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A UNIFIED CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS INVOLVING THE HURWITZ-LERCH ZETA FUNCTION

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. Making use of convolution product, we introduce a unified class of analytic functions with negative coefficients. Also, we obtain the coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $TP_\mu^\lambda(\alpha,\beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $P_\mu^\lambda(\alpha,\beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$ are determined. Relevant connections of the results with various known results are also considered.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C},\ |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.2}$$

we recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (cf., e.g., [29,p. 121]).

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| = 1)$$

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where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, ...\}); \mathbb{N} := \{1, 2, 3, ...\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [5], Ferreira and Lopez [6], Garg et al. [8], Lin and Srivastava [11], Lin et al. [12], and others. Srivastava and Attiya [28] (see also Raducanu and Srivastava [18], and Prajapat and Goyal [15]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z) \tag{1.4}$$

$(z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1 + b)^\mu [\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U). \tag{1.5}$$

We recall here the following relationships (given earlier by [15], [18]) which follow easily by using (1.1), (1.4) and (1.5)

$$\mathcal{J}_b^\mu f(z) = z + \sum_{n=2}^{\infty} C_n(b, \mu) a_n z^n \tag{1.6}$$

where

$$C_n(b, \mu) = \left(\frac{1 + b}{n + b} \right)^\mu \tag{1.7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}$.

(1) For $\mu = 0$

$$\mathcal{J}_b^0(f)(z) := f(z). \tag{1.8}$$

(2) For $\mu = 1$

$$\mathcal{J}_b^1(f)(z) := \int_0^z \frac{f(t)}{t} dt := \mathcal{L}_b f(z). \tag{1.9}$$

(3) For $\mu = 1$ and $b = \nu (\nu > -1)$

$$\mathcal{J}_\nu^1(f)(z) := \frac{1 + \nu}{z^\nu} \int_0^z t^{1-\nu} f(t) dt := \mathcal{F}_\nu(f)(z). \tag{1.10}$$

(4) For $\mu = \sigma (\sigma > 0)$ and $b = 1$

$$\mathcal{J}_1^\sigma(f)(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n + 1} \right)^\sigma a_n z^n = \mathcal{I}^\sigma(f)(z), \tag{1.11}$$

where $\mathcal{L}_b(f)$ and $\mathcal{F}_\nu$ are the integral operators introduced by Alexandor [1] and Bernardi [3], respectively, and $\mathcal{I}^\sigma(f)$ is the Jung-Kim-Srivastava integral operator [13] closely related to some multiplier transformation studied by Fleet [7]. Making use of the operator $\mathcal{J}_b^\mu$, we introduce a new subclass of analytic functions with negative coefficients and discuss some some usual properties of the geometric function theory of this generalized function class.

For $\lambda \geq 0$, $-1 \leq \alpha < 1$ and $\beta \geq 0$, we let $P_\mu^\lambda(\alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the inequality

$$\mathrm{Re}\ \left\{ \frac{z(\mathcal{J}_b^\mu f)'(z) + \lambda z^2 (\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z) + \lambda z (\mathcal{J}_b^\mu f)'(z)} - \alpha \right\} > \beta \left| \frac{z(\mathcal{J}_b^\mu f)'(z) + \lambda z^2 (\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z) + \lambda z (\mathcal{J}_b^\mu f)'(z)} - 1 \right| \tag{1.12}$$

where $z \in U$, $\mathcal{J}_b^\mu f(z)$ is given by (1.6) . We further let $TP_\mu^\lambda(\alpha, \beta) = P_\mu^\lambda(\alpha, \beta) \cap T$, where

$$T \ := \ \left\{ f \in A : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U \right\} \tag{1.13}$$

is a subclass of A introduced and studied by Silverman [21].

In particular, for $0 \leq \lambda \leq 1$, the class $TP_\mu^\lambda(\alpha, \beta)$ provides a transition from $k-$uniformly starlike functions to $k-$uniformly convex functions.By suitably specializing the values of μ, α, β and λ the class $TP_\mu^\lambda(\alpha, \beta)$ reduces to the various subclasses introduced and studied in [2, 4, 21, 26, 27]. As illustrations, we present few following examples:

Example 1: If $\mu = 0$ and $\lambda = 1$, then

$$TP_0^1(\alpha, \beta) \equiv UCT(\alpha, \beta) := \left\{ f \in T : \mathrm{Re}\ \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right|, \ \ z \in U \right\}. \tag{1.14}$$

A function in $UCT(\alpha, \beta)$ is called $\beta-$uniformly convex of order α, $0 \leq \alpha < 1$. This class was introduced in [4]. We also observe that

$$UST(\alpha, 0) \equiv T^*(\alpha), \qquad UCT(\alpha, 0) \equiv C(\alpha)$$

are, respectively, well-known subclasses of starlike functions of order α and convex functions of order α. Indeed it follows from (1.16) and (1.14) that

$$f \in UCT(\alpha, \beta) \Leftrightarrow zf' \in TS_p(\alpha, \beta). \tag{1.15}$$

For $\lambda = 0$ and different choices of μ we can state various subclasses of S.

Example 2: If $\mu = 0$, then

$$TP_0^0(\alpha, \beta) \equiv TS_p(\alpha, \beta) := \left\{ f \in T : \mathrm{Re}\ \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|, \ \ z \in U \right\} \tag{1.16}$$

A function in $TS_p(\alpha, \beta)$ is called $\beta-$uniformly starlike of order α, $0 \leq \alpha < 1$. This class was introduced in [4]. We also note that the classes $TS_p(\alpha, 0)$ and $TS_p(0, 0)$ were first introduced in [21].

Example 3: If $\mu = 1$ and $f(z)$ is as defined in (1.9), then

$$TP_1^0(\alpha, \beta) \equiv T\mathcal{L}_b(\alpha, \beta) := \left\{ f \in T : \mathrm{Re}\ \left(\frac{z(\mathcal{L}_b f(z))'}{\mathcal{L}_b f(z)} - \alpha \right) > \beta \left| \frac{z(\mathcal{L}_b f(z))'}{\mathcal{L}_b f(z)} - 1 \right|, \ \ z \in U \right\},$$

where $\mathcal{L}_b f(z)$ is defined by $\mathcal{L}_b f(z) := z - \sum\limits_{n=2}^{\infty} \left(\frac{1+b}{n+b} \right) a_n z^n$.

Example 4: If $\mu = 1, b = \nu (\nu > -1)$ and $f(z)$is as defined in (1.10), then

$$TP_1^0(\alpha, \beta) \equiv T\mathcal{F}_\nu(\alpha, \beta) := \left\{ f \in T : \mathrm{Re}\ \left(\frac{\mathcal{F}_\nu f(z)}{\mathcal{F}_\nu f(z)} - \alpha \right) > \beta \left| \frac{\mathcal{F}_\nu f(z)}{\mathcal{F}_\nu f(z)} - 1 \right|, \ \ z \in U \right\},$$

where $\mathcal{F}_\nu f(z)$ is given by $\mathcal{F}_\nu f(z) := z - \sum\limits_{n=2}^{\infty} \left(\frac{1+\nu}{n+\nu}\right) a_n z^n$.

Example 5: If $\mu = \sigma(\sigma > 0)$, $b = 1$ and $f(z)$ is defined in (1.11), then

$$TP_\sigma^1(\alpha,\beta) \quad \equiv \quad \mathcal{I}^\sigma(\alpha,\beta) := \left\{ f \in T : \mathrm{Re}\ \left(\frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - \alpha \right) > \beta \left| \frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - 1 \right|, \ z \in U \right\},$$

where $\mathcal{I}^\sigma f(z)$ is defined by $\mathcal{I}^\sigma f(z) := z - \sum\limits_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n$.

We remark that the classes of uniformly convex and uniformly starlike functions were introduced by Goodman [9, 10], and later generalized by and others [4, 16, 17, 19, 20, 26, 27].

The main object of this paper is to study the coefficient bounds, extreme points and radius of starlikeness for functions belong to the generalized class $TP_\mu^\lambda(\alpha,\beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $P_\mu^\lambda(\alpha,\beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f_k'(z)$ are determined.

2. Coefficient Bounds

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $P_\mu^\lambda(\alpha,\beta)$ and $TP_\mu^\lambda(\alpha,\beta)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $P_\mu^\lambda(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]|a_n||C_n(b,\mu)| \le 1-\alpha, \tag{2.1}$$

$0 \le \lambda \le 1$, $-1 \le \alpha < 1$, $\beta \ge 0$.

Proof. It sufficies to show that

$$\beta \left| \frac{z(\mathcal{J}_b^\mu f)'(z) + \lambda z^2 (\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z) + \lambda z (\mathcal{J}_b^\mu f)'(z)} - 1 \right| - \mathrm{Re} \left\{ \frac{z(\mathcal{J}_b^\mu f)'(z) + \lambda z^2 (\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z) + \lambda z (\mathcal{J}_b^\mu f)'(z)} - 1 \right\} \le 1-\alpha$$

We have

$$\begin{aligned}
& \beta \left| \frac{z(\mathcal{J}_b^\mu f)'(z) + \lambda z^2 (\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z) + \lambda z (\mathcal{J}_b^\mu f)'(z)} - 1 \right| - \mathrm{Re} \left\{ \frac{z(\mathcal{J}_b^\mu f)'(z) + \lambda z^2 (\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z) + \lambda z (\mathcal{J}_b^\mu f)'(z)} - 1 \right\} \\
\le \quad & (1+\beta) \left| \frac{z(\mathcal{J}_b^\mu f)'(z) + \lambda z^2 (\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z) + \lambda z (\mathcal{J}_b^\mu f)'(z)} - 1 \right| \\
\le \quad & \frac{(1+\beta) \sum\limits_{n=2}^{\infty} (n-1)[1+\lambda(n-1)]|a_n||C_n(b,\mu)|}{1 - \sum\limits_{n=2}^{\infty} [1+\lambda(n-1)]|a_n||C_n(b,\mu)|}.
\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]|a_n||C_n(b,\mu)| \le 1-\alpha,$$

and hence the proof is complete. □

Theorem 2.2. *A necessary and sufficient condition for $f(z)$ of the form (1.13) to be in the class $TP_\mu^\lambda(\alpha,\beta)$, $-1 \le \alpha < 1, 0 \le \lambda \le 1, \beta \ge 0$ is that*

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]\ a_n C_n(b,\mu) \le 1-\alpha, \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f \in P_\mu^\lambda(\alpha,\beta)$ and z is real then

$$\frac{1-\sum\limits_{n=2}^{\infty} n[1+\lambda(n-1)]a_n C_n(b,\mu)z^{n-1}}{1-\sum\limits_{n=2}^{\infty}[1+\lambda(n-1)]a_n C_n(b,\mu)z^{n-1}} - \alpha \ge \beta \left| \frac{\sum\limits_{n=2}^{\infty}(n-1)[1+\lambda(n-1)]|a_n||C_n(b,\mu)|}{1-\sum\limits_{n=2}^{\infty}[1+\lambda(n-1)]|a_n||C_n(b,\mu)|} \right|$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]\ a_n C_n(b,\mu) \le 1-\alpha.$$

□

In view of the Examples 1 to 5 in Section 1 and Theorem 2.2, we have following corollaries for the classes defined in these examples.

Corollary 2.3. [4] *A necessary and sufficient condition for $f(z)$ of the form (1.13) to be in the class $UST(\alpha,\beta)$, $-1 \le \alpha < 1, \beta \ge 0$ is that*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\ a_n \le 1-\alpha,$$

Corollary 2.4. [4] *A necessary and sufficient condition for $f(z)$ of the form (1.13) to be in the class $UCT(\alpha,\beta)$, $-1 \le \alpha < 1, \beta \ge 0$ is that*

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)]\ a_n \le 1-\alpha,$$

Corollary 2.5. *A necessary and sufficient condition for $f(z)$ of the form (1.13) to be in the class $T\mathcal{L}_b(\alpha,\beta)$, $-1 \le \alpha < 1, \beta \ge 0$ is that*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\left(\frac{1+b}{n+b}\right)a_n \le 1-\alpha.$$

Corollary 2.6. *A necessary and sufficient condition for $f(z)$ of the form (1.13) to be in the class $T\mathcal{F}_\nu(\alpha,\beta)$, $-1 \le \alpha \le 1$ and $\beta \ge 0$ is that*

$$\sum_{n=2}^{\infty}[n(\beta+1)-(\alpha+\beta)]\ \left(\frac{1+\nu}{n+\nu}\right)a_n \le 1-\alpha.$$

Corollary 2.7. *A necessary and sufficient condition for $f(z)$ of the form (1.13) to be in the class $\mathcal{I}^\sigma(\alpha,\beta)$, $-1 \le \alpha < 1, \beta \ge 0$ is that*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\left(\frac{2}{n+1}\right)^\sigma a_n \le 1-\alpha.$$

When $\beta = 0$ and $\lambda = 1$ with $\mu = 0$, Theorem 2.2 gives the following interesting result.

Corollary 2.8. [21] *If $f \in \mathcal{T}$, then $f \in \mathcal{C}(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha.$$

Corollary 2.9. *If $f \in TP_\mu^\lambda(\alpha,\beta)$, then*

$$a_n \leq \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)}, \; n \geq 2, \tag{2.3}$$

where $0 \leq \lambda \leq 1$, $-1 \leq \alpha < 1$ and $\beta \geq 0$. Equality in (2.3) holds for the function

$$f(z) = z - \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)}z^n. \tag{2.4}$$

Similarly many known results can be obtained as particular cases of the following theorems, so we omit stating the particular cases for the following theorems.

3. Closure Properties

Theorem 3.1. *Let*

$$\begin{aligned} f_1(z) &= z \text{ and} \\ f_n(z) &= z - \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)}z^n, \qquad n \geq 2. \end{aligned} \tag{3.1}$$

Then $f \in TP_\mu^\lambda(\alpha,\beta)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z), \qquad \omega_n \geq 0, \quad \sum_{n=1}^{\infty} \omega_n = 1. \tag{3.2}$$

Proof. Suppose $f(z)$ can be written as in (3.2). Then

$$f(z) = z - \sum_{n=2}^{\infty} \omega_n \frac{1-\alpha}{[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)}z^n.$$

Now,

$$\sum_{n=2}^{\infty} \omega_n \frac{[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)(1-\alpha)}{(1-\alpha)[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)} = \sum_{n=2}^{\infty} \omega_n = 1-\omega_1 \leq 1.$$

Thus $f \in TP_\mu^\lambda(\alpha,\beta)$. Conversely, let us have $f \in TP_\mu^\lambda(\alpha,\beta)$. Then by using (2.3), we set

$$\omega_n = \frac{[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)}{1-\alpha}a_n, \quad n \geq 2$$

and $\omega_1 = 1 - \sum_{n=2}^{\infty} \omega_n$. Then we have $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$ and hence this completes the proof of Theorem 3.1. □

Theorem 3.2. *The class $TP_\mu^\lambda(\alpha,\beta)$ is a convex set.*

Proof. Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j = 1,2 \tag{3.3}$$

be in the class $TP_\mu^\lambda(\alpha,\beta)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \eta f_1(z) + (1-\eta)f_2(z), \quad 0 \leq \eta \leq 1,$$

is in the class $TP_\mu^\lambda(\alpha,\beta)$. Since

$$h(z) = z - \sum_{n=2}^{\infty}[\eta a_{n,1} + (1-\eta)a_{n,2}]z^n,$$

an easy computation with the aid of Theorem 2.2 gives,

$$\begin{aligned}\sum_{n=2}^{\infty}&(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]\eta C_n(b,\mu)a_{n,1}\\ &+\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)](1-\eta)C_n(b,\mu)a_{n,2}\\ &\leq \eta(1-\alpha)+(1-\eta)(1-\alpha)\\ &\leq 1-\alpha,\end{aligned}$$

which implies that $h \in TP_\mu^\lambda(\alpha,\beta)$. Hence $TP_\mu^\lambda(\alpha,\beta)$ is convex. □

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TP_\mu^\lambda(\alpha,\beta)$.

Theorem 3.3. *Let the function $f(z)$ defined by (1.13)belong to the class $TP_\mu^\lambda(\alpha,\beta)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \left[\frac{(1-\delta)[n(\beta+1)-(\alpha+\beta)](1+\lambda(n-1))C_n(b,\mu)}{n(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{3.4}$$

The result is sharp, with extremal function $f(z)$ given by (3.1).

Proof. Given $f \in T$, and f is close-to-convex of order δ, we have

$$|f'(z) - 1| < 1 - \delta. \tag{3.5}$$

For the left hand side of (3.5) we have

$$|f'(z)-1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_n|z|^{n-1} < 1.$$

Using the fact, that $f \in TP_\mu^\lambda(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]C_n(b,\mu)}{(1-\alpha)}a_n \leq 1,$$

We can say (3.5) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]C_n(b,\mu)}{(1-\alpha)}a_n$$

Or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\delta)(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]C_n(b,\mu)}{n(1-\alpha)}\right]$$

which completes the proof. □

Theorem 3.4. *If $f \in TP_{\mu}^{\lambda}(\alpha,\beta)$, then*

(i) *f is starlike of order $\delta(0 \le \delta < 1)$ in the disc $|z| < r_2$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, $(|z| < r_2 \ ; \ 0 \le \delta < 1)$, where*

$$r_2 = \inf_{n\ge 2}\left[\left(\frac{1-\delta}{n-\delta}\right)\frac{(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]C_n(b,\mu)}{(1-\alpha)}\right]^{\frac{1}{n-1}} . \tag{3.6}$$

(ii) *f is convex of order δ $(0 \le \delta < 1)$ in the unit disc $|z| < r_3$, that is $Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, $(|z| < r_3; 0 \le \delta < 1)$, where*

$$r_3 = \inf_{n\ge 2}\left[\left(\frac{1-\delta}{n(n-\delta)}\right)\frac{(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]C_n(b,\mu)}{(1-\alpha)}\right]^{\frac{1}{n-1}} . \tag{3.7}$$

Each of these results are sharp for the extremal function $f(z)$ given by (3.1).

Proof. (i) Given $f \in T$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta. \tag{3.8}$$

For the left hand side of (3.8) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\ |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n\ |z|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in TP_{\mu}^{\lambda}(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]}{(1-\alpha)}a_nC_n(b,\mu) \le 1.$$

We can say (3.8) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} < \frac{(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]C_n(b,\mu)}{(1-\alpha)}$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\delta}{n-\delta}\right)\frac{(1+\lambda(n-1))[n(\beta+1)-(\alpha+\beta)]C_n(b,\mu)}{(1-\alpha)}\right]$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i). □

4. Partial Sums

Following the earlier works by Silverman [22] and Silvia [23] on partial sums of analytic functions. We consider in this section partial sums of functions in the class $P_\mu^\lambda(\alpha,\beta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f_k'(z)$.

Theorem 4.1. *Let $f(z) \in P_\mu^\lambda(\alpha,\beta)$. Define the partial sums $f_1(z)$ and $f_k(z)$, by*

$$f_1(z) = z; \text{ and } f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \ (k \in N/1) \tag{4.1}$$

Suppose also that

$$\sum_{n=2}^{\infty} d_n |a_n| \leq 1,$$

where

$$d_n := \frac{(1+\lambda(n-1))[n(\alpha+\beta)-(\alpha+\beta)]C_n(b,\mu)}{(1-\alpha)}. \tag{4.2}$$

Then $f \in P_\mu^\lambda(\alpha,\beta)$. Furthermore,

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{d_{k+1}} \quad z \in U, k \in N \tag{4.3}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{d_{k+1}}{1+d_{k+1}}. \tag{4.4}$$

Proof. For the coefficients d_n given by (4.2) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{4.5}$$

Therefore we have

$$\sum_{n=2}^{k} |a_n| + d_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} d_n |a_n| \leq 1 \tag{4.6}$$

by using the hypothesis (4.2). By setting

$$\begin{aligned} g_1(z) &= d_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{d_{k+1}}\right)\right\} \\ &= 1 + \frac{d_{k+1}\sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} a_n z^{n-1}} \end{aligned} \tag{4.7}$$

and applying (4.6), we find that

$$\begin{aligned} \left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\leq \frac{d_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|}{2 - 2\sum\limits_{n=2}^{n}|a_n| - d_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|} \\ &\leq 1, \quad z \in U, \end{aligned} \tag{4.8}$$

which readily yields the assertion (4.3) of Theorem 4.1. In order to see that

$$f(z) = z + \frac{z^{k+1}}{d_{k+1}} \tag{4.9}$$

gives sharp result, we observe that for $z = re^{i\pi/k}$ that $\frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{d_{k+1}} \to 1 - \frac{1}{d_{k+1}}$ as $z \to 1^-$. Similarly, if we take

$$\begin{aligned} g_2(z) &= (1 + d_{k+1})\left\{\frac{f_k(z)}{f(z)} - \frac{d_{k+1}}{1 + d_{k+1}}\right\} \\ &= 1 - \frac{(1 + d_{n+1}) \sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} a_n z^{n-1}} \end{aligned} \tag{4.10}$$

and making use of (4.6), we can deduce that

$$\left|\frac{g_2(z) - 1}{g_2(z) + 1}\right| \leq \frac{(1 + d_{k+1}) \sum\limits_{n=k+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{k} |a_n| - (1 - d_{k+1}) \sum\limits_{n=k+1}^{\infty} |a_n|} \tag{4.11}$$

which leads us immediately to the assertion (4.4) of Theorem 4.1.

The bound in (4.4) is sharp for each $k \in N$ with the extremal function $f(z)$ given by (4.9). The proof of the Theorem 4.1, is thus complete. □

Theorem 4.2. *If $f(z)$ of the form (1.1) satisfies the condition (2.1). Then*

$$Re\left\{\frac{f'(z)}{f'_k(z)}\right\} \geq 1 - \frac{k+1}{d_{k+1}}. \tag{4.12}$$

Proof. By setting

$$\begin{aligned} g(z) &= d_{k+1}\left\{\frac{f'(z)}{f'_k(z)} - \left(1 - \frac{k+1}{d_{k+1}}\right)\right\} \\ &= \frac{1 + \frac{d_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} na_n z^{n-1} + \sum\limits_{n=2}^{\infty} na_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} na_n z^{n-1}} \\ &= 1 + \frac{\frac{d_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} na_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} na_n z^{n-1}}. \end{aligned}$$

$$\left|\frac{g(z) - 1}{g(z) + 1}\right| \leq \frac{\frac{d_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} n|a_n|}{2 - 2\sum\limits_{n=2}^{k} n|a_n| - \frac{d_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} n|a_n|}. \tag{4.13}$$

Now

$$\left|\frac{g(z) - 1}{g(z) + 1}\right| \leq 1$$

if

$$\sum_{n=2}^{k} n|a_n| + \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \leq 1 \tag{4.14}$$

since the left hand side of (4.14) is bounded above by $\sum\limits_{n=2}^{k} d_n|a_n|$ if

$$\sum_{n=2}^{k}(d_n - n)|a_n| + \sum_{n=k+1}^{\infty} d_n - \frac{d_{k+1}}{k+1} n|a_n| \geq 0, \tag{4.15}$$

and the proof is complete. The result is sharp for the extremal function $f(z) = z + \frac{z^{k+1}}{c_{k+1}}$. □

Theorem 4.3. *If $f(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{d_{k+1}}{k+1+d_{k+1}}. \tag{4.16}$$

Proof. By setting

$$\begin{aligned} g(z) &= [(k+1)+d_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{d_{k+1}}{k+1+d_{k+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n a_n z^{n-1}}{1+\sum\limits_{n=2}^{k} n a_n z^{n-1}} \end{aligned}$$

and making use of (4.15), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|} \leq 1,$$

which leads us immediately to the assertion of the Theorem 4.3. □

5. Integral Means Inequalities

In 1925, Littlewood [14] proved the following subordination theorem.

Lemma 5.1. *If the functions f and g are analytic in U with $g \prec f$, then for $\rho > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\rho} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\rho} d\theta. \tag{5.1}$$

In [21], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [24] and settled in [25], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\rho} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\rho} d\theta,$$

for all $f \in T$, $\rho > 0$ and $0 < r < 1$. In [25], he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

In the following theorem we obtain integral means inequalities for the functions in the family $TP_\mu^\lambda(\alpha,\beta)$. By taking appropriate choices of the parameters we obtain the integral means inequalities for several known as well as new subclasses.

Applying Lemma 5.1, Theorem 2.1 and Theorem 2.9, we prove the following result.

Theorem 5.2. *Suppose $f \in TP_\mu^\lambda(\alpha,\beta)$, $\rho > 0$, $0 \le \alpha < 1$, $\beta \ge 0$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\alpha}{(2-\alpha)(1+\lambda)C_2(b,\mu)}z^2.$$

where $C_2(b,\mu)$ is given by (1.7). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\rho \, d\theta \le \int_0^{2\pi} |f_2(z)|^\rho \, d\theta. \tag{5.2}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (5.2) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^\rho d\theta \le \int_0^{2\pi} \left|1 - \frac{1-\alpha}{(2-\alpha)(1+\lambda)C_2(b,\mu)}z\right|^\rho d\theta.$$

By Lemma 5.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\alpha}{(2-\alpha)(1+\lambda)C_2(b,\mu)}z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\alpha}{(2-\alpha)(1+\lambda)C_2(b,\mu)}w(z), \tag{5.3}$$

and using (2.2), we obtain

$$\begin{aligned} |w(z)| &= \left|\sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)}{1-\alpha} \; a_n C_n(b,\mu) z^{n-1}\right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)}{1-\alpha} |a_n| \\ &\le |z|, \end{aligned}$$

where $C_n(b,\mu)$ is given by (1.7). Which completes the proof by Theorem 5.2. $\square$

In view of the Examples 1 to 5 in Section 1 and Theorem 5.2, we can deduce the integral means inequalities for the classes defined in the above stated examples.

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