On a deformed Riemannian extension of affine Szabó connections

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Abstract. In this paper, we exhibit example of Szabó metrics of neutral signature, which is obtained by the deformed Riemannian extension.

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1 Introduction

Let M be an n-dimensional manifold with a torsion free affine connection ∇ and let T^*M be its cotangent bundle. In [17], Patterson and Walker introduced the notion of Riemannian extensions and showed how a pseudo-Riemannian metric can be given to the $2n$-dimensional cotangent bundle of an n-dimensional manifold with given non-Riemannian structure. They shows that Riemannian extension provides a solution of the general problem of embedding a manifold M carrying a given structure in a manifold N carrying another structure, the embedding being carried out in such a way that the structure on N induces in a natural way the given structure on M. The Riemannian extension can be constructed with the help of the coefficients of the torsion free affine connection.

In [11], the authors generalized the Riemannian extension to the so-called deformed Riemannian extensions. In this paper, we shall consider some of the geometric aspects of deformed Riemannian extensions and we will investigate the spectral geometry of the Szabó operator on M and on T^*M.

Our paper is organized as follows. In the section 2, we recall some basic definitions and results on the classical Riemann extension and the deformed Riemannian extension developed in the book [11]. In section 3, we recall some results on affine Szabó manifolds. Finally in section 4, we construct example of pseudo-Riemannian Szabó metrics of signature $(2, 2)$, using the deformed Riemannian extensions, whose Szabó operator is nilpotent.

Throughout this paper, all manifolds, tensors fields and connections are always assumed to be differentiable of class $\mathcal{C}^\infty$.

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2 Deformed Riemannian extensions

Let (M,∇) be an n-dimensional affine manifold and T^*M be its cotangent bundle. Let $\pi : T^*M \to M$ be the natural projection defined by

$$\pi(p,\omega) = p \in M \quad \text{and} \quad (p,\omega) \in T^*M.$$

A system of local coordinates $(U,u_i), i = 1,\dots,n$ around $p \in M$ induces a system of local coordinates $(\pi^{-1}(U), u_i, u_{i'} = \omega_i)$, $i' = n+i = n+1,\dots,2n$ around $(p,\omega) \in T^*M$, where $u_{i'} = \omega_i$ are components of covectors ω in each cotangent space T^*_pM, $p \in U$ with respect to the natural coframe $\{du^i\}$. If we use the notation $\partial_i = \dfrac{\partial}{\partial u_i}$ and $\partial_{i'} = \dfrac{\partial}{\partial \omega_i}$, then at each point $(p,\omega) \in T^*M$, its follows that

$$\{(\partial_1)_{(p,\omega)},\dots,(\partial_n)_{(p,\omega)},(\partial_{1'})_{(p,\omega)},\dots,(\partial_{n'})_{(p,\omega)}\},$$

is a basis for the cotangent space $(T^*M)_{(p,\omega)}$.

For each vector field X on M, define a function $\iota X : T^*M \longrightarrow \mathbb{R}$ by

$$\iota X(p,\omega) = \omega(X_p).$$

This function is locally expressed by,

$$\iota X(u_i,u_{i'}) = u_{i'}X^i,$$

for all $X = X^i\partial_i$. Vector fields on T^*M are characterized by their actions on functions ιX. The complete lift X^C of a vector field X on M to T^*M is characterized by the identity

$$X^C(\iota Z) = \iota[X,Z], \quad \text{for all} \quad Z \in \Gamma(TM).$$

Moreover, since a $(0,s)$-tensor field on M is characterized by its evaluation on complete lifts of vector fields on M, for each tensor field T of type $(1,1)$ on M, we define a 1-form ιT on T^*M which is characterized by the identity

$$\iota T(X^C) = \iota(TX).$$

For more details on the geometry of cotangent bundle, we refer to the book of Yano and Ishihara [20].

Let ∇ be a torsion free affine connection on an n-dimensional affine manifold M. The Riemannian extension g_∇ is the pseudo-Riemannian metric on T^*M of neutral signature (n,n) characterized by the identity [11]

$$g_\nabla(X^C,Y^C) = -\iota(\nabla_XY + \nabla_YX).$$

In the locally induced coordinates $(u_i,u_{i'})$ on $\pi^{-1}(U) \subset T^*M$, the Riemannian extension is expressed by

$$g_\nabla = \begin{pmatrix} -2u_{k'}f^k_{ij} & \delta^j_i \\ \delta^j_i & 0 \end{pmatrix},$$

with respect to $\{\partial_1,\dots,\partial_n,\partial_{1'},\dots,\partial_{n'}\}(i,j,k = 1,\dots,n; k' = k+n)$, where f^k_{ij} are the coefficients of the torsion free affine connection ∇ with respect to (U,u_i) on M.

Riemannian extensions were originally defined by Patterson and Walker [17] and further studied by Afifi [1], thus relating pseudo-Riemannian properties of T^*M with the affine structure of the base manifold (M, ∇). Moreover, Riemannian extensions were also considered by Garcia-Rio et al. [10] in relation to Osserman manifolds (see also [5]). Since Riemannian extensions provide a link between affine and pseudo-Riemannian geometries, some properties of the affine connection ∇ can be investigated by means of the corresponding properties of the Riemannian extension g_∇. For instance, (M, ∇) is locally symmetric if and only if (T^*M, g_∇) is locally symmetric. Furthermore (M, ∇) is projectively flat if and only if (T^*M, g_∇) is locally conformally flat (see [4] for more details and references therein). For Riemannian extensions, also see [15, 16, 19].

In [4], the authors introduced a deformation of the Riemannian extension above by means of a symmetric $(0, 2)$-tensor field ϕ on M; more precisely, they consider the cotangent bundle T^*M equipped with the metric $g_\nabla + \pi^*\phi$, which is called the deformed Riemannian extension.

Let ϕ be a symmetric $(0, 2)$-tensor field on an affine manifold (M, ∇) and let π be the natural projection from the cotangent bundle T^*M to M. The deformed Riemannian extension $g_{\nabla,\phi}$ is the metric of neutral signature (n, n) on the cotangent bundle given by:

$$g_{\nabla,\phi} = g_\nabla + \pi^*\phi.$$

Let f_{ij}^k be the coefficients of the torsion free affine connection ∇ and let ϕ_{ij} be the local components of the symmetric $(0, 2)$-tensor field ϕ. In local coordinates the deformed Riemannian extension [11] is given by

$$g_{(\nabla,\phi)} = \begin{pmatrix} \phi_{ij}(u) - 2u_{k'} f_{ij}^k & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix},$$

with respect to $\{\partial_1, \ldots, \partial_n, \partial_{1'}, \ldots, \partial_{n'}\}, (i, j, k = 1, \ldots, n; k' = k + n)$. Equivalently,

$$g_{(\nabla,\phi)}(\partial_i, \partial_j) = \phi_{ij}(u) - 2u_{k'} f_{ij}^k; \quad g_{(\nabla,\phi)}(\partial_i, \partial_{j'}) = \delta_i^j; \quad g_{(\nabla,\phi)}(\partial_{i'}, \partial_{j'}) = 0$$

Note that the crucial terms $g_{(\nabla,\phi)}(\partial_i, \partial_j)$ now no longer vanish on the 0-section. The Walker distribution is the kernel of the projection from T^*M:

$$\mathcal{D} = \ker\{\pi^*\} = \mathrm{Span}\{\partial_{i'}\}.$$

The tensor ϕ plays an essential role. Even if the underlying connection is flat, the deformed Riemannian extension need not be flat [11]. Deformed Riemannian extensions have nilpotent Ricci operator and hence, they are Einstein if and only if they are Ricci flat. They can be used to construct non-flat Ricci flat pseudo-Riemannian manifolds [4]. For deformed Riemannian extensions, also see [2, 3, 6].

The classical and deformed Riemannian extensions provide a link between the affine geometry of (M, ∇) and the neutral signature metric on T^*M. Some properties of the torsion free affine connection ∇ can be investigated by means of the corresponding properties of the classical and deformed Riemannian extensions.

3 The affine Szabó manifolds

Let (M, ∇) be an affine manifold and $X \in \Gamma(T_pM)$. The affine Szabó operator [13] $\mathcal{S}^{\nabla}(X)$ with respect to X and $p \in M$ is given by $\mathcal{S}^{\nabla}(X) : T_pM \to T_pM$ such that

$$\mathcal{S}^{\nabla}(X)Y = (\nabla_X \mathcal{R}^{\nabla})(Y, X)X,$$

for any vector field Y. The affine Szabó operator satisfies $\mathcal{S}^{\nabla}(X)X = 0$ and $\mathcal{S}^{\nabla}(\beta X) = \beta^3 \mathcal{S}(X)$, for $\beta \in \mathbb{R}^*$ and $X \in T_pM$. If $Y = \partial_m$, for $m = 1, 2, \dots, n$ and $X = \sum_i \alpha_i \partial_i$, one gets

$$\mathcal{S}^{\nabla}(X)\partial_m = \sum_{i,j,k=1}^{n} \alpha_i \alpha_j \alpha_k (\nabla_i \mathcal{R}^{\nabla})(\partial_m, \partial_j)\partial_k,$$

where $\nabla_i = \nabla_{\partial_i}$.

Definition 3.1. [7] Let (M, ∇) be a smooth affine manifold and $p \in M$.

(i) (M, ∇) is called affine Szabó at $p \in M$ if the affine Szabó operator $\mathcal{S}^{\nabla}$ has the same characteristic polynomial for every vector field X on M.

(ii) (M, ∇) is called affine Szabó if (M, ∇) is affine Szabó at each $p \in M$.

Theorem 3.1. [7] *Let (M, ∇) be an n-dimensional affine manifold and $p \in M$. Then (M, ∇) is affine Szabó at $p \in M$ if and only if the characteristic polynomial of the affine Szabó operator $\mathcal{S}^{\nabla}$ is $P_\lambda[\mathcal{S}^{\nabla}(X)] = \lambda^n$, for every $X \in T_pM$.*

We have a complete description of affine Szabó surfaces.

Theorem 3.2. [7] *Let (M, ∇) be a two-dimensional smooth affine manifold. Then (M, ∇) is affine Szabó at $p \in M$ if and only if the Ricci tensor of (M, ∇) is cyclic parallel at $p \in M$.*

Let Σ be a surface endowed the torsion free affine connection ∇ given by

$$\nabla_{\partial_1}\partial_1 = f_1(u_1)\partial_2 \quad \text{and} \quad \nabla_{\partial_1}\partial_2 = f_2(u_1)\partial_2.$$

The non zero component of the curvature tensor $\mathcal{R}$ of the torsion free affine connection ∇ is

$$\mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_1 = a\partial_2,$$

where $a = \partial_1 f_2 + f_2^2$. The non zero component of the Ricci tensor Ric is

$$\text{Ric}(\partial_1, \partial_1) = -a$$

Let $X = \sum_i \alpha_i \partial_i$, $i = 1, 2$ be a vector on M, then, the affine Szabó operator is given by

$$(\nabla_X \mathcal{R}^{\nabla})(\partial_1, X)X = A\partial_2, \;\; (\nabla_X \mathcal{R}^{\nabla})(\partial_2, X)X = B\partial_2,$$

where the coefficients A and B are given by

$$A = \alpha_1^2 \alpha_2 \partial_1 a + \alpha_1 \alpha_2^2 \partial_2 a$$
$$B = -\alpha_1^3 \partial_1 a - \alpha_1^2 \alpha_2 \partial_2 a.$$

The matrix associated to $\mathcal{S}^\nabla(X)$ with respect to the basis $\{\partial_1, \partial_2\}$ is given by

$$(\mathcal{S}^\nabla(X)) = \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix}.$$

Its characteristic polynomial is given by $P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^2 - \lambda B$. Since (M, ∇) is affine Szabó, by Theorem 3.1, 0 is the only eigenvalue of the affine Szabó operator $\mathcal{S}^\nabla(X)$. Therefore, $\mathrm{trace}(\mathcal{S}^\nabla(X)) = B = 0$, which implies that

$$\partial_1 a = 0, \quad \text{and} \quad \partial_2 a = 0.$$

The converse is obvious. We have the following:

Theorem 3.3. *Let (Σ, ∇) be an affine surface endowed with the torsion free affine connection ∇ given by $\nabla_{\partial_1}\partial_1 = f_1(u_1)\partial_2$ and $\nabla_{\partial_1}\partial_2 = f_2(u_1)\partial_2$. Then (Σ, ∇) is affine Szabó if and only if $\partial_1 a = 0$ and $\partial_2 a = 0$, where $a = \partial_1 f_2 + f_2^2$.*

An n-dimensional affine manifold (M, ∇) is a generalized affine plane wave manifold if

$$\nabla_{\partial_i}\partial_j = \sum_{k>\max(i,j)} \Gamma_{ij}^k(u_1, \dots, u_{k-1})\partial_k.$$

If $n = 2$, then the only possible non-zero coefficient of the torsion free affine connection could be $\Gamma_{11}^2(u_1)$. Now, we have the following observation:

Theorem 3.4. *Let (Σ, ∇) be a generalized affine plane wave surface. Then (Σ, ∇) is affine Szabó.*

In higher dimensions the situation is however more involved where the cyclic parallelism of the Ricci tensor is a necessary but not sufficient condition for an affine connection to be Szabó. Affine Szabó connections are of interest not only in affine geometry, but also in the study of pseudo-Riemannian Szabó metrics since they provide some nice examples without Riemannian analogue by means of the Riemannian extensions. See [8, 9] for more details.

4 The deformed Riemannian extensions of an affine Szabó manifold

A pseudo-Riemannian manifold (M, g) is said to be Szabó if the Szabó operators $\mathcal{S}(X) = (\nabla_X R)(\cdot, X)X$ has constant eigenvalues on the unit pseudo-sphere bundles $S^\pm(TM)$. Any Szabó manifold is locally symmetric in the Riemannian [18] and the Lorentzian [12] setting but the higher signature case supports examples with nilpotent Szabó operators (cf. [14] and the references therein). Next, we will use the deformed Riemannian construction to exhibit a four-dimensional Szabó metric.

Let $M = \mathbb{R}^2$ and ∇ be the torsion free connection defined by

$$\nabla_{\partial_1}\partial_1 = f_1(u_1)\partial_2 \quad \text{and} \quad \nabla_{\partial_1}\partial_2 = f_2(u_1)\partial_2. \tag{4.1}$$

The deformed Riemannian extension of the torsion free affine connection defined by (4.1) is the pseudo-Riemannian metric tensor on $\mathbb{R}^4$ of signature $(2,2)$ given by

$$\begin{aligned} g_{\nabla,\phi} &= (\phi_{11}-2u_4f_1)du_1\otimes du_1+\phi_{22}du_2\otimes du_2\\ &\quad+(\phi_{12}-2u_4f_2)(du_1\otimes du_2+du_1\otimes du_2)\\ &\quad+(du_1\otimes du_3+du_3\otimes du_1+du_2\otimes du_4+du_4\otimes du_2). \end{aligned} \tag{4.2}$$

Further assume that f_1 and f_2 satisfies

$$\partial_1 a=0 \quad \text{and} \quad \partial_2 a=0, \tag{4.3}$$

where $a=\partial_1 f_2+f_2^2$. Then the non-zero Christoffel symbols of $g_{\nabla,\phi}$ are given by

$$\begin{aligned} \tilde{\Gamma}_{11}^2 &= f_1, \quad \tilde{\Gamma}_{12}^2=f_2,\\ \tilde{\Gamma}_{14}^3 &= -f_1, \quad \tilde{\Gamma}_{14}^4=-f_2, \quad \tilde{\Gamma}_{24}^3=-f_2,\\ \tilde{\Gamma}_{11}^3 &= \frac{1}{2}\partial_1\phi_{11}-\phi_{12}f_1+2u_4f_1f_2-u_4\partial_1 f_1,\\ \tilde{\Gamma}_{11}^4 &= \partial_1\phi_{12}-\frac{1}{2}\partial_2\phi_{11}-\phi_{22}f_1-2u_4\partial_1 f_2,\\ \tilde{\Gamma}_{12}^3 &= \frac{1}{2}\partial_2\phi_{11}-\phi_{12}f_2+2u_4f_2f_2, \quad \tilde{\Gamma}_{12}^4=\frac{1}{2}\partial_1\phi_{22}-\phi_{22}f_2,\\ \tilde{\Gamma}_{22}^3 &= \partial_2\phi_{12}-\frac{1}{2}\partial_1\phi_{22}, \quad \tilde{\Gamma}_{22}^4=\frac{1}{2}\partial_2\phi_{22}. \end{aligned}$$

The only nonvanishing covariant derivatives are given by

$$\begin{aligned} \tilde{\nabla}_{\partial_1}\partial_1 &= f_1\partial_2+\Big(\frac{1}{2}\partial_1\phi_{11}-\phi_{12}f_1+2u_4f_1f_2-u_4\partial_1 f_1\Big)\partial_3\\ &\quad+\Big(\partial_1\phi_{12}-\frac{1}{2}\partial_2\phi_{11}-\phi_{22}f_1-2u_4\partial_1 f_2\Big)\partial_4,\\ \tilde{\nabla}_{\partial_1}\partial_2 &= f_2\partial_2+\Big(\frac{1}{2}\partial_2\phi_{11}-\phi_{12}f_2+2u_4f_2f_2\Big)\partial_3\\ &\quad+\Big(\frac{1}{2}\partial_1\phi_{22}-\phi_{22}f_2\Big)\partial_4,\\ \tilde{\nabla}_{\partial_1}\partial_4 &= -f_1\partial_3-f_2\partial_4, \quad \tilde{\nabla}_{\partial_2}\partial_2=\Big(\partial_2\phi_{12}-\frac{1}{2}\partial_1\phi_{22}\Big)\partial_3+\Big(\frac{1}{2}\partial_2\phi_{22}\Big)\partial_4,\\ \tilde{\nabla}_{\partial_2}\partial_4 &= -f_2\partial_3. \end{aligned}$$

It follows that the only nonvanishing components of the curvature tensor of $(\mathbb{R}^4, g_{\nabla,\phi})$ are given by

$$\begin{aligned} \tilde{R}(\partial_1,\partial_4)\partial_1 &= a\partial_4, \quad \tilde{R}(\partial_1,\partial_4)\partial_2=-a\partial_3, \quad \tilde{R}(\partial_1,\partial_2)\partial_4=-a\partial_3,\\ \tilde{R}(\partial_1,\partial_2)\partial_1 &= a\partial_2+a\Big[2u_4f_2-\phi_{12}\Big]\partial_3+\Big[\frac{1}{2}\partial_1^2\phi_{22}-\partial_2\partial_1\phi_{12}+\frac{1}{2}\partial_2^2\phi_{11}\\ &\quad-\phi_{22}\partial_1 f_2-f_2\partial_1\phi_{22}+\frac{1}{2}f_1\partial_2\phi_{22}\Big]\partial_4,\\ \tilde{R}(\partial_1,\partial_2)\partial_2 &= \Big[\partial_1\partial_2\phi_{12}-\frac{1}{2}\partial_1^2\phi_{22}-\frac{1}{2}\partial_2^2\phi_{11}-\phi_{22}f_2^2-\frac{1}{2}f_1\partial_2\phi_{22}+f_2\partial_1\phi_{22}\Big]\partial_3, \end{aligned}$$

where $a = \partial_1 f_2 + f_2^2$. Putting

$$\tilde{R}(\partial_1, \partial_2)\partial_1 = a_2\partial_2 + a_3\partial_3 + a_4\partial_4 \quad \text{and} \quad \tilde{R}(\partial_1, \partial_2)\partial_2 = b_3\partial_3,$$

where a_2, a_3, a_4 and b_3 are the components of the curvature tensor. Now, let $X = \sum_{i=1}^4 \alpha_i\partial_i$ be a vector field on $\mathbb{R}^4$, then Szabó operator are:

$$\begin{aligned}
(\nabla_X\tilde{R})(\partial_1, X)X &= S_{31}\partial_3 + S_{41}\partial_4,\\
(\nabla_X\tilde{R})(\partial_2, X)X &= S_{32}\partial_3 + S_{42}\partial_4,\\
(\nabla_X\tilde{R})(\partial_3, X)X &= 0 \quad \text{and} \quad (\nabla_X\tilde{R})(\partial_4, X)X = 0,
\end{aligned}$$

where

$$\begin{aligned}
S_{31} &= \alpha_2^3\Big[\partial_2 b_3 + 2a\Gamma_{22}^4\Big] + \alpha_1^2\alpha_2\Big[\partial_1 a_3 - b_3\Gamma_{11}^2 + a\Gamma_{11}^4 - a_3\Gamma_{12}^2 + a_2\Gamma_{12}^3 + a_4\Gamma_{14}^3\Big]\\
&\quad +\alpha_1\alpha_2^2\Big[\partial_2 a_3 + \partial_1 b_3 - 3b_3\Gamma_{12}^2 + 3a\Gamma_{12}^4 + a_2\Gamma_{22}^3 + a_4\Gamma_{24}^3\Big]\\
&\quad +\alpha_1\alpha_2\alpha_4\Big[\partial_4 a_3 + 3a\Gamma_{12}^2 + 3a\Gamma_{14}^4 + 2a\Gamma_{14}^3\Big] + \alpha_1^2\alpha_4\Big[a\Gamma_{11}^2 + a\Gamma_{14}^3\Big],\\
S_{32} &= \alpha_1^3\Big[-\partial_1 a_3 - a\Gamma_{12}^3 - a\Gamma_{11}^4 + a_4 f_1 + a_3 f_2 + b_3 f_1\Big]\\
&\quad +\alpha_1^2\alpha_2\Big[-\partial_1 b_3 - \partial_2 a_3 - a\Gamma_{22}^3 + 3b_3 f_2 + a_4 f_2 - 3a\Gamma_{12}^4\Big]\\
&\quad +\alpha_1^2\alpha_4\Big[-\partial_4 a_3 + 2af_2\Big] + \alpha_1\alpha_2^2\Big[-\partial_2 b_3 - 2a\Gamma_{22}^4\Big],\\
S_{41} &= \alpha_1^2\alpha_2\Big[\partial_1 a_4 - a_4\Gamma_{12}^2 + a_4\Gamma_{14}^4\Big] + \alpha_1\alpha_2^2\Big[\partial_2 a_4\Big],\\
S_{42} &= \alpha_1^3\Big[2a_4 f_2 - \partial_1 a_4\Big] - \alpha_1^2\alpha_2\partial_2 a_4.
\end{aligned}$$

Hence, the matrix associated to the Szabó operator with respect to the basis $\{\partial_1, \dots, \partial_4\}$ is given by

$$(\mathcal{S}(X)) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ S_{31} & S_{32} & 0 & 0\\ S_{41} & S_{42} & 0 & 0 \end{pmatrix}. \tag{4.4}$$

It follows from the expression (4.4) of the Szabó operator $\mathcal{S}(X)$, where X is a nonnull vector field on $\mathbb{R}^4$, that its characteristic polynomial

$$P_\lambda(\mathcal{S}(X)) = \det(S(X) - \lambda \mathrm{Id}_4) = \lambda^4.$$

Thus all eigenvalues are zero. This proves that $(g_{\nabla,\phi})$ is Szabó. Hence we have the following

Theorem 4.1. *Let $M = \mathbb{R}^2$ and ∇ be the torsion free connection defined by $\nabla_{\partial_1}\partial_1 = f_1(u_1)\partial_2$ and $\nabla_{\partial_1}\partial_2 = f_2(u_1)\partial_2$. Assume that f_1 and f_2 satisfies $\partial_1 a = 0$ and $\partial_2 a = 0$, where $a = \partial_1 f_2 + f_2^2$. Then the pseudo-Riemannian metric $g_{(\nabla,\phi)}$ on the cotangent bundle T^*M of neutral signature $(2,2)$ defined by setting*

$$\begin{aligned}
g_{\nabla,\phi} &= (\phi_{11} - 2u_4 f_1)du_1 \otimes du_1 + \phi_{22}du_2 \otimes du_2\\
&\quad +(\phi_{12} - 2u_4 f_2)(du_1 \otimes du_2 + du_1 \otimes du_2)\\
&\quad +(du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2).
\end{aligned} \tag{4.5}$$

is Szabó for any symmetric $(0,2)$*-tensor field* ϕ.

Remark 4.1. Condition (4.3) on the coefficients f_1 and f_2 of the torsion free affine connection helps us to simplify the calculation. Also, such a condition is equivalent to $(T^*M, g_{\nabla,\phi})$ being Szabó.

By using the result of [11, p.35, Lemma 1.36], we have

Corollary 4.2. *The Szabó metric defined by* (4.5) *is Einstein if and only if* $a = 0$, *where* $a = \partial_1 f_2 + f_2^2$.

Remark 4.2. If $a = 0$, then $f_2(u_1) = \dfrac{1}{u_1} + c$, where c is constant.

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