u,n],v)$ after change $u\leftrightarrow v$. Substituting the above into (4.7), we find $g(\nabla_u\, n, v) = \langle\mathcal{A}(u),\, v\rangle$, where $\mathcal{A}$ is given in (4.4)–(4.5). In particular,

$$
\begin{aligned}
\langle 2\,\mathcal{A}(u),\beta^{\sharp\top}_i\rangle \;=\; &-2\,\rho\, c_1\langle\bar A^*(\beta^{\sharp\top}_i),u\rangle - 2\,\rho\gamma^j_1\langle\overline{\mathrm{Def}}_{\beta^\sharp_i}(\beta^{\sharp\top}_j),u\rangle\\
&+\; n(\rho)\beta_i(u) + \beta_j(u)\beta_i(U^j) + U^j(u)b^\top_{ij}.
\end{aligned}
$$

By Lemma 2.2 and $g(\nabla_u\, n, v) = -\phi\, g(A^g(u),v)$, see (4.1), we get (4.3). □

The elementary symmetric functions $\sigma_k(A)$ of a $m\times m$-matrix A (or a linear transformation) are defined by equality $\det(\mathrm{id}+tA) = \sum_{i\le m}\sigma_k(A)\,t^k$ and are called mean curvatures in the case of shape operator. Thus, $\sigma_0(A)=1,\ \sigma_1(A)=\mathrm{Tr}\,A,\ \ldots,\ \sigma_m(A)=\det A$.

Corollary 4.2 (The mean curvature of $\mathcal{D}$)**.** *Let conditions of Proposition 4.1 are satisfied. Then*

$$
\begin{aligned}
\rho\,\phi\sigma_1(A^g) \;=\; &\rho\, c_1\sigma_1(\bar A) - \frac{m}{2}\,n(\rho) + \rho\gamma^i_1(\overline{\mathrm{div}}\,\beta^\sharp_i - \beta_i(\bar Z) + N(\beta_i(N)))\\
&-\; \beta_j(U^j) - \gamma^{ij}_3\langle\mathcal{A}(\beta^{\sharp\top}_i),\, \beta^\sharp_j\rangle, \qquad (4.8)
\end{aligned}
$$

where U^j *are given in* (4.5) *and*

$$\begin{aligned}\langle\mathcal{A}(\beta_i^{\sharp\top}),\,\beta_j^{\sharp}\rangle \;=\; \rho\, c_1\langle\bar A^*(\beta_i^{\sharp\top}),\beta_j^{\sharp\top}\rangle + \rho\gamma_1^k\big(\beta_k^{\sharp\top}(b_{ij}^{\top})/2 - \beta_k(N)\langle\bar A^*(\beta_i^{\sharp\top}),\beta_j^{\sharp}\rangle\big)\\ - b_{ij}^{\top}(\frac12\, n(\rho) + \beta_k(U^k))\,. \qquad (4.9)\end{aligned}$$

Proof. Let $\{e_i\}$ be a local g-orthonormal frame of $\mathcal{D}$. We calculate

$$\langle\overline{\rm Def}_{\beta_k^\sharp}(\beta_i^{\sharp\top}),\beta_j^{\sharp\top}\rangle \;=\; \frac12\,\langle\bar\nabla b_{ij}^{\top},\,\beta_k^{\sharp\top}\rangle - \beta_k(N)\langle\bar A^*(\beta_i^{\sharp\top}),\beta_j^{\sharp}\rangle,$$

see (4.4)–(4.5). Tracing of (4.3), we obtain

$$\rho\phi\sigma_1(A^g) = -\sigma_1(\mathcal{A}) - \gamma_3^{ij}\langle\mathcal{A}(\beta_j^{\sharp\top}),\,\beta_i^{\sharp}\rangle.$$

Then, using

$${\rm Tr}\,(\overline{\rm Def}_{\beta_i^\sharp})^{\top}_{|T\mathcal{F}} = \overline{\rm div}\,\beta_i^{\sharp} - \beta_i(\bar Z) + N(\beta_i(N)),$$

(4.9) and Lemma 2.2, we get (4.8)–(4.9). □

Example 4.1. (i) One may ask the question: "When $\mathcal{D}$ is totally geodesic with respect to g, i.e., $A^g = 0$?" In this case, when $\bar\nabla\beta_i = 0$ and $\beta_i(N) = 0$, by Proposition 4.1, $\bar A$ has a special form

$$\bar A = W^i\otimes\beta_i + \omega^i\otimes\beta_i^{\sharp},$$

for some vector fields W^i and 1-forms ω^i. If $p = 1$ then, necessarily, ${\rm rank}\,\bar A \le 2$.

In next corollary and proposition, for simplicity, we assume that $\mathcal{D}$ is integrable and $p = 1$.

Corollary 4.3 (The second mean curvature)**.** *If* $p = 1$ *and* $\bar\nabla\beta^\sharp = 0$ *then*

$$\begin{aligned}(\rho\,\phi)^2\sigma_2(A^g) &= (\rho\, c_1)^2\sigma_2(\bar A) + \frac18\, m(m-1)\, n(\rho)^2 - \frac12\,(m-1)\, c_1\rho\, n(\rho)\,\sigma_1(\bar A)\\ &+ \frac14\,\beta(U)\langle 2\,\gamma_3\mathcal{A}(\beta^{\sharp\top}) + U,\,\beta^\sharp\rangle - \frac14\,(b^2 - \beta(N)^2)\langle 2\,\gamma_3\mathcal{A}(\beta^{\sharp\top}) + U,\,U\rangle\\ &+ (\frac{m-1}{2} n(\rho) - \rho c_1\sigma_1(\bar A))\langle\gamma_3\mathcal{A}(\beta^{\sharp\top}) + U,\beta^\sharp\rangle + \rho c_1\langle\gamma_3\mathcal{A}(\beta^{\sharp\top}) + U,\bar A(\beta^{\sharp\top})\rangle, \qquad (4.10)\end{aligned}$$

where $\mathcal{A} = -\rho\, c_1\bar A + {\rm Sym}(U\otimes\beta^{\top})$ *and* U *is given in* (4.5)*.*

Proof. By conditions, $\overline{\rm Def}_{\beta^\sharp} = 0$. Thus, by Proposition 4.1,

$$\rho\,\phi A^g = \rho\, c_1\bar A - \frac12\, n(\rho)\,{\rm id}^{\top} - A_1 - A_2,$$

where $A_1 = \frac12\, U\otimes\beta^{\top}$ and $A_2 = (\frac12\, U^\flat + \gamma_3(\beta\circ\mathcal{A}))\otimes\beta^{\sharp\top}$ are rank 1 matrices (thus $\sigma_2(A_i) = 0$) and

$$\mathcal{A} = -\rho\, c_1\bar A + \frac12\, n(\rho)\,{\rm id}^{\top} + {\rm Sym}(U\otimes\beta^{\top})$$

is symmetric. Applying the identity

$$\sigma_2(\sum\nolimits_i P_i) = \sum\nolimits_i\sigma_2(P_i) + \sum\nolimits_{i<j}\big(\sigma_1(P_i)\sigma_1(P_j) - \sigma_1(P_iP_j)\big),$$

to matrices $P_1 = \rho\, c_1 \bar A$, $P_2 = -\frac12\, n(\rho)\,\mathrm{id}^\top$, $P_3 = -A_1$ and $P_4 = -A_2$, and using equalities $\langle(\beta\circ\mathcal{A})^\sharp,\, u\rangle = \langle\mathcal{A}(u^\top),\, \beta^\sharp\rangle$ and $\sigma_2(\mathrm{id}^\top) = m(m-1)/2$, we get

$$\begin{aligned}
&(\rho\,\phi)^2\sigma_2(A^g) = (\rho\, c_1)^2\sigma_2(\bar A) + m(m-1)\, n(\rho)^2/8\\
&-\frac12\,(m-1)\,c_1\rho\, n(\rho)\,\sigma_1(\bar A) + \sigma_1(A_1)\,\sigma_1(A_2) - \sigma_1(A_1A_2)\\
&+\big((m-1)\,n(\rho)/2 - \rho\, c_1\sigma_1(\bar A)\big)\,\sigma_1(A_1+A_2) + \rho\, c_1\sigma_1(\bar A(A_1+A_2)),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1(A_1) &= \beta(U)/2, \quad \sigma_1(A_2) = \langle 2\,\gamma_3\mathcal{A}(\beta^{\sharp\top}) + U,\, \beta^\sharp\rangle/2,\\
\sigma_1(A_1A_2) &= (b^2-\beta(N)^2)\langle 2\gamma_3\mathcal{A}(\beta^\sharp) + U,\, U\rangle/4,\\
\sigma_1(\bar A(A_1+A_2)) &= \langle\gamma_3\mathcal{A}(\beta^{\sharp\top}) + U,\, \bar A(\beta^{\sharp\top})\rangle.
\end{aligned}$$

From the above (4.10) follows. □

In next proposition, we express Z through $\bar Z$, see (4.2), and invariants of $\mathcal{D}$ with respect to a.

Proposition 4.4. *Let g be a new Riemannian metric determined by an integrable distribution $\mathcal{D}$, a 1-form β and a function $\phi(s)$ on (M,a) with conditions* (2.4), (2.10). *Then*

$$\rho Z = \mathcal{Z} + \gamma_3\beta(\mathcal{Z})\,\beta^{\sharp\top},$$

where the vector field $\mathcal{Z}$ is given by

$$\begin{aligned}
\mathcal{Z} &= \big[p_1\bar\nabla^\top(\gamma_1/\phi) + p_2\bar\nabla^\top(c_1/\phi(s))\big]\phi(s)^{-1} + \big[p_3\bar Z + p_4\bar A(\beta^{\sharp\top}) + p_5\bar\nabla^\top(\beta(N))\big]\phi^{-2},\\
p_1 &= c_1\left((4\rho_1\gamma_1 - \rho_0 + 3\rho_1 s\gamma_1^2)b^2 - \rho + c_1^2\rho_1 s\right)\beta(N) - \rho_1(2s\gamma_1+1)c_1^2\beta(N)^2\\
&\quad -\ \rho_1(s\gamma_1+1)b^2c_1^2 + \gamma_1(\rho_0 - 2\gamma_1\rho_1 - \gamma_1^2\rho_1 s)b^4 + \gamma_1\rho b^2,\\
p_2 &= (\rho_0 - 2\rho_1 s\gamma_1^2 - 3\rho_1\gamma_1)\,c_1\beta(N)^2 + \big(\gamma_1(2\gamma_1\rho_1 + \gamma_1^2\rho_1 s - \rho_0)b^2\\
&\quad +\ \rho_1(2+3s\gamma_1)\,c_1^2 - \gamma_1\rho\big)\beta(N) - c_1^3\rho_1 s + (\rho - \gamma_1\rho_1(s\gamma_1+1)b^2)c_1,\\
p_3 &= \gamma_1(3\gamma_1\rho_1 + 2\gamma_1^2\rho_1 s - \rho_0)c_1\beta(N)^3 + ((\rho_0 - 5\rho_1 s\gamma_1^2 - 5\rho_1\gamma_1)c_1^2 + \gamma_1^2\rho\\
&\quad +\ \gamma_1^2(\rho_0 - 2\gamma_1\rho_1 - \gamma_1^2\rho_1 s)b^2)\beta(N)^2 + (2\rho_1(1+2s\gamma_1)c_1^3\\
&\quad +\ \gamma_1 c_1((3\gamma_1\rho_1 + 2\gamma_1^2\rho_1 s - \rho_0)b^2 - 2\rho))\beta(N) - c_1^4\rho_1 s + (\rho - \gamma_1\rho_1(s\gamma_1+1)b^2)c_1^2,\\
p_4 &= \gamma_1(\rho_0 - 2\gamma_1^2\rho_1 s - 3\gamma_1\rho_1)\,c_1\beta(N)^2 + \gamma_1 c_1((\rho_0 - 2\gamma_1\rho_1 - \gamma_1^2\rho_1 s)b^2 + \rho)\\
&\quad +\ [(4\rho_1\gamma_1 - \rho_0 + 3\rho_1 s\gamma_1^2)\,c_1^2 + \gamma_1^2(2\gamma_1\rho_1 + \gamma_1^2\rho_1 s - \rho_0)b^2 - \gamma_1^2\rho]\beta(N) - \rho_1(s\gamma_1+1)\,c_1^3,\\
p_5 &= \gamma_1[c_1^3\rho_1 s - \rho_1(2s\gamma_1+1)c_1^2\beta(N) + c_1(\gamma_1\rho_1(1+\gamma_1 s)b^2 - \rho)].
\end{aligned}$$

Moreover, if $\beta^\sharp$ is tangent to $\mathcal{D}$ and $b = \mathrm{const}$ then

$$\begin{aligned}
&\mathcal{Z} = \phi^{-2}\big\{\, c_1^2[\,\rho - c_1^2\rho_1 s - \gamma_1\rho_1(s\gamma_1+1)\,b^2]\bar Z\\
&+c_1[\gamma_1\rho - \rho_1(s\gamma_1+1)\,c_1^2 + \gamma_1(\rho_0 - 2\,\gamma_1\rho_1 - \gamma_1^2\rho_1 s)b^2]\bar A(\beta^\sharp)\big\}.
\end{aligned}$$

Proof. Extend $X\in T_x\mathcal{F}$ onto a neighborhood of a point $x\in M$ with the property $(\bar\nabla_Y X)^\top = 0$ for any $Y\in T_xM$. By formula (4.6), we obtain at x:

$$g(Z,X) = g([X,\nu],\nu). \tag{4.11}$$

Using equalities $\nu=\phi^{-1}(c_1N-\gamma_1\beta^\sharp)$ and $[X,fY]=X(f)Y+f[X,Y]$ we get

$$
\begin{aligned}
g([X,\nu],\nu) \;&=\; (c_1/\phi)\,X(c_1/\phi)\,g(N,N) - X(c_1\gamma_1/\phi^2)\,g(N,\beta^\sharp)\\
&+\; (\gamma_1/\phi)X(\gamma_1/\phi)\,g(\beta^\sharp,\beta^\sharp) + (c_1/\phi)^2 g([X,N],N)\\
&-\; (\gamma_1c_1/\phi^2)\,[\,g([X,\beta^\sharp],N)+g([X,N],\beta^\sharp)\,] + (\gamma_1/\phi)^2 g([X,\beta^\sharp],\beta^\sharp).
\end{aligned}
\tag{4.12}
$$

To compute first three terms in (4.12), by (2.2) for $p=1$,

$$
g(u,v)=\rho\langle u,v\rangle+\rho_0\beta(u)\beta(v)+\rho_1(\beta(u)\langle n,v\rangle+\beta(v)\langle n,u\rangle-\beta(n)\langle n,u\rangle\langle n,v\rangle),
\tag{4.13}
$$

and Lemma 2.1, we find

$$
\begin{aligned}
g(\beta^\sharp,\beta^\sharp) \;&=\; \rho\,b^2+\rho_0b^4+2\rho_1b^2s-\rho_1s^3,\\
g(N,\beta^\sharp) \;&=\; (\rho+\rho_0b^2+\rho_1s)\,\beta(N)+\rho_1(b^2-s^2)\langle n,N\rangle,\\
g(N,N) \;&=\; \rho+\rho_0\beta(N)^2+2\,\rho_1\beta(N)\langle n,N\rangle-\rho_1s\langle n,N\rangle^2.
\end{aligned}
$$

To compute last four terms in (4.12), we will use

$$
\begin{aligned}
&[X,\beta^\sharp]=[X,\beta^{\sharp\top}]+X(\beta(N))N+\beta(N)\big(\langle Z,X\rangle\,N-\bar A(X)\big),\\
&[X,N]=\bar\nabla_XN-\bar\nabla_NX=-\bar A(X)-\langle\bar\nabla_NX,\,N\rangle\,N=\langle\bar Z,\,X\rangle\,N-\bar A(X),
\end{aligned}
$$

and by (4.13) and Lemma 2.1, obtain the equalities

$$
\begin{aligned}
g([X,N],\,\beta^\sharp) \;&=\; (\rho+\rho_0b^2+\rho_1s)\langle[X,N],\beta^\sharp\rangle+\rho_1(b^2-s^2)\langle[X,N],n\rangle,\\
g([X,\beta^\sharp],\,\beta^\sharp) \;&=\; (\rho+\rho_0b^2+\rho_1s)\langle[X,\beta^\sharp],\beta^\sharp\rangle+\rho_1(b^2-s^2)\langle[X,\beta^\sharp],n\rangle,\\
g([X,N],\,N) \;&=\; \rho\langle[X,N],\,N\rangle+(\rho_0\beta(N)+\rho_1\langle n,N\rangle)\langle[X,N],\beta^\sharp\rangle\\
&+\;\rho_1(\beta(N)-s\langle n,N\rangle)\langle[X,N],n\rangle,\\
g([X,\beta^\sharp],\,N) \;&=\; \rho\langle[X,\beta^\sharp],\,N\rangle+(\rho_0\beta(N)+\rho_1\langle n,N\rangle)\langle[X,\beta^\sharp],\beta^\sharp\rangle\\
&+\;\rho_1(\beta(N)-s\langle n,N\rangle)\langle[X,\beta^\sharp],n\rangle.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&g([X,\nu],\nu)=(c_1/\phi)\,X(c_1/\phi)\,[\rho+\rho_0\beta(N)^2+2\,\rho_1\beta(N)\langle n,N\rangle\\
&\;-\;\rho_1s\langle n,N\rangle^2]-X(\gamma_1c_1/\phi^2)\,[(\rho+\rho_0b^2+\rho_1s)\,\beta(N)+\rho_1(b^2\\
&\;-\;s^2)\langle n,N\rangle]+(\gamma_1/\phi)X(\gamma_1/\phi)\,[\rho\,b^2+\rho_0b^4+2\rho_1b^2s-\rho_1s^3]\\
&\;+\;(c_1\phi)^2[\rho\langle[X,N],\,N\rangle+(\rho_0\beta(N)+\rho_1\langle n,N\rangle)\beta([X,N])\\
&\;+\;\rho_1(\beta(N)-s\langle n,N\rangle)\langle n,[X,N]\rangle]-(\gamma_1c_1/\phi^2)[\rho\langle[X,\beta^\sharp],N\rangle+(\rho_0\beta(N)\\
&\;+\;\rho_1\langle n,N\rangle)\beta([X,\beta^\sharp])+\rho_1(\beta(N)-s\langle n,N\rangle)\langle n,[X,\beta^\sharp]\rangle]\\
&\;+\;(\gamma_1c_1/\phi^2)\,[(\rho+\rho_0b^2+\rho_1s)\beta([X,N])+\rho_1(b^2-s^2)\langle n,[X,N]\rangle]\\
&\;+\;(\gamma_1^2/\phi^2)\,[(\rho+\rho_0b^2+\rho_1s)\beta([X,\beta^\sharp])+\rho_1(b^2-s^2)\langle n,[X,\beta^\sharp]\rangle].
\end{aligned}
$$

Note that $\langle n, N\rangle = c_1 - \gamma_1\beta(N)$ and $\beta(n) = c_1\beta(N) - \gamma_1 b^2$, see (2.5), and

$$\begin{aligned}
\langle [X,N],\, N\rangle &= \langle \bar Z, X\rangle,\\
\langle [X,N], \beta^\sharp\rangle &= \langle \beta(N)\bar Z - \bar A(\beta^{\sharp\top}),\, X\rangle,\\
\langle [X,N],\, n\rangle &= c_1\langle [X,N],\, N\rangle - \gamma_1\langle [X,N],\, \beta^\sharp\rangle\\
&= \langle (c_1{-}\gamma_1\beta(N))\bar Z + \gamma_1\bar A(\beta^{\sharp\top}),\, X\rangle,\\
\langle [X,\beta^\sharp],\, N\rangle &= \langle \bar\nabla(\beta(N)) + \beta(N)\bar Z,\, X\rangle,\\
\langle [X,\beta^\sharp], \beta^\sharp\rangle &= b\, X(b) - \langle \bar\nabla_{\beta^\sharp} X, \beta^\sharp\rangle = \langle b\,\bar\nabla b + \beta(N)^2\bar Z - \beta(N)\bar A(\beta^{\sharp\top}), X\rangle,\\
\langle [X,\beta^\sharp],\, n\rangle &= c_1\langle [X,\beta^\sharp],\, N\rangle - \gamma_1\langle [X,\beta^\sharp],\, \beta^\sharp\rangle\\
&= \langle (c_1\beta(N) - \gamma_1\beta(N)^2)\bar Z - \gamma_1 b\,\bar\nabla b + \gamma_1\beta(N)\bar A(\beta^{\sharp\top}), X\rangle.
\end{aligned}$$

By (4.11), $g(Z,\, X) = \langle \mathcal{Z},\, X\rangle$. With the help of Lemma 2.2 we complete the proof. □

5 The Reeb type integral formula

In this section we apply results in Sections 1 − 4 to prove a new integral formula for a closed Riemannian manifold with a set of linearly independent 1-forms and a codimension one distribution, which generalizes the Reeb's integral formula (0.1).

Theorem 5.1. *Let g be a new Riemannian metric determined by $\mathcal{D} = \ker\omega$, 1-forms β_i $(1 \le i \le p)$ on a closed Riemannian manifold (M, a) and a function $\phi(s)$, where $s = (s_1, \dots, s_p)$, with conditions* (2.4), (2.10). *Then*

$$\begin{aligned}
&\int_M \mu_g(n)(\rho\,\phi)^{-1}\big\{\rho\, c_1\sigma_1(\bar A) - (m/2)\, n(\rho) + \rho\,\gamma_1^i(\,\beta_i(\bar Z) - N(\beta_i(N))) + \beta_i(U^i)\\
&\quad - \gamma_3^{ij}\langle \mathcal{A}(\beta_i^{\sharp\top}), \beta_j^{\sharp\top}\rangle - \rho\,\phi(\beta_i^\sharp(\gamma_1^i\phi) + \gamma_1^i\phi\beta_i^\sharp(\log\mu_g(n)))\big\}\,\mathrm{d\,vol}_a = 0. \qquad (5.1)
\end{aligned}$$

Proof. For the metric g the Reeb's integral formula (0.1) reads

$$\int_M H_\beta\ \mathrm{d\,vol}_g = 0\,. \qquad (5.2)$$

By (5.2), we have

$$\int_M \mu_g(n)\,\sigma_1(A^g)\,\mathrm{d\,vol}_a = 0.$$

Corollary 4.2 and using $f^i\,\overline{\mathrm{div}}\,\beta_i^\sharp = \overline{\mathrm{div}}\,(f^i\beta_i^\sharp) - \beta_i^\sharp(f^i)$ with $f^i = \mu_g(n)\,\gamma_1^i/\phi$, yield (5.1). □

The integral formula (5.1) holds when all 1-forms are defined outside a closed submanifold of codimension ≥ 2 under convergence of some integrals, see discussion in [7, 16]. The singular case is important since many manifolds admit no codimension-one distributions or foliations, while all of them admit non-vanishing 1-forms outside some "set of singularities".

Corollary 5.2. *In conditions of Theorem 5.1 for $p = 1$, let b and $\beta(N)$ be constant. Then*

$$\int_M \langle q_1\bar A(\beta^{\sharp\top}) + q_2\bar Z,\ \beta^\sharp\rangle\,\mathrm{d\,vol}_a = 0, \qquad (5.3)$$

where the constants q_1 *and* q_2 *are given by*

$$q_1 \;=\; -\rho(\rho+(b^2-\beta(N)^2)\gamma_2)^{-1}(c_1\rho_1\gamma_1(1+s\,\gamma_1)+\gamma_2(c_1-\beta(N)\gamma_1)),$$
$$q_2 \;=\; \gamma_1\rho-c_1\rho_1\rho(\rho+(b^2-\beta(N)^2)\gamma_2)^{-1}(1+s\,\gamma_1)(c_1-\beta(N)\gamma_1).$$

Proof. If b and $\beta(N)$ are constant, that is $\beta^\sharp$ and its $\mathcal{D}^\perp$-component have constant lengths, then $s,\rho,\rho_i,\gamma_i,c_1$ and $\phi(s),\mu_g(n)$ are also constant. In this case, (5.1) yields (5.3). □

There are topological obstructions to the existence of codimension one totally geodesic and Riemannian foliations on a closed Riemannian manifold, see [4, 6]. For such foliations we get

Corollary 5.3. *In conditions of Theorem 5.1 for* $p=1$*, let* b *and* $\beta(N)$ *be constant. (i) If* $\bar A=0$ *and* $q_2\ne 0$ *then either* $\beta(\bar Z)\equiv 0$ *or* $\beta(\bar Z)_x\cdot\beta(\bar Z)_{x'}<0$ *for some points* $x\ne x'$*. (ii) If* $\bar Z=0$ *and* $q_1\ne 0$ *then either* $\langle\bar A(\beta^{\sharp\top}),\,\beta^\sharp\rangle\equiv 0$ *or* $\langle\bar A(\beta^{\sharp\top}),\,\beta^\sharp\rangle_x\cdot\langle\bar A(\beta^{\sharp\top}),\beta^\sharp\rangle_{x'}<0$ *for some points* $x\ne x'$*.*

Example 5.1. (i) For Randers metric ($p=1$), by (5.1) we get, see [13],

$$\begin{aligned}&\int_M (c\,c_1)^{m+1}c^{-1}\Big((c\,c_1)\,\sigma_1(\bar A)-\frac{m+2}{2}\,(N+c_1^{-1}\beta^\sharp)(c\,c_1)+c_1N(c)\\ &\quad-(c_1-c)\big[N(c)+\langle c^{-1}\bar A(\beta^{\sharp\top})+\bar Z,\beta^\sharp\rangle\big]\Big)\,{\rm d\,vol}_a=0,\end{aligned}\tag{5.4}$$

which is the Reeb formula when $\beta=0$. If $\beta(N)=0$ then (5.4) reads

$$\int_M c^{2m+1}\big(c^2\sigma_1(\bar A)-(m+1)\,c\,N(c)-(m+2)\,\beta^\sharp(c)\big)\,{\rm d\,vol}_a=0.$$

If b and $\beta(N)\ne 0$ are constant then (5.4) reads $\int_M\langle\bar A(\beta^{\sharp\top})+c\,\bar Z,\,\beta^\sharp\rangle\,{\rm d\,vol}_a=0$, see also (5.3) with $q_1=c^{-1}c_1(c-c_1)$ and $q_2=c_1(c-c_1)$.

(ii) For Kropina metric, if $\beta(N)=0$ then $\mu_g(n)=(2/b)^{2m+2}$, and

$$\gamma_1=-\sqrt2/(2b),\quad \gamma_2=0,\quad c_1=1/\sqrt2,$$
$$s=b/\sqrt2,\quad \rho=4/b^2,\quad \rho_0=12/b^4,\quad \rho_1=-8\sqrt2/b^3.$$

Hence, by Proposition 4.1 for $p=1$, $\sigma_1(A^g)=\frac b2\,\sigma_1(\bar A)-\frac12\,\overline{\rm div}\,\beta^\sharp+\frac{m}{\sqrt2}\,n(b)+\frac{1}{2\,b}\,\beta^\sharp(b)$, and, we get integral formula

$$\int_M\Big(\frac2b\Big)^{2m+2}\Big\{b\,\sigma_1(\bar A)+\sqrt2\,m\,n(b)-\frac{2m+1}{b}\,\beta^\sharp(b)\Big\}\,{\rm d\,vol}_a=0\,,$$

which for $b=\rm const$ reduces to (0.1) for metric a.

(iii) The following application of (5.3) (when b and $\beta(N)$ are constant) seems to be interesting. Let $\bar Z=0$, $q_1\ne0$ and α-unit vector field $X\in\mathfrak{X}_\mathcal{D}$ be an eigenvector of $\bar A$ with an eigenvalue $\lambda: M\setminus\Sigma\to\mathbb{R}$. Then $\beta^\sharp=\varepsilon' X+\varepsilon N$, where $\varepsilon=\rm const\in(0,\delta_0)$ and $\varepsilon'=\rm const\in(0,\sqrt{1-\varepsilon^2})$, obeys (5.3). Thus, $\int_M\lambda\ {\rm d\,vol}_a=0$. Consequently, either $\lambda\equiv0$ on M or $\lambda(x)\,\lambda(x')<0$ for some points $x\ne x'$. Furthermore, this implies Reeb formula (0.1) for $\langle\cdot,\cdot\rangle$:

$$\int_M\sigma_1(\bar A)\,{\rm d\,vol}_a=\sum\nolimits_i\int_M\lambda_i\,{\rm d\,vol}_a=0.$$

6 The counterpart of Reeb integral formula

In this section we assume for simplicity that $\mathcal{D}$ is integrable and $p = 1$, and use (α, β)-metrics.

The counterpart of the Reeb integral formula for the second mean curvature reads

$$\int_M (2\,\sigma_2(\bar A) - \overline{\mathrm{Ric}}_{N,N})\,\mathrm{d}\,\mathrm{vol}_a = 0. \tag{6.1}$$

Here $\overline{\mathrm{Ric}}_{N,N} = \mathrm{Tr}_a(u \to \bar R_{N,u}\,N)$ is the Ricci curvature of a in the N-direction. The proof of (6.1), see e.g. [11], is based on the Divergence theorem applied to

$$\overline{\mathrm{div}}\,(\sigma_1(\bar A)\,N + \bar Z) = \overline{\mathrm{Ric}}_{N,N} - 2\,\sigma_2(\bar A).$$

We will generalize (6.1) for codimension one foliations with general (α, β)-metrics on M. In this case, the volume form of g with μ_g given in (3.6) obeys

$$\mathrm{d}\,\mathrm{vol}_g = \mu_g(n)\,\mathrm{d}\,\mathrm{vol}_a\,. \tag{6.2}$$

Let $\mathrm{Ric}^g_{\nu,\nu} = \mathrm{Tr}_g(u \to R^g_{\nu,u}\,\nu)$ be the Ricci curvature of g in the ν-direction, where $R^g_{u,v} = [\nabla_v, \nabla_u] - \nabla_{[v,u]}$ is the curvature tensor derived using the Levi-Civita connection of g. The *Chern connection* D^ν is torsion free and almost metric, it is determined by

$$g(D^\nu_u\,v, w) - g(\nabla_u\,v, w) = C_\nu(D^\nu_w\,\nu, u, v) - C_\nu(D^\nu_u\,\nu, v, w) - C_\nu(D^\nu_v\,\nu, u, w), \tag{6.3}$$

see [14], for any vector fields u, v, w, where $g(\nabla_u\,v,\,w)$ is given in (4.6).

The difference $\mathcal{T} = D^\nu - \nabla$ is called the *contorsion tensor*. It is a symmetric tensor because both connections, ∇ and D^ν, are torsion-free. By (6.3), $D^\nu_\nu\,\nu = \nabla_\nu\,\nu$ holds; hence, $\mathcal{T}_\nu\,\nu = 0$ (thus, ν is geodesic for F if and only if it is geodesic for g).

Comparing the curvature $R^D_{u,v} = [D^\nu_v, D^\nu_u] - D^\nu_{[v,u]}$ of D^ν with $R^g_{u,v}$, we find

$$R^D_{\nu,u} - R^g_{\nu,u} = (\nabla_u \mathcal{T})_\nu - (\nabla_\nu \mathcal{T})_u - [\mathcal{T}_\nu,\,\mathcal{T}_u], \quad u \in TM. \tag{6.4}$$

In [5], the Ricci curvature $\mathrm{Ric}^D_y = \mathrm{Tr}_g(u \to R^D_{y,u}\,y)$ of (α, β)-metric is expressed through $\overline{\mathrm{Ric}}_y$ of α; in particular, $\bar\nabla\beta = 0$ provides $\mathrm{Ric}^D_y = \overline{\mathrm{Ric}}_y$ $(y \ne 0)$.

Let $C^\sharp_\nu$ be a $(1,1)$-tensor g-dual to the symmetric bilinear form $C_\nu(Z, \cdot\,, \cdot)$:

$$g(C^\sharp_\nu(u), v) = C_\nu(Z, u, v), \quad u, v \in TM.$$

Note that $A^g + C^\sharp_\nu$ is the shape operator of the leaves with respect to D^ν, see [13]. By (6.3), we get

$$\mathcal{T}_\nu = -C^\sharp_\nu, \quad \mathrm{Tr}\,\mathcal{T}_\nu = -\sigma_1(C^\sharp_\nu) = -I_\nu(Z). \tag{6.5}$$

Unlike Theorem 5.1, the following theorem contains non-Riemannian quantities.

Theorem 6.1. *Let g be a new metric determined by a codimension-one foliation $\mathcal{F}$ ($T\mathcal{F} = \mathcal{D}$), a 1-form β on (M, a), and a function $\phi(s)$ with the conditions* (2.4), (2.10) *and* $\bar\nabla\beta^\sharp=0$. *Then*

$$
\begin{aligned}
&\int_M\Big\{\Big[\,(c_1\rho)^2\,(\underline{2\,\sigma_2(\bar A)-\overline{\rm Ric}_{\,N,N}})+\frac14\,m(m-1)\,n(\rho)^2-(m-1)\,c_1\rho\,n(\rho)\,\sigma_1(\bar A)\\
&+\frac12\,\beta(U)\langle 2\gamma_3\mathcal{A}(\beta^{\sharp\top})+U,\beta^\sharp\rangle-\frac12(b^2-\beta(N)^2)\langle 2\gamma_3\mathcal{A}(\beta^{\sharp\top})+U,U\rangle-\big(2\rho\,c_1\sigma_1(\bar A)\\
&-(m-1)\,n(\rho)\big)\,\langle\gamma_3\mathcal{A}(\beta^{\sharp\top})+U,\,\beta^\sharp\rangle+2\,\rho\,c_1\langle\gamma_3\mathcal{A}(\beta^{\sharp\top})+U,\,\bar A(\beta^{\sharp\top})\rangle\Big](\rho\,\phi(s))^{-2}\\
&-I_\nu((A^g+C^\sharp_\nu+\sigma_1(A^g)\,{\rm id})Z)-2\sigma_1(A^gC^\sharp_\nu)-\sigma_1((C^\sharp_\nu)^2)\Big\}\,\mu_g(n)\,{\rm d\,vol}_a=0\,, \qquad (6.6)
\end{aligned}
$$

where A^g, $\mathcal{A}$ *and* U *are given in Proposition 4.1,* Z *is given in Proposition 4.4 and* $\mu_g(n)$ *is given in* (3.6) *with* $y=n$ *and* $s=\beta(n)$.

Proof. We will use the adjoint (1,2)-tensor $\mathcal{T}^*$ defined by

$$g(\mathcal{T}^*_u v,w)=g(\mathcal{T}_u w,v)$$

for $u,v,w\in TM$. Note that $\mathcal{T}^*_\nu\,\nu=0$ and define ${\rm Tr}_g\,\mathcal{T}^*=\sum_i\mathcal{T}^*_{b_i}\,b_i$ – the trace of $\mathcal{T}^*$ with respect to g. Assuming $(\nabla_\nu\,b_i)^\top=0$ and $(\nabla_{b_i}\,\nu)^\perp=0$ at a point $x\in M$, calculate at x:

$$
\begin{aligned}
\sum\nolimits_i g((\nabla_i\,\mathcal{T})_\nu\,\nu,b_i)\,&=\,2\sum\nolimits_i g(\mathcal{T}^*_\nu\,b_i,\,A^g(b_i))=2\sigma_1(C^\sharp_\nu A^g),\\
\sum\nolimits_i g((\nabla_\nu\,\mathcal{T})_i\,\nu,b_i)\,&=\,{\rm div}_g({\rm Tr}_g\,\mathcal{T}^*),\quad \sum\nolimits_i g([\mathcal{T}_i,\,\mathcal{T}_\nu]\,\nu,b_i)=-\sigma_1((C^\sharp_\nu)^2),
\end{aligned}
$$

using the symmetry $\mathcal{T}_i\,\nu=\mathcal{T}_\nu\,b_i$. Then, applying (6.4) we get

$$
\begin{aligned}
{\rm Ric}^D_{\nu,\nu}-{\rm Ric}^g_{\nu,\nu}\,&=\,\sum\nolimits_i[\,g((\nabla_i\mathcal{T})_\nu\,\nu,b_i)-g((\nabla_\nu\mathcal{T})_i\,\nu,b_i)+g([\mathcal{T}_i,\,\mathcal{T}_\nu]\,\nu,b_i)\,]\\
&=\,2\sigma_1(C^\sharp_\nu A^g)-\sigma_1((C^\sharp_\nu)^2)-{\rm div}^\perp_g({\rm Tr}^\top\,\mathcal{T}^*). \qquad (6.7)
\end{aligned}
$$

From (6.7) and

$${\rm div}^\perp_g({\rm Tr}_g\,\mathcal{T}^*)={\rm div}_g(({\rm Tr}_g\,\mathcal{T}^*)^\perp)-g({\rm Tr}_g\,\mathcal{T}^*,\,\sigma_1(A^g)\,\nu-Z)$$

we obtain

$$
\begin{aligned}
{\rm div}_g(({\rm Tr}_g\,\mathcal{T}^*)^\perp)\,&=\,{\rm Ric}^g_{\nu,\nu}-{\rm Ric}^D_{\nu,\nu}\\
&+\,g({\rm Tr}_g\,\mathcal{T}^*,\sigma_1(A^g)\,\nu-Z)-2\sigma_1(A^gC^\sharp_\nu)-\sigma_1((C^\sharp_\nu)^2). \qquad (6.8)
\end{aligned}
$$

Then, using (6.3) and (6.5), we find

$$
\begin{aligned}
g({\rm Tr}_g\,\mathcal{T}^*,\,\nu)&=-\sum\nolimits_i C_\nu(D^\nu_\nu\,\nu,b_i,b_i)=-\sigma_1(C^\sharp_\nu)=-I_\nu(Z),\\
g({\rm Tr}_g\,\mathcal{T}^*,\,u)&=-\sum\nolimits_i C_\nu(D^\nu_u\,\nu,b_i,b_i)=I_\nu((A^g+C^\sharp_\nu)(u))
\end{aligned}
$$

for $u\in\mathcal{D}$. By the above we obtain

$$g({\rm Tr}_g\,\mathcal{T}^*,\,\sigma_1(A^g)\,\nu-Z)=-I_\nu((A^g+C^\sharp_\nu+\sigma_1(A^g)\,{\rm id})Z).$$

By conditions, $b = \mathrm{const}$ and $\bar R(X,Y)\beta^\sharp = 0$ $(X, Y \in TM)$. Using

$$\mathrm{Ric}^D_{n,n} = \overline{\mathrm{Ric}}_{\,n,n} = c_1^2\, \overline{\mathrm{Ric}}_{\,N,N} + \gamma_1^2\, \overline{\mathrm{Ric}}_{\,\beta^\sharp,\beta^\sharp} - 2\, c_1\gamma_1 \sum\nolimits_i \langle \bar R(N, b_i)\beta^\sharp, b_i\rangle$$

and $\mathrm{Ric}^D_{\nu,\nu} = \phi^{-2}\,\mathrm{Ric}^D_{n,n}$, we find

$$\mathrm{Ric}^D_{\nu,\nu} = (c_1/\phi)^2\, \overline{\mathrm{Ric}}_{N,N}.$$

By the above, (6.1) and (6.2) for g, using (6.8) and Corollary 4.3, we find (6.6). □

Corollary 6.2. *In conditions of Theorem 6.1, let* $\beta(N) = \mathrm{const}$, $\bar Z = 0$ *and* $q_3 \ne 0$, *where*

$$\begin{aligned} q_3 &= \frac{q\rho(4\,\rho\, c_1 - (b^2-\beta(N)^2)q) - 4\rho^2 c_1^2\gamma_2}{4(\rho + (b^2-\beta(N)^2)\gamma_2)},\\ q &= \rho_1 c_1\gamma_1(1 + s\,\gamma_1) - (\rho_0 - \rho_1\gamma_1)\,(c_1 - \beta(N)\gamma_1) - \gamma_1\gamma_2\beta(N). \end{aligned}$$

Then $\bar A(\beta^{\sharp\top}) = 0$, *hence* $\mathrm{rank}(\bar A) < m$. *If* $\mathcal F$ *is totally umbilical then* $\mathcal F$ *is totally geodesic.*

Proof. By conditions, $s, \rho, \rho_i, \gamma_i, c_1$ are constant (since b and $\beta(N)$ are constant) and $\mathrm{Ric}^D_{\nu,\nu} = \mathrm{Ric}^g_{\nu,\nu}$. Hence, see (6.8),

$$\int_M \big\{ g(\mathrm{Tr}_g\, \mathcal T^*,\, \sigma_1(A^g)\,\nu - Z) - 2\sigma_1(A^g C^\sharp_\nu) - \sigma_1((C^\sharp_\nu)^2)\big\}\, \mathrm d\,\mathrm{vol}_g = 0.$$

Thus, (6.6) and (6.1) yield

$$\begin{aligned} &\int_M \Big\{ \frac14\, \beta(U)\langle 2\,\gamma_3\mathcal A(\beta^{\sharp\top}) + U,\, \beta^\sharp\rangle - \frac14\,(b^2-\beta(N)^2)\langle 2\,\gamma_3\mathcal A(\beta^{\sharp\top}) + U,\, U\rangle \\ &\quad -\rho\, c_1\sigma_1(\bar A)\langle \gamma_3\mathcal A(\beta^{\sharp\top}) + U,\, \beta^\sharp\rangle + \rho\, c_1\langle \gamma_3\mathcal A(\beta^{\sharp\top}) + U,\, \bar A(\beta^{\sharp\top})\rangle\Big\}\, \mathrm d\,\mathrm{vol}_a = 0, \end{aligned} \tag{6.9}$$

where, in view of $\bar\nabla^\top_n\, \beta^{\sharp\top} = -\gamma_1\beta(N)\bar A(\beta^{\sharp\top})$, we have

$$U = q\bar A(\beta^{\sharp\top}), \quad \mathcal A = -\rho\, c_1\bar A + q\,\mathrm{Sym}(\bar A(\beta^{\sharp\top})\otimes\beta^\top).$$

If $\beta(N) = \mathrm{const}$ then $\beta(\bar Z) = 0$ and $\langle \bar A(\beta^{\sharp\top}), \beta^{\sharp\top}\rangle = 0$:

$$\begin{aligned} 0 &= \langle \bar\nabla_N\, \beta^\sharp,\, N\rangle = \langle \bar\nabla_N\,(\beta^{\sharp\top} + \beta(N)N),\, N\rangle = -\langle \beta^\sharp,\, \bar Z\rangle,\\ 0 &= \langle \bar\nabla_{\beta^{\sharp\top}}\beta^\sharp,\, N\rangle = \langle \bar\nabla_{\beta^{\sharp\top}}(\beta^{\sharp\top} + \beta(N)N),\, N\rangle = -\langle \bar A(\beta^{\sharp\top}), \beta^\sharp\rangle. \end{aligned}$$

By (6.9),

$$\int_M q_3\|\bar A(\beta^{\sharp\top})\|^2_\alpha\, \mathrm d\,\mathrm{vol}_a = 0,$$

and $q_3 \ne 0$ yields $\bar A(\beta^{\sharp\top}) \equiv 0$. If $\mathcal F$ is totally umbilical then $0 = \langle \bar A(\beta^{\sharp\top}), \beta^{\sharp\top}\rangle = \|\beta^{\sharp\top}\|^2_a\sigma_1(\bar A)$, hence $\sigma_1(\bar A) = 0$. By the above, $\bar A = 0$ on M. □

Example 6.1. For Randers metric, we obtain $q_3 = \frac14\, c^2c_1^2((c - 2\,c_1)^2 - 1)$ with $c_1 = c + \beta(N)$ and $c = \sqrt{1 - b^2 + \beta(N)^2}$. For Kropina metric, we have $q_3 = -\frac{1}{16}\,\beta(N)(16\,c_1 s^3 + b^2\beta(N) - \beta(N)^3)\, s^{-10}$ with $s = \sqrt{b(b - \beta(N))/2}$.

Let $k_1 \le k_2 \le \ldots \le k_m$ be the eigenvalues of A^g. One can consider the integral

$$U_{\mathcal F} = \int_M \sum\nolimits_{i<j} (k_i - k_j)^2 \,\mathrm{d}\,\mathrm{vol}_g,$$

which measures "how far from g-umbilicity" is a foliation $\mathcal F$, see [6] for Riemannian case. Put

$$\mu_{\min} = \min_{y\in TM\setminus\{0\}} \mu_{g_y}(y).$$

Theorem 6.3. *Let g be a new Riemannian metric determined by a codimension-one foliation $\mathcal F$, a 1-form β on (M,a), and a function ϕ with conditions* (2.4), (2.10), $\bar\nabla\beta = 0$, $\beta(N) = \mathrm{const}$ *and* $\overline{\mathrm{Ric}}_{N,N} \le -r < 0$. *Then*

$$U_{\mathcal F} \ge m\, r\, (c_1/\phi(s))^{\,2} \mu_{\min} \mathrm{Vol}_a(M). \tag{6.10}$$

In particular, if $c_1 \ne 0$ then $\mathcal F$ is nowhere g-totally umbilical.

Proof. One may show that

$$\sum\nolimits_{i<j} (k_i - k_j)^2 = (m-1)\,\sigma_1^2(A^g) - 2\,m\,\sigma_2(A^g).$$

Hence, and by (6.1) for g we obtain

$$U_{\mathcal F} \ge -m \int_M 2\,\sigma_2(A^g)\,\mathrm{d}\,\mathrm{vol}_g = -m\int_M \mathrm{Ric}^g_{\nu,\nu}\,\mathrm{d}\,\mathrm{vol}_g\,.$$

By conditions, $\mathrm{Ric}^g_{\nu,\nu} = (c_1/\phi(s))^{\,2}\,\overline{\mathrm{Ric}}_{\,N,N}$, and $s,\rho,\rho_i,\gamma_i,c_1,\phi(s),\mu_g(\nu)$ are constant. Thus,

$$U_{\mathcal F} \ge -m\,(c_1/\phi(s))^{\,2} \mu_{\min} \int_M \overline{\mathrm{Ric}}_{\,N,N}\,\mathrm{d}\,\mathrm{vol}_a,$$

which reduces to (6.10) since our assumption $\overline{\mathrm{Ric}}_{N,N} \le -r < 0$. □

Following [3] for Riemannian case, define the *energy of a vector field ν* by

$$\mathcal E(\nu) = \frac{m+1}{2}\,\mathrm{Vol}_g(M) + \frac12 \int_M \|D\nu\|_g^2\,\mathrm{d}\,\mathrm{vol}_g\,.$$

By (6.1) for g and the inequality $\|D\nu\,\|_g^2 \ge \frac{2}{m}\,\sigma_2(A^g)$, see [3], we get the following.

Theorem 6.4. *Let g be a new Riemannian metric determined by a codimension-one foliation $\mathcal F$, a 1-form β on (M,a), and a function ϕ with conditions* (2.4), (2.10), $\bar\nabla\beta^\sharp = 0$ *and* $\beta(N) = \mathrm{const}$. *Then for a unit g-normal ν,*

$$\mathcal E(\nu) \ge \mu_{\min}\Big(\frac{m+1}{2}\,\mathrm{Vol}_a(M) + \frac{c_1^2}{2\,m\,\phi^2}\int_M \overline{\mathrm{Ric}}_{\,N,N}\,\mathrm{d}\,\mathrm{vol}_a\Big).$$

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