A study on $N(k)-$contact metric manifolds

Halil İbrahim Yoldaş and Erol Yaşar

Abstract. In this paper, we study $N(k)-$contact metric manifolds. Firstly, we characterize the $N(k)-$contact metric manifold endowed with a concircular vector field. Then, we discuss $N(k)-$contact metric manifolds admitting quasi-Yamabe solitons and obtain some necessary conditions for an $N(k)-$contact metric manifold to be Sasakian. Finally, we investigate $N(k)-$contact metric manifolds satisfying the curvature conditions $R.h = 0$, $h.R = 0$, $R.Q = 0$ and $Q.R = 0$.

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1 Introduction

The first study on $N(k)-$contact metric manifolds was given by Tanno in [21]. In this paper, Tanno obtained that if the structure vector field ξ belongs to the $k-$nullity distribution on an Einstein compact Riemannian manifold M of dimension $2n+1 \geq 5$, then $k = 1$ and M is Sasakian. Then, Blair et al. extended $N(k)-$contact metric manifolds to the $(k, \mu)-$contact metric manifolds in 1995 [5]. Further, $N(k)-$contact metric manifolds and $(k, \mu)-$contact metric manifolds have been studied by many mathematicians (e.g., see [10]-[12], [16], [17], [20] and [22]).

The notion of Yamabe soliton in Riemannian geometry was introduced by Hamilton as special solutions of the Yamabe flow [13]. Yamabe solitons naturally arise as limits of dilations of singularities in the Yamabe flow. A Yamabe soliton is a Riemannian manifold (M, g) if it admits a vector field V such that

$$(\pounds_V g)(X, Y) = (\lambda - r)g(X, Y), \tag{1.1}$$

where $\pounds_V g$ is the Lie-derivative of the metric tensor g in the direction vector field V, which is called soliton field of the Yamabe soliton (M, g), λ is a real number, r is the scalar curvature of M, and X, Y are the vector fields on M. A Yamabe soliton which satisfies (1.1) is denoted by (M, g, V, λ). Also, a Yamabe soliton is called a gradient if the soliton field V is the gradient of a smooth function $-\beta$ (i.e.,

$V = -\nabla\beta$) and is called shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively. If $\pounds_V g = 0$ and $\pounds_V g = \rho g$, then the soliton field V is said to be Killing and conformal Killing, respectively, where ρ is a function.

In [9], Chen and Deshmukh introduced the notion of quasi-Yamabe soliton (as a generalization of Yamabe soliton) on a Rimannian manifold (M, g), as follows:

$$(\pounds_V g)(X, Y) = (\lambda - r)g(X, Y) + \mu V^\star(X)V^\star(Y), \tag{1.2}$$

where $V^\star$ is the dual $1-$form of V, λ is a real number and μ is a smooth function on M. The vector field V is also called soliton field for the quasi-Yamabe soliton. A quasi-Yamabe soliton is denoted by (M, g, V, λ, μ).

On the other hand, a vector field v on a Riemannian manifold (M, g) is called concircular if it satisfies

$$\nabla_X v = fX \tag{1.3}$$

for any $X \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M and f is a smooth function on M. If f is equal to one in (1.3), then v is called a concurrent vector field. Moreover, if f vanishes identically in (1.3) the vector field v is called a parallel vector field [6]. Concircular vector fields play an important role in the theory of projective and conformal transformations. The integral curves of concircular vector fields are geodesics. Therefore, such vector fields are known as geodesic fields in literature [18]. There has been several works about concircular and concurrent vector fieds in literature (see [6]-[8], [14], [15], [19] and [24]).

The present paper is organized as follows:

Section 1 is concerned with introduction. In section 2, we give some basic notions about almost contact metric manifolds and $N(k)-$contact metric manifolds. In section 3, we deal with $N(k)-$contact metric manifolds endowed with a concircular vector field and obtain some results about this manifold. In section 4, we investigate $N(k)-$contact metric manifolds admitting quasi-Yamabe soliton and give some characterizations for such a manifold. In last section, we study $N(k)-$contact metric manifolds satisfying certain curvature conditions.

2 Preliminaries

In this section, we recall some fundamental notations and formulas of almost contact metric manifolds from [2] and [3].

A differentiable manifold M of dimension $(2n+1)$ is said to be an almost contact metric manifold if it admits an almost contact metric structure (φ, ξ, η, g) and the Riemannian metric g satisfies the following relations:

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi) \tag{2.1}$$

and

$$g(\varphi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\varphi X, Y) = -g(X, \varphi Y) \tag{2.2}$$

for any $X, Y \in \Gamma(TM)$, where ξ is a vector field of type $(0, 1)$, (which is so-called the characteristic vector field), $1-$ form η is the $g-$dual of ξ of type $(1, 0)$ and φ is a tensor field of type $(1, 1)$ on M.

On the other hand, in [2], D.E. Blair defined the fundamental $2-$form Φ of M as follows:

$$\Phi(X,Y)=g(X,\varphi Y)$$

for any $X,Y\in\Gamma(TM)$. Furthermore, an almost contact metric manifold M is called a contact metric manifold if it satisfies

$$\Phi(X,Y)=d\eta(X,Y).$$

The Nijenhuis tensor field of φ is defined by

$$N_\varphi(X,Y)=[\varphi X,\varphi Y]+\varphi^2[X,Y]-\varphi[X,\varphi Y]-\varphi[\varphi X,Y]$$

for all $X,Y\in\Gamma(TM)$. If M is an almost contact metric manifold and the Nijenhuis tensor of φ satisfies

$$N_\varphi+2d\eta\otimes\xi=0$$

then, M is called a normal contact metric manifold. A normal contact metric manifold M is called Sasakian. An almost contact metric manifold M is Sasakian if and only if

$$(\nabla_X\varphi)Y=g(X,Y)\xi-\eta(Y)X).$$

For a Sasakian manifold, we also have

$$\begin{aligned}\nabla_X\xi &= -\varphi X,\\ R(X,Y)\xi &= \eta(Y)X-\eta(X)Y,\end{aligned}$$

where ∇ and R are the Levi-Civita connection and the Riemannian curvature tensor on M, respectively.

The $(k,\mu)-$nullity distribution on contact metric manifolds was introduced by Blair et al. and defined in [5]

$$\begin{aligned}N(k,\mu):p\to N_p(k,\mu) &= \{Z\in T_pM|R(X,Y)Z\\ &= (kI+\mu h)(g(Y,Z)X-g(X,Z)Y)\},\end{aligned}\tag{2.3}$$

where $(k,\mu)\in\mathbb{R}^2$, I is an identity map and h is the tensor field of type $(1,1)$ defined by $h=\frac{1}{2}\pounds_\xi\varphi$. This tensor field satisfy

$$h\xi=0,\qquad h\varphi+\varphi h=0,\qquad \nabla_X\xi=-\varphi X-\varphi hX,\tag{2.4}$$

and

$$g(hX,Y) = g(X,hY),\tag{2.5}$$

$$\eta(hX) = 0.\tag{2.6}$$

A contact metric manifold M is called a $(k,\mu)-$contact metric manifold, if ξ belongs to $(k,\mu)-$nullity distribution $N(k,\mu)$. If μ vanishes identically in (2.3), then the

$(k,\mu)-$nullity distribution $N(k,\mu)$ reduces to $k-$nullity distribution $N(k)$ and is given by [21]

$$\begin{aligned} N(k): p \rightarrow N_p(k) &= \{Z \in T_pM | R(X,Y)Z \\ &= k(g(Y,Z)X - g(X,Z)Y)\}. \end{aligned}$$

If $\xi \in N(k)$, then a contact metric manifold M is called an $N(k)-$contact metric manifold [21]. Also, if $k=1$, an $N(k)-$contact metric manifold is Sasakian. If $k=0$, then the manifold is locally isometric to the product $E^{n+1} \times S^4$ for $n>1$ and flat for $n=1$ [4]. For an $N(k)-$contact metric manifold, the followings are satisfied [4]:

$$h^2 = (k-1)\varphi^2, \tag{2.7}$$
$$(\nabla_X \varphi)Y = g(X+hX,Y)\xi - \eta(Y)hX, \tag{2.8}$$
$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{2.9}$$
$$R(\xi,X)Y = k(g(X,Y)\xi - \eta(Y)X), \tag{2.10}$$
$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad +[2nk - 2(n-1)]\eta(X)\eta(Y), \quad n \geq 1 \end{aligned} \tag{2.11}$$
$$S(X,\xi) = 2nk\eta(X), \tag{2.12}$$
$$Q\xi = 2nk\xi, \tag{2.13}$$
$$r = 2n(2n-2+k), \tag{2.14}$$

where r stands for the scalar curvature, S is the Ricci tensor and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

On the other hand, a Riemannian manifold (M,g) is called $\eta-$Einstein if there exists two real constants a and b such that the Ricci tensor field S of M satisfies

$$S = ag + b\eta \otimes \eta.$$

If the constants b and a are equal to zero, then M is called Einstein and a special type of $\eta-$Einstein, respectively ([1], [23]). Also, on a Riemannian manifold (M,g), we have the followings

$$\begin{aligned} & g(\nabla\beta, X) = X(\beta), \\ & H^\beta(X,Y)) = g(\nabla_X(\nabla\beta), Y), \end{aligned} \tag{2.15}$$

where $\nabla\beta$ and H^β are the gradient of a function β on M and the Hessian of β, respectively [8].

Example 2.1. [11] We consider the three-dimensional manifold

$$M = \{(x,y,z) \in \mathbb{R}^3, (x,y,z) \neq (0,0,0)\},$$

where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$. Let e_1, e_2 and e_3 be the linearly independent vector fields in $\mathbb{R}^3$ which satisfies

$$[e_1,e_2] = (1+\lambda)e_3, \quad [e_1,e_3] = -(1-\lambda)e_2 \quad \textit{and} \quad [e_2,e_3] = 2e_1,$$

where λ is a real number. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_i, e_i) &= 1 \\ g(e_i, e_j) &= 0 \quad for \quad i \neq j. \end{aligned}$$

Also, let η, φ be the $1-$ form and the $(1,1)-$tensor field, respectively defined by

$$\eta(Z, e_1) = 1, \quad \varphi(e_2) = e_3, \quad \varphi(e_3) = -e_2, \quad \varphi(e_1) = 0$$

for any $Z \in \Gamma(TM)$. Furthermore,

$$he_1 = 0, \quad he_2 = \lambda e_2, \quad and \quad he_3 = -\lambda e_3.$$

On the other hand, using Koszul's formula for the Riemannian metric g, we have:

$$\begin{aligned} &\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \\ &\nabla_{e_3} e_2 = -(1-\lambda) e_1, \qquad \nabla_{e_3} e_1 = (1-\lambda) e_2 \\ &\nabla_{e_2} e_1 = -\nabla_{e_2} e_3 = -(1+\lambda) e_3. \end{aligned}$$

Therefore, $(M, \varphi, \xi, \eta, g)$ is a $3-$dimensional a contact metric manifold. Using the above equations, one has

$$\begin{aligned} &R(e_1, e_2)e_3 = 0, && R(e_1, e_3)e_2 = 0, && R(e_2, e_3)e_1 = 0, \\ &R(e_1, e_2)e_2 = (1-\lambda^2)e_1, && R(e_1, e_2)e_1 = -(1-\lambda^2)e_2, \\ &R(e_1, e_3)e_3 = (1-\lambda^2)e_1, && R(e_1, e_3)e_1 = -(1-\lambda^2)e_3, \\ &R(e_2, e_3)e_3 = -(1-\lambda^2)e_2, && R(e_2, e_3)e_2 = (1-\lambda^2)e_3. \end{aligned}$$

Hence, The manifold M is a $3-$dimensional $N(k)-$contact metric manifold.

3 $N(k)-$contact metric manifolds endowed with a concircular vector field

In this section, we deal with an $N(k)-$contact metric manifold endowed with a concircular vector field and obtain some important characterizations such a manifold.

Now, we begin to this section with the following:

Proposition 3.1. *Let M be an $N(k)-$contact metric manifold. Then, the characteristic vector field ξ cannot be the gradient $\nabla\beta$ of a function β on M.*

Proof. Let us assume that the structure vector field ξ is the gradient $\nabla\beta$ of a function β on M, that is, $\nabla\beta = \xi$. From (2.4), the Hessian H^β of β satisfies

$$\begin{aligned} H^\beta(X, Y) &= g(\nabla_X(\nabla\beta), Y) \\ &= g(\nabla_X \xi, Y) \\ &= -g(\varphi X, Y) - g(\varphi h X, Y) \end{aligned} \tag{3.1}$$

for any $X, Y \in \Gamma(TM)$. Since the Hessian H^β of β is a symmetric in X and Y, from (3.1) one has

$$-g(\varphi X, Y) - g(\varphi h X, Y) = -g(\varphi Y, X) - g(\varphi h Y, X). \tag{3.2}$$

From (2.2), (2.4) and (3.2), we get

$$2g(\varphi X, Y) = 0$$

and hence

$$d\eta(X, Y) = 0. \tag{3.3}$$

Removing X, Y in equation (3.3), we have

$$d\eta = 0.$$

Since $d\eta \neq 0$, this is a contradiction. Therefore, the proof is completed. □

Now, we shall give an important theorem of this section.

Theorem 3.2. *Let M be an $N(k)-$contact metric manifold. Then, the characteristic vector field ξ is not concircular on M.*

Proof. Let us assume that the structure vector field ξ is a concircular on M. Then, we write

$$\nabla_X \xi = fX \tag{3.4}$$

for any $X \in \Gamma(TM)$. With the help of (2.4) and (3.4), one has

$$fX = -\varphi X - \varphi hX. \tag{3.5}$$

Taking the inner product of (3.5) with vector field φY, we get

$$fg(X, \varphi Y) = -g(\varphi X, \varphi Y) - g(\varphi hX, \varphi Y). \tag{3.6}$$

Also, interchanging the roles of X and Y in (3.6) gives

$$fg(Y, \varphi X) = -g(\varphi Y, \varphi X) - g(\varphi hY, \varphi X). \tag{3.7}$$

Adding (3.6) and (3.7) and using (2.1), (2.2), (2.4)-(2.6), we have

$$g(\varphi X, \varphi Y) = -g(hX, Y). \tag{3.8}$$

Replacing X by hX in (3.8), we write

$$g(\varphi hX, \varphi Y) = -g(h^2 X, Y). \tag{3.9}$$

From (2.1), (2.2), (2.6) and (2.7) we get

$$-g(\varphi^2 hX, Y) = -g((k-1)\varphi^2 X, Y).$$

and hence

$$g(hX, Y) = (k-1)g(\varphi X, \varphi Y). \tag{3.10}$$

Combining (3.8) with (3.10) yields

$$kg(hX, Y) = 0 \tag{3.11}$$

Again, replacing X by hX in (3.11) and making use of (2.2), (2.7) one has

$$k(k-1)g(\varphi X, \varphi Y) = 0,$$

equivalent to

$$k(k-1)d\eta(\varphi X, Y) = 0.$$

Since $d\eta \neq 0$, $k = 0$ or $k = 1$. If $k = 1$, then $h = 0$. From (3.8), one can write

$$d\eta(\varphi X, Y) = g(\varphi X, \varphi Y) = -g(hX, Y) = 0.$$

This is a contradiction. Therefore, we have $k = 0$. If we use (2.2) and (3.8) in (3.6), we get

$$fg(X, \varphi Y) = fd\eta(X, Y) = 0,$$

which implies that $f = 0$. Then, from (2.4) and (3.4) we find that

$$0 = (\pounds_\xi g)(X, Y) = 2g(hX, \varphi Y)$$

Since $h \neq 0$, this is a contradiction. Thus, the vector field ξ is not concircular on M, which completes the proof of the theorem. □

Next theorem is the final result of this section.

Theorem 3.3. *Let M be an $N(k)-$contact metric manifold endowed with a concircular vector field v. If the vector field hv is a concircular on M, then M is either locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$, or M is Sasakian.*

Proof. Let hv be a concircular vector field on M. Then, we have

$$\nabla_X hv = fX \tag{3.12}$$

for any $X \in \Gamma(TM)$, where f is a function on M. If we take the inner product of (3.12) with ξ, one has

$$g(\nabla_X hv, \xi) = f\eta(X). \tag{3.13}$$

Also, using the fact that $g(hv, \xi) = 0$ and from (2.4) we write

$$\begin{aligned} g(\nabla_X hv, \xi) &= -g(hv, \nabla_X \xi) \\ &= g(hv, \varphi X) + g(hv, \varphi hX), \end{aligned}$$

then the equation (3.13) becomes

$$g(hv, \varphi X) + g(hv, \varphi hX) = f\eta(X). \tag{3.14}$$

Replacing X by hX in (3.14) and using (2.1), (2.6), (2.7), we obtain

$$g(hv, \varphi hX) - (k-1)g(hv, \varphi X) = 0. \tag{3.15}$$

Putting $X = hX$ in (3.15) and (2.7) yields

$$-(k-1)g(hv, \varphi X) - (k-1)g(hv, \varphi hX) = 0. \tag{3.16}$$

From (3.15) and (3.16), we find

$$-(k-1)g(hv, \varphi X) - (k-1)^2 g(hv, \varphi X) = 0.$$

and we further infer

$$k(k-1)d\eta(hv, X) = 0.$$

Since $d\eta \neq 0$, $k = 0$ or $k = 1$. Thus, we get the desired result. □

4 $N(k)-$contact metric manifolds admitting quasi-Yamabe solitons

In this section, we characterize $N(k)-$contact metric manifolds admitting a quasi-Yamabe soliton defined by (1.2), and obtain some necessary conditions for such manifolds to be Sasakian. We begin with the following:

Theorem 4.1. *Let M be an $N(k)-$contact metric manifold. If M admits a quasi-Yamabe soliton as its soliton field ξ, then M has either constant scalar curvature, or M is Sasakian.*

Proof. It follows from the definition of the Lie derivative and from (2.2), (2.4), that we have

$$(\pounds_\xi g)(X, Y) = 2g(hX, \varphi Y), \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$. Since M is a quasi-Yamabe soliton with soliton field ξ, from (1.2) and (4.1) one has

$$2g(hX, \varphi Y) = (\lambda - r)g(X, Y) + \mu\eta(X)\eta(Y). \tag{4.2}$$

Also, if we replace X by hX in (4.2) and use (2.1), (2.6), (2.7), we get

$$-2(k-1)g(X, \varphi Y) = (\lambda - r)g(hX, Y). \tag{4.3}$$

By interchanging the roles of X and Y in (4.3), we obtain

$$-2(k-1)g(Y, \varphi X) = (\lambda - r)g(hY, X). \tag{4.4}$$

Then (2.2), (2.5), (4.3) and (4.4) yield

$$0 = 2(\lambda - r)g(hX, Y). \tag{4.5}$$

Again, replacing X by hX in (4.5) and using the fact that $h^2 = (k-1)\varphi^2$, we get

$$0 = (k-1)(\lambda - r)g(\varphi X, \varphi Y),$$

which implies that

$$0 = (k-1)(\lambda - r)d\eta(\varphi X, Y).$$

Since $d\eta \neq 0$, $k = 1$ or $\lambda = r$. This completes the proof of the theorem. □

Using the equality (4.2), we can state the following.

Corollary 4.2. *Let M be an $N(k)-$contact metric manifold admitting a quasi-Yamabe soliton as its soliton field the structure vector field ξ. If M has constant scalar curvature $r = \lambda$, then the structure vector field ξ is a Killing on M.*

Now, we shall give the main theorem of this section.

Theorem 4.3. *Let M be an $N(k)-$contact metric manifold. If M admits a quasi-Yamabe soliton whose the soliton field V is a pointwise collinear with ξ, then soliton field V is either a constant multiple of ξ or M is Sasakian.*

Proof. Let V be a pointwise collinear with the structure vector field ξ, that is, $V = b\xi$, where b is a function on M. Then, from (1.2), we have

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) = (\lambda - r)g(X,Y) + \mu b^2\eta(X)\eta(Y) \tag{4.6}$$

for any $X, Y \in \Gamma(TM)$. From (2.2), (2.4) and (4.6), one has

$$X(b)\eta(Y) + Y(b)\eta(X) + 2g(hX, \varphi Y) = (\lambda - r)g(X,Y) + \mu b^2\eta(X)\eta(Y). \tag{4.7}$$

By replacing X by hX in (4.7) and using (2.1), (2.6), (2.7) we infer

$$hX(b)\eta(Y) - 2(k-1)g(X, \varphi Y) = (\lambda - r)g(hX, Y). \tag{4.8}$$

Interchanging the roles of X and Y in (4.8), we write

$$hY(b)\eta(X) - 2(k-1)g(Y, \varphi X) = (\lambda - r)g(hY, X). \tag{4.9}$$

Adding (4.8) and (4.9) and from (2.2), we get

$$hX(b)\eta(Y) + hY(b)\eta(X) = 2(\lambda - r)g(hX, Y). \tag{4.10}$$

By putting $Y = \xi$ in (4.10) and making use of (2.6), (2.15) gives

$$g(\nabla b, hX) = 0. \tag{4.11}$$

Since the Riemannian metric g is non-degenere, the equation (4.11) implies that $\nabla b = 0$ or $hX = 0$, which completes the proof. □

Proposition 4.4. *Let M be an $N(k)-$contact metric manifold admitting a quasi-Yamabe soliton whose soliton field V is a pointwise collinear with the structure vector field ξ. If M has constant scalar curvature $r = \lambda$, then the soliton field V is the gradient ∇b of a function b provided $\mu.b = 2$, where λ and μ are defined by (1.2).*

Proof. Since M has constant scalar curvature $r = \lambda$, then from (4.7) we have

$$X(b)\eta(Y) + Y(b)\eta(X) + 2g(hX, \varphi Y) = \mu b^2 \eta(X)\eta(Y). \tag{4.12}$$

for any $X, Y \in \Gamma(TM)$. If we take ξ instead of X and Y in (4.12) and use (2.1), (2.4), then we get

$$\xi(b) = \frac{1}{2}\mu b^2. \tag{4.13}$$

Also, by subsituting $Y = \xi$ in (4.12) and using (4.13), we get

$$X(b) = \frac{1}{2}\mu b^2 \eta(X)$$

equivalently,

$$g(\nabla b, X) = g(\frac{1}{2}\mu b^2 \xi, X). \tag{4.14}$$

Removing X in equation (4.14), one has

$$\nabla b = \frac{1}{2}\mu b^2 \xi.$$

Using the fact that $\mu.b = 2$ in the above equation, we obtain

$$\nabla b = b\xi$$

and hence

$$\nabla b = V.$$

This is the desired result. Therefore, the proof is completed. □

5 $N(k)-$contact metric manifolds satisfying the curvature conditions $R.h = 0$, $h.R = 0$, $R.Q = 0$ and $Q.R = 0$

In this section, we investigate $N(k)-$contact metric manifolds satisfying certain curvature conditions and give some characterization theorems which classify these manifolds.

The first result of this section is the following:

Theorem 5.1. *Let M be an $N(k)-$contact metric manifold such that the condition $R.h = 0$ is satisfied. Then, M is either locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$, or M is Sasakian.*

Proof. Let us assume that an $N(k)-$contact metric manifold satisfies the condition $(R(X, Y).h)Z = 0$, that is,

$$R(X, Y)hZ - h(R(X, Y)Z) = 0 \tag{5.1}$$

for any $X, Y, Z \in \Gamma(TM)$, where R denotes the Riemann curvature tensor and h denotes the tensor field defined by $h = \frac{1}{2}\pounds_\xi \varphi$. By setting $X = \xi$ in (5.1), we have

$$R(\xi, Y)hZ - h(R(\xi, Y)Z) = 0 \tag{5.2}$$

Also, by virtue of (2.4),(2.6), (2.10) and (5.2), we get

$$k(g(Y, hZ)\xi) - k\eta(Z)hY)) = 0. \tag{5.3}$$

By replacing Z by hZ in (5.3) and using (2.2) (2.6), (2.7), we obtain

$$-k(k-1)g(\varphi Y, \varphi Z) = 0$$

and hence

$$k(k-1)d\eta(\varphi Y, Z) = 0.$$

Since $d\eta \neq 0$, $k = 0$ or $k = 1$. Thus, the proof is completed. □

Theorem 5.2. *Let M be an $N(k)-$contact metric manifold such that the condition $h.R = 0$ is satisfied. Then, M is either locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$, or M is Sasakian.*

Proof. Let us assume that an $N(k)-$contact metric manifold satisfies the condition $(h.R)(X, Y)Z = 0$, namely

$$h(R(X, Y)Z) - R(hX, Y)Z - R(X, hY)Z - R(X, Y)hZ = 0 \tag{5.4}$$

for any $X, Y, Z \in \Gamma(TM)$. Putting $X = \xi$ in (5.4) and using (2.4) lead to

$$h(R(\xi, Y)Z) - R(\xi, hY)Z - R(\xi, Y)hZ = 0. \tag{5.5}$$

Furthermore, if we employ (2.10) in (5.5) and use (2.4)-(2.6), then the equation (5.5) becomes

$$2kg(hY, Z) = 0. \tag{5.6}$$

By replacing Y by hY in (5.6) and by making use of (2.2), (2.7), the equation (5.6) reduces to

$$2k(k-1)g(\varphi Y, \varphi Z) = 0,$$

that is,

$$k(k-1)d\eta(\varphi Y, Z) = 0.$$

Since $d\eta \neq 0$, $k = 0$ or $k = 1$. This result ends the proof of the theorem. □

Theorem 5.3. *Let M be an $N(k)-$contact metric manifold such that the condition $R.Q = 0$ is satisfied. Then, M is either locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$, or M is an Einstein.*

Proof. Let us suppose that an $N(k)-$contact metric manifold satisfies the condition $(R(X,Y).Q)Z = 0$, that is,

$$R(X,Y)QZ - Q(R(X,Y)Z) = 0 \tag{5.7}$$

for any $X, Y, Z \in \Gamma(TM)$, where Q stands for the Ricci operator defined by $S(X,Y) = g(QX,Y)$. Substitution of $X = \xi$ in (5.7) gives

$$R(\xi,Y)QZ - Q(R(\xi,Y)Z) = 0. \tag{5.8}$$

Moreover, by virtue of (2.10) and (5.8), we write

$$k(g(Y,QZ)\xi - \eta(QZ)Y) - Q(k(g(Y,Z)\xi - \eta(Z)Y)) = 0. \tag{5.9}$$

Taking the inner product of (5.9) with the vector field T and using (2.1), (2.12), we have

$$\begin{aligned} kS(Y,Z)\eta(T) - 2nk^2\eta(Z)g(Y,T) - 2nk^2g(Y,Z)\eta(T) \\ +k\eta(Z)S(Y,T) = 0. \end{aligned} \tag{5.10}$$

Putting $T = \xi$ in (5.10) and making use of (2.1), (2.12), we derive

$$kS(Y,Z) - 2nk^2g(Y,Z) = 0.$$

Therefore, we have

$$k(S(Y,Z) - 2nkg(Y,Z)) = 0$$

which implies that

$$k = 0$$

or

$$S(Y,Z) = 2nkg(Y,Z).$$

Hence, we get the requested result. □

Theorem 5.4. *Let M be an $N(k)-$contact metric manifold such that the condition $Q.R = 0$ is satisfied. Then, M is either locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$, or M is a special type of $\eta-$Einstein.*

Proof. Let us assume that an $N(k)-$contact metric manifold satisfies the condition $(Q.R)(X,Y)Z = 0$, namely

$$Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ = 0 \tag{5.11}$$

for any $X, Y, Z \in \Gamma(TM)$. Substituting $X = \xi$ in (5.11), one has

$$Q(R(\xi,Y)Z) - R(Q\xi,Y)Z - R(\xi,QY)Z - R(\xi,Y)QZ = 0. \tag{5.12}$$

For the first and second term of (5.12), using (2.10) and (2.13) we have

$$Q(R(\xi,Y)Z) = 2nk^2 g(Y,Z)\xi - k\eta(Z)QY, \tag{5.13}$$

$$R(Q\xi,Y)Z = 2nk^2 g(Y,Z)\xi - 2nk^2\eta(Z)Y. \tag{5.14}$$

For the third and fourth term of (5.12), after using (2.10) and (2.12), we derive

$$R(\xi,QY)Z = kS(Y,Z)\xi - k\eta(Z)QY, \tag{5.15}$$

$$R(\xi,Y)QZ = kS(Y,Z)\xi - 2nk^2\eta(Z)Y. \tag{5.16}$$

If we use the equations (5.13)-(5.16) in (5.12), we obtain

$$-2kS(Y,Z)\xi + 4nk^2\eta(Z)Y = 0. \tag{5.17}$$

Also, taking the inner product of (5.13) with ξ, we get

$$4nk^2\eta(Z)\eta(Y) - 2kS(Y,Z) = 0$$

which yields

$$-2k(S(Y,Z) - 2nk\eta(Z)\eta(Y)) = 0.$$

Hence,

$$k = 0$$

or

$$S(Y,Z) = 2nk\eta(Z)\eta(Y)$$

which completes the proof of the theorem. □

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Author's address:

Halil İbrahim Yoldaş and Erol Yaşar,
Department of Mathematics, Faculty of Science and Arts,
Mersin University, 33343, Mersin, Turkey.
E-mail: hibrahimyoldas@gmail.com, eroly69@gmail.com