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A RELATION-THEORETIC $(F, \mathcal{R})$-CONTRACTION PRINCIPLE WITH APPLICATIONS TO MATRIX EQUATIONS

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Abstract. In this paper, we introduce certain notions namely: $(F, \mathcal{R})$-contraction, T-orbital transitivity and orbit $\mathcal{R}$-continuity and utilize the same to prove a relation-theoretic contraction principle under $(F, \mathcal{R})$-contraction in a metric space endowed with a binary relation $\mathcal{R}$. We also furnish some examples to demonstrate the utility of our main results. As applications, we apply our main results to nonlinear matrix equations.

1. Introduction

The tremendous applications of fixed point theory had always inspired the growth of this domain. In 1922, Banach formulated his most simple but very natural result which is now popularly referred as Banach contraction principle. This principle is a very popular tool for guaranteeing the existence and uniqueness of solution of a multitude problems arising in several domains of Mathematics and Physical Sciences. In the course of last several decades, this principle has been extended and generalized in many directions with several applications in various directions.

In 2012, Wardowski [27] initiated the idea of F-contraction with a view to consider a new class of nonlinear contractions which in turn generalizes Banach contraction principle. Thereafter, many authors generalized and improved F-contraction in different ways (see[7, 8, 9, 10, 11, 12, 13, 15, 14, 19, 23, 24, 26, 28] and references cited therein). One of these extensions is F_R-contraction due to Sawangsup et al. [23], in which the authors established some relation-theoretic fixed point results by using the idea of F-contraction.

In this paper, we introduce the notions of $(F, \mathcal{R})$-contraction, T-orbital transitivity and orbit $\mathcal{R}$-continuity and utilize the same to present some existence and uniqueness of fixed point results for a self-mapping T defined on a metric space (X, d) endowed with a binary relation $\mathcal{R}$. In our results, the binary relation $\mathcal{R}$ is T-orbitally transitive, so that we restrict the set of pairs of points for which the contractivity condition must hold as the binary relation is not essentially required to be reflexive, antisymmetric or transitive. However, we also replace the

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completeness and continuity conditions by relatively weaker assumptions namely: $\mathcal{R}$-increasingly precompleteness of an appropriate subspace and orbit $\mathcal{R}$-continuity. We adopt some examples to exhibit the utility of our results. Finally, we apply our results to prove the existence and uniqueness of solution of a certain class of nonlinear matrix equations.

2. Relation-theoretic notions and auxiliary results

From now on, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{R}$ stands for the set of all real numbers. In the sequel, X is a nonempty set and $T : X \to X$. For brevity, we write Tx instead of $T(x)$, $\{x_n\} \to x$ whenever $\{x_n\}$ converges to x and for all n one means that for all $n \in \mathbb{N}_0$. A point $x \in X$ is said to be *a fixed point of* T if $Tx = x$ ($Fix(T)$ denotes the set of all such points). Let $x_0 \in X$, a sequence $\{x_n\} \subseteq X$ defined by $x_{n+1} = T^n x_0 = Tx_n$, for all n, is called a *Picard sequence* based on x_0. Recall that a sequence $\{x_n\}$ in a metric space (X, d) is said to be *asymptotically regular* if $\{d(x_{n+1}, x_n)\} \to 0$.

A nonempty subset $\mathcal{R}$ of $X \times X$ is said to be *a binary relation on* X. Trivially, $X \times X$ is always a binary relation on X known as *universal relation* and denoted by $\mathcal{R}_X$. Throughout this work, $\mathcal{R}$ stands for a binary relation defined on X. For simplicity, we write $x\mathcal{R}y$ whenever $(x, y) \in \mathcal{R}$ and $x\mathcal{R}^{\neq}y$ whenever $x\mathcal{R}y$ and $x \neq y$. Observe that $\mathcal{R}^{\neq}$ is also a binary relation on X such that $\mathcal{R}^{\neq} \subseteq \mathcal{R}$. The points x and y are said to be $\mathcal{R}$*-comparable* if $x\mathcal{R}y$ or $y\mathcal{R}x$, this is denoted by $[x, y] \in \mathcal{R}$. A binary relation $\mathcal{R}$ is said to be: *amorphous* if it has no specific property at all; *reflexive* if $x\mathcal{R}x$ for all $x \in X$; *transitive* if $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$ for all $x, y, z \in X$; T*-transitive* if it is transitive on TX; *antisymmetric* if $x\mathcal{R}y$ and $y\mathcal{R}x$ imply $x = y$ for all $x, y \in X$; *preorder* if it is reflexive and transitive and *partial order* if it is reflexive, transitive and antisymmetric. Following [17], the inverse or dual relation of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$ and defined by $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$. The symmetric closure of $\mathcal{R}$ is denoted by $\mathcal{R}^s$ and defined by $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.

Definition 2.1. [16] *For $x, y \in X$, a path of length p ($p \in \mathbb{N}$) in $\mathcal{R}$ from x to y is a finite sequence $\{u_0, u_1, ..., u_p\} \subseteq X$ such that $u_0 = x$, $u_p = y$ and $u_i\mathcal{R}u_{i+1}$ for each $i \in \{0, 1, ..., p-1\}$.*

Definition 2.2. [2] *A subset $E \subseteq X$ is said to be $\mathcal{R}$-connected if for each $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y.*

Definition 2.3. [25] *A sequence $\{x_n\} \subseteq X$ is said to be: $\mathcal{R}$-nondecreasing if $x_n\mathcal{R}x_{n+1}$ for all n; $\mathcal{R}$-increasing if $x_n\mathcal{R}^{\neq}x_{n+1}$ for all n.*

Here it can be pointed out that Alam and Imdad [2] used the term "$\mathcal{R}$-preserving" instead of "$\mathcal{R}$-nondecreasing".

As usual, the set $O(x) = \{x, Tx, T^2x, ...\}$ is called the orbit of x under T. Now, we introduce the notion of T-orbital transitivity as follows:

Definition 2.4. *A binary relation $\mathcal{R}$ on a nonempty set X is said to be* T-orbitally transitive *if it is transitive on $O(x)$ for all $x \in X$.*

Remark. *Transitivity $\Rightarrow$ T-transitivity $\Rightarrow$ T-orbital transitivity, the converse is not true in general.*

Example 2.5. *Take* $X=\{0,\frac{1}{2},\frac{1}{2^2},...\}$ *and define a binary relation* $\mathcal{R}$ *on* X *as follows:*

$$x\mathcal{R}y \iff x>y>0 \text{ or } (x,y)\in\{(0,0),(0,\tfrac{1}{4}),(0,\tfrac{1}{2^n}):n\geq 4\}.$$

Define $T:X\to X$ *by:* $Tx=\frac{1}{2}x$ *for all* $x\in X$. *Then* $\mathcal{R}$ *is* T*-orbitally transitive which is not* T*-transitive. To see this, observe that* $(0,\frac{1}{4}),(\frac{1}{4},\frac{1}{8})\in\mathcal{R}$ *and* $(0,\frac{1}{8})\notin\mathcal{R}$.

Definition 2.6. [1] *If* $Tx\mathcal{R}Ty$ *for all* $x,y\in X$ *such that* $x\mathcal{R}y$, *then* $\mathcal{R}$ *is called* T-closed.

Here it can be pointed out that the notion $\mathcal{R}$ is T-closed is equivalent to say that T is $\mathcal{R}$-nondecreasing used by Roldán and Shahzad [22].

Definition 2.7. *A sequence* $\{x_n\}$ *is said to be:* a $(T,\mathcal{R})$-Picard sequence *if it is a Picard sequence and* $x_n\mathcal{R}x_{n+1}$ *for all* n; a $(T,\mathcal{R})$-increasing-Picard sequence *if it is a Picard sequence and* $x_n\mathcal{R}^{\varkappa}x_{n+1}$ *for all* n.

Definition 2.8. [6] *Let* (X,d) *be a metric space. A self-mapping* T *on* X *is said to be* an orbitally continuous *if for each* $x,u\in X$ *and any sequence* $\{n_i\}$ *of positive integers with* $\lim_{i\to\infty}T^{n_i}x=u\in X$, *we have* $\lim_{i\to\infty}TT^{n_i}x=Tu$.

Definition 2.9. [22] *Let* (X,d) *be a metric space endowed with a binary relation* $\mathcal{R}$. *A self-mapping* T *on* X *is said to be* $\mathcal{R}$-continuous *if* $\{Tx_n\}\to Tx$ *for all sequence* $\{x_n\}\subseteq X$ *such that* $\{x_n\}\to x$ *and* $x_n\mathcal{R}x_m$ *for all* n,m *with* $n<m$.

Now, we introduce the notion of orbital $\mathcal{R}$-continuity as follows:

Definition 2.10. *Let* (X,d) *be a metric space endowed with a binary relation* $\mathcal{R}$. *A self-mapping* T *on* X *is said to be* an orbitally $\mathcal{R}$-continuous *if for all* $x,u\in X$ *and any sequence* $\{n_i\}$ *of positive integers, we have*

$$\{T^{n_i}x\}\to u \text{ and } T^{n_i}x\mathcal{R}T^{n_{i+1}}x \text{ (for all } i\in\mathbb{N}) \text{ imply } \{TT^{n_i}x\}\to Tu.$$

The following implications are obvious:

$$\begin{array}{ccc} \textit{Continuity} & \Longrightarrow & \textit{orbital continuity} \\ \Downarrow & & \Downarrow \\ \mathcal{R}\textit{-continuity} & \Longrightarrow & \textit{orbitally } \mathcal{R}-\textit{continuity}. \end{array}$$

Lemma 2.11. [21] *Let* (X,d) *be a metric space and* $\{x_n\}$ *a sequence in* X. *If* $\{x_n\}$ *is not Cauchy in* X, *then there exist* $\epsilon_0>0$ *and two subsequences* $\{x_{n(k)}\}$ *and* $\{x_{m(k)}\}$ *of* $\{x_n\}$ *such that* $k\leq n(k)\leq m(k)$, $d(x_{n(k)},x_{m(k)-1})\leq\epsilon_0<d(x_{n(k)},x_{m(k)})$ $\forall k\in\mathbb{N}_0$. *Moreover, if* $\{x_n\}$ *is asymptotically regular, then*

$$\lim_{k\to\infty}d(x_{n(k)},x_{m(k)})=\lim_{k\to\infty}d(x_{n(k)-1},x_{m(k)-1})=\epsilon_0.$$

Definition 2.12. [25] *Let* (X,d) *be a metric space. A subset* $B\subseteq X$ *is said to be* precomplete *if each Cauchy sequence* $\{x_n\}\subseteq B$ *converges to some* $x\in X$.

Definition 2.13. [3] *Let* (X,d) *be a metric space endowed with a binary relation* $\mathcal{R}$. *A subset* $B\subseteq X$ *is said to be* $(\mathcal{R},d)$-increasingly precomplete *if each* $\mathcal{R}$*-increasing Cauchy sequence* $\{x_n\}\subseteq B$ *converges to some* $x\in X$.

Remark. *Every precomplete subset of* X *is* $(\mathcal{R},d)$*-increasingly precomplete whatever the binary relation* $\mathcal{R}$.

Definition 2.14. *(see* [22]*) Let* (X, d) *be a metric space equipped with a binary relation* $\mathcal{R}$*. A subset* $B \subseteq X$ *is said to be* $(\mathcal{R}, d)$-increasingly regular *if for every* $\mathcal{R}$*-increasing sequence* $\{x_n\} \subseteq X$ *such that* $\{x_n\} \to x \in X$*, we have* $x_n \mathcal{R} x$ *for all* n*.*

3. $(F, \mathcal{R})$-contraction and auxiliary results

In 2012 Wardowski [27] introduced F-contraction as follows:

Definition 3.1. [27] *Let* $\mathcal{F}$ *be the family of all functions* $F : (0, \infty) \to \mathbb{R}$ *which satisfy the following conditions:*

(F_1) *F is strictly increasing;*
(F_2) *for every sequence* $\{\beta_n\} \subset (0, \infty)$*,*

$$\lim_{n\to\infty} \beta_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\beta_n) = -\infty;$$

(F_3) *there exists* $k \in (0, 1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$*.*

Definition 3.2. [27] *Let* (X, d) *be a metric space. A self-mapping* T *on* X *is said to be* an F-contraction *if there exists* $\tau > 0$ *and* $F \in \mathcal{F}$ *such that*

$$[d(Tx, Ty) > 0] \Longrightarrow [\tau + F(d(Tx, Ty)) \leq F(d(x, y))], \ \forall x, y \in X.$$

Wardowski [27] proved that every F-contraction mapping on a complete metric space has a unique fixed point. Thereafter, Piri and Kumam [19] replaced condition (F_3) by the continuity of F and proved a theorem which is analogous to Wardowski′s theorem. In 2016 Durmaz et al. [7] proved order-theoretic fixed point results using F-contraction. Very recently, Sawangsup et al. [23] introduced the notion of F_R-contraction and utilized the same to prove a relation-theoretic fixed point results.

We observe that (F_1) can be withdrawn and all the related results can survive without it. In fact condition (F_1) is used only to show that the F-contraction mapping is contractive and hence continuous. We notice that the continuity of the F-contraction mappings is coming by making use of (F_2).

Inspired by the above mentioned articles, we introduce the notion of $(F, \mathcal{R})$-contraction as follows:

Definition 3.3. *Let* (X, d) *be a metric space. A self-mapping* T *on* X *is said to be* an $(F, \mathcal{R})$-contraction *if there exists* $\tau > 0$ *such that*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \ \ \text{for all } x, y \in X \text{ with } x\mathcal{R}^{\varkappa} y \text{ and } Tx\mathcal{R}^{\varkappa} Ty, \quad (3.1)$$

where $F : (0, \infty) \to \mathbb{R}$ *is a continuous mapping such that, for every sequence* $\{\beta_n\} \subset (0, \infty)$*, we have*

$$\lim_{n\to\infty} \beta_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\beta_n) = -\infty. \quad (3.2)$$

Remark. *Observe that in Definition 3.3 the condition* (F_1) *is absence.*

Example 3.4. *(see* [27, 19]*) Let us define* $F_i : (0, \infty) \to \mathbb{R}$*,* $i = 1, 2, 3, 4$ *by:*

(i) $F_1(\beta) = \ln \beta$*;*
(ii)$F_2(\beta) = -\frac{1}{\beta}$*;*
(iii)$F_3(\beta) = \beta - \frac{1}{\beta}$*;*

Clearly, the functions F_1, F_2 *and* F_3 *are continuous beside satisfying (3.2). Thus, each mapping* $T : X \to X$ *satisfying (3.1) with* F_1, F_2 *or* F_3 *is an* $(F, \mathcal{R})$*-contraction.*

Example 3.5. *Let $F:(0,\infty)\to\mathbb{R}$ be given by: $F(\alpha)=\ln\left(\frac{\alpha}{3}+\sin\alpha\right)$. It is clear that F is continuous beside satisfying (3.2). However, it dose not satisfy F_1. Thus, each mapping $T:X\to X$ satisfying (3.1) with such F is an $(F,\mathcal{R})$-contraction.*

The following proposition immediate due to the symmetricity of d.

Proposition 3.6. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and $T:X\to X$. Then for each continuous mapping $F:(0,\infty)\to\mathbb{R}$ satisfying (3.2), the following are equivalent:*

$$\tau+F(d(Tx,Ty))\le F(d(x,y)) \ \text{ for all } x,y\in X \text{ such that } (x,y)\in\mathcal{R};$$

$$\tau+F(d(Tx,Ty))\le F(d(x,y)) \ \text{ for all } x,y\in X \text{ such that } [x,y]\in\mathcal{R}.$$

Proposition 3.7. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and $T:X\to X$. If T is $(F,\mathcal{R})$-contraction, $\mathcal{R}$ is T-orbitally transitive and X is $(\mathcal{R},d)$-increasingly regular, then T is orbitally $\mathcal{R}^{\varkappa}$-continuous.*

Proof. Let $x,u\in X$ and $\{n_i\}$ be a sequence of positive integers. Assume that $\{T^{n_i}x\}\to u$ and $T^{n_i}x\mathcal{R}^{\varkappa}T^{n_{i+1}}x$ for all $i\in\mathbb{N}$. Then, we have $T^{n_i}x\mathcal{R}^{\varkappa}u$ for all i (due to $(\mathcal{R},d)$-increasing regularity of X). As $\mathcal{R}$ is T-orbitally transitive, we obtain $TT^{n_i}x\mathcal{R}^{\varkappa}Tu$ for all $i\in\mathbb{N}$. Applying (3.1), we have (for all $i\in\mathbb{N}$)

$$\tau+F(d(TT^{n_i}x,Tx))\le F(d(T^{n_i}x,x)),$$

implying thereby $F(d(TT^{n_i}x,Tx))<F(d(T^{n_i}x,x))$. Since $\{T^{n_i}x\}\to x$, so, on letting $i\to\infty$ and using (3.2), we obtain $\lim_{i\to\infty}d(TT^{n_i}x,Tx)=0$. Thus, T is orbitally $\mathcal{R}^{\varkappa}$-continuous. □

Proposition 3.8. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and $T:X\to X$. If T is $(F,\mathcal{R})$-contraction, $Fix(T)$ is non-empty and $\mathcal{R}^s$-connected, then T has a unique fixed point.*

Proof. On contrary, let us assume that there exist $x,y\in Fix(T)$ such that $x\neq y$. Then there exists a path in $\mathcal{R}^s$ (say $\{u_0,u_1,...,u_p\}\subseteq Fix(T)$) of some finite length p from x to y (with $u_i\neq u_{i+1}$ for each i, $(0\le i\le k-1)$, otherwise $x=y$, a contradiction) so that

$$u_0=x,\ u_p=y \text{ and } [u_i,u_{i+1}]\in\mathcal{R} \text{ for each } i,\ (0\le i\le p-1).$$

As $u_i\in Fix(T)$, $Tu_i=u_i$ for each $i\in\{0,1,...,p\}$. Hence, on using (3.1), we obtain $\tau+F(u_i,u_{i+1})\le F(u_i,u_{i+1})$, for all i $(0\le i\le k-1)$ which is a contradiction. □

Proposition 3.9. *Let $\mathcal{R}$ be a binary relation on a non-empty set X and $T:X\to X$. If $\mathcal{R}$ is T-closed and there exists $x_0\in X$ such that $x_0\mathcal{R}Tx_0$, then there exists a $(T,\mathcal{R})$-Picard sequence based at the initial point x_0.*

Proof. Since $x_0\in X$ and T is self-mapping on X, one can find $x_1\in X$ such that $x_1=Tx_0$. Hence, we have $x_0\mathcal{R}x_1$ and as T is $\mathcal{R}$-closed, we have $Tx_0\mathcal{R}Tx_1$. Similarly, there exists $x_2\in X$ such that $x_2=Tx_1$ and $x_1\mathcal{R}x_2$. Thus, inductively, we can construct a sequence $\{x_n\}\subseteq X$ such that $x_{n+1}=Tx_n$ and $x_n\mathcal{R}x_{n+1}$ for all n. □

4. Fixed point results

In this section, we present our main fixed point results as follows:

Theorem 4.1. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and T a self-mapping on X such that $\mathcal{R}$ is T-orbitally transitive. Suppose that the following conditions are satisfied:*

(a) *there exists a $(T, \mathcal{R})$-Picard sequence;*
(b) *TX is $\mathcal{R}$-increasingly precomplete;*
(c) *T is an $(F, \mathcal{R})$-contraction;*
(d) *T is orbitally $\mathcal{R}^{\ne}$-continuous.*

Then T has a fixed point. Indeed, if $\{x_n\}$ is any $(T, \mathcal{R})$-Picard sequence, then either $\{x_n\}$ contains a fixed point of T or $\{x_n\}$ converges to a fixed point of T.

Before giving the proof, let us highlight the improvements accomplished in the result which are described in the following lines:

- TX is taken to be $\mathcal{R}$-increasingly precomplete, which is relatively weaker than the following conditions:
  (1) TX is precomplete;
  (2) X or TX is complete;
  (3) there exists a complete subset $Y \subseteq X$ such that $TX \subseteq Y \subseteq X$;
  (4) X is complete and TX is closed.
  Observe that if any one of these four conditions holds, then TX is $(\mathcal{R}, d)$-increasingly precomplete;
- T is hypothesized to be orbitally $\mathcal{R}$-continuous. Indeed, orbit $\mathcal{R}$-continuity is weaker as compare to orbital continuity as well as $\mathcal{R}$-continuity;
- $\mathcal{R}$ is considered to be T-orbitally transitive. In fact, T-orbital transitivity is weaker than transitivity as well as T-transitivity.

Proof. Observe that hypothesis (a) guarantees the existence of a $(T, \mathcal{R})$-Picard sequence, i.e., there exists a sequence $\{x_n\} \subseteq X$ such that $x_{n+1} = Tx_n$ and $x_n \mathcal{R} x_{n+1}$ for all n. Denote $\beta_n = d(x_{n+1}, x_n)$ for all n. If there exists $n_0 \in \mathbb{N}_0$ such that $\beta_{n_0} = 0$, then $x_{n_0} = Tx_{n_0}$ and the result is established. Assume that $\beta_n > 0$ (i.e., $x_{n+1} \neq x_n$) for all n. Then $\{x_n\}$ is $\mathcal{R}$-increasing sequence. On using condition (c), for all n, we have

$$F(\beta_n) \leq F(\beta_{n-1}) - \tau \leq F(\beta_{n-2}) - 2\tau \leq ... \leq F(\beta_0) - n\tau,$$

which on letting $n \to \infty$ gives rise $\lim_{n\to\infty} F(\beta_n) = -\infty$, which together with (3.2) imply that

$$\lim_{n\to\infty} \beta_n = 0. \tag{4.1}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence via contradiction. To do so, assume that $\{x_n\}$ is not Cauchy sequence, then Lemma 2.11 and equation (4.1) guarantee the existence of $\epsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $k \leq n(k) \leq m(k), \quad d(x_{n(k)}, x_{m(k)-1}) \leq \epsilon_0 < d(x_{n(k)}, x_{m(k)}) \ \ \forall k \in \mathbb{N}_0$ and

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon_0. \tag{4.2}$$

As $\mathcal{R}$ is T-orbitally transitive, we obtain $x_{n(k)-1} \mathcal{R}^{\ne} x_{m(k)-1}$ and $Tx_{n(k)-1} \mathcal{R}^{\ne} Tx_{m(k)-1}$. Hence, applying (3.1), we have

$$\tau + F(d(Tx_{n(k)-1}, Tx_{m(k)-1})) \leq F(d(x_{n(k)-1}, x_{m(k)-1})). \tag{4.3}$$

As F is continuous, on letting $n \to \infty$ in (4.3) and using (4.2), we obtain $\tau + F(\epsilon_0) \leq F(\epsilon_0)$, a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence which is also $\mathcal{R}$-increasing. As $x_{n+1} = Tx_n$ for all n, $\{x_n\}_{n\geq 1} \subseteq TX$. Since TX is $\mathcal{R}$-increasingly precomplete, there exists $z \in X$ such that $\{x_n\} \to z$.

Finally, we prove that z is a fixed point of T. As $\{T^n x_0 = x_n\} \to z$, $T^n x_0 \mathcal{R}^{\#} T^{n+1} x_0$ and T is orbitally $\mathcal{R}^{\#}$-continuous, we obtain $\{TT^n x_0 = x_{n+1}\} \to Tz$. Now, owing to the uniqueness of the limit, we obtain $Tz = z$, i.e., z is a fixed point of T. This concludes the proof. □

Next, we present analogous theorem for Theorem 4.1 using $(\mathcal{R}, d)$-increasing regularity.

Theorem 4.2. *Conclusions of Theorem 4.1 remain true if condition* (d) *is replaced by the following:*
(e) *X is $(\mathcal{R}, d)$-increasingly regular.*

Proof. This theorem is immediate in view of Proposition 3.7 and Theorem 4.1. □

Now, we present a corresponding uniqueness result as follows:

Theorem 4.3. *If in addition to the hypotheses of Theorem 4.1(or Theorem 4.2), we assume that $Fix(T)$ is $\mathcal{R}^s$-connected, then the fixed point of T is unique.*

Proof. This theorem is immediate in view of Theorem 4.1 (or Theorem 4.2) and Proposition 3.8. □

The following examples exhibit that Theorems 4.1 and 4.3 are a genuine extension of all relevant results specially due to Wardowski [27], Piri and Kumam [19], Durmaz et al. [7] and Sawangsup et al. [23].

Example 4.4. *Let $X = [0, \infty)$ endowed with the usual metric. Consider a sequence $\{\pi_n\} \subseteq X$ defied by $\pi_n = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$. Define a binary relation $\mathcal{R}$ on X by: $\mathcal{R} = \{(\pi_1, \pi_1), (\pi_i, \pi_{i+1}) : i \geq 1\}$. Define a mapping $T : X \to X$ as follows:*

$$Tx = \begin{cases} x, & if\ 0 \leq x \leq 2; \\ \pi_1, & if\ 2 \leq x \leq \pi_2; \\ \pi_i + \left(\frac{\pi_{i+1} - \pi_i}{\pi_{i+2} - \pi_{i+1}}\right)(x - \pi_{i+1}), & if\ \pi_{i+1} \leq x \leq \pi_{i+2},\ \ i = 1, 2, \dots\ . \end{cases}$$

Then for the function F_3 given in Example 3.4, T is $(F, \mathcal{R})$-contraction for $\tau = 6$. Observe that if $x\mathcal{R}^{\#}y$ and $Tx\mathcal{R}^{\#}Ty$, then $x = \pi_i$, $y = \pi_{i+1}$ for some $i \in \mathbb{N} - \{1\}$. Further, for all $n, m \in \mathbb{N}$ such that $m > n > 1$, we have

$$6 + |T(\pi_m) - T(\pi_n)| - \frac{1}{|T(\pi_m) - T(\pi_n)|} \leq |\pi_m - \pi_n| - \frac{1}{|\pi_m - \pi_n|}.$$

Therefore, $6 + F(d(Tx, Ty)) \leq F(d(x, y))$ for all $x, y \in X$ such that $x\mathcal{R}^{\#}y$ and $Tx\mathcal{R}^{\#}Ty$. Hence, T is an $(F, \mathcal{R})$-contraction. Moreover, by a routine calculation one can show that all the hypotheses of Theorem 4.1 are satisfied ensuring the existence of a fixed point of T. Furthermore, $Fix(T)$ is not $\mathcal{R}^s$-connected as there is no path in $\mathcal{R}^s$ joining the fixed points 0 and 1 so that the uniqueness condition is not satisfied. Notice that T has infinitely many fixed points.

Here it can be pointed out that in the context of the present example fixed point results of Wardowski [27] and Piri and Kumam [19] are not applicable as the Wardowski′s F-contractive condition dose not hold for each $\tau > 0$ and for any arbitrary function F. Indeed, for each $x, y \in [0, 2]$ with $d(Tx, Ty) > 0$ we get $x \neq y$ and for any $\tau > 0$ we have

$$\tau + F(d(Tx, Ty)) = \tau + F(d(x, y)) > F(d(x, y)).$$

Example 4.5. *Take $X = [0, \infty)$ endowed with the usual metric. Define a binary relation $\mathcal{R}$ on X by:*

$$x\mathcal{R}y \iff (x, y) \in \{(0, 0), (n, n + 2) : n \in \mathbb{N}\}.$$

Define a mapping $T : X \to X$ by:

$$Tx = \begin{cases} \frac{x}{2}, & if\ 0 \leq x < 1; \\ 2, & if\ x \text{ is an odd number in } [1, \infty); \\ 3, & if\ x \text{ is an even number in } [1, \infty); \\ 4, & if\ \ x \text{ is non-integer in } [1, \infty). \end{cases}$$

Then for F_1 given in Example 3.4, T is $(F, \mathcal{R})$-contraction with any $\tau > 0$. Moreover, by a routine calculation one can show that all the hypotheses of Theorem 4.3 are satisfied. Observe that T has a unique fixed point (namely $x = 0$).

Here it can be pointed out that in the context of the present example $\mathcal{R}$ is not transitive, hence results of Durmaz et al. [7] and Sawangsup et al. [23] are not applicable. Furthermore, fixed point results of Wardowski [27] and Piri and Kumam [19] are not applicable as the Wardowski′s F-contractive condition dose not hold for each $\tau > 0$ and for any arbitrary function F. Indeed, for $x = 3$ and $y = 4$ we get $Tx \neq Ty$ and for any $\tau > 0$ we have

$$\tau + F(d(Tx, Ty)) = \tau + F(1) > F(1).$$

5. Applications to nonlinear matrix equations

In what follows we require the following notations:
Let us denote $\mathcal{M}(n) :=$ set of all $n \times n$ complex matrices, $\mathcal{H}(n) :=$ set of all Hermitian matrices in $\mathcal{M}(n)$, $\mathcal{P}(n) :=$ set of all positive definite matrices in $\mathcal{M}(n)$ and $\mathcal{H}^+(n) :=$ set of all positive semidefinite matrices in $\mathcal{M}(n)$. For $X \in \mathcal{P}(n)$ ($X \in \mathcal{H}^+(n)$), we write $X \succ 0$ ($X \succeq 0$). Furthermore, $X \succ Y$ ($X \succeq Y$) means $X - Y \succ 0$ ($X - Y \succeq 0$). The symbol $\|.\|$ stands for the spectral norm of a matrix A defined by $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A, where A^* is the conjugate transpose of A. Also, $\|A\|_{tr} = \sum_{k=1}^{n} s_k(A)$, where $s_k(A)$ ($1 \leq k \leq n$) are the singular values of $A \in \mathcal{M}(n)$. Here, $(\mathcal{H}(n), \|.\|_{tr})$ is complete metric space (for more details see [20, 5, 4]). Moreover, the binary relation $\preceq$ on $\mathcal{H}(n)$ defined by: $X \preceq Y \Leftrightarrow Y \succeq X$ for all $X, Y \in \mathcal{H}(n)$ is a T-orbitally transitive w.r.t any self-mapping T on $\mathcal{H}(n)$, in fact it is a transitive relation.

In this section, we apply our results to prove the existence and uniqueness of a solution of the nonlinear matrix equation

$$X = H + \sum_{k=1}^{m} A_k^* \mathcal{Q}(X) A_k, \tag{5.1}$$

where H is a Hermitian positive definite matrix and $\mathcal{Q}$ is a continuous order preserving[1] mapping from $\mathcal{H}(n)$ into $\mathcal{P}(n)$ such that $\mathcal{Q}(0)=0$, A_k are arbitrary $n\times n$ matrices and A_k^* their conjugates.

The following lemmas are needed in the sequel.

Lemma 5.1. [20] *If $A\succeq 0$ and $B\succeq 0$ are $n\times n$ matrices, then $0\leq tr(AB)\leq \|A\|tr(B)$.*

Lemma 5.2. [18] *If $A\in\mathcal{H}(n)$ such that $A\prec I_n$, then $\|A\|<1$.*

Theorem 5.3. *Consider the matrix equation (5.1). Assume that there exist two positive real numbers τ and c such that:*

(i) *for every $X,Y\in\mathcal{H}(n)$ such that $X\preceq Y$ with $\sum_{k=1}^{n}A_k^*\mathcal{Q}(X)A_k\neq\sum_{k=1}^{n}A_k^*\mathcal{Q}(Y)A_k$, we have $\left|tr\big(\mathcal{Q}(Y)-\mathcal{Q}(X)\big)\right|\leq\frac{|tr(Y-X)|}{c\left(1+\tau|tr(Y-X)|\right)}$;*
(ii) *$\sum_{k=1}^{m}A_kA_k^*\prec cI_n$ and $\sum_{k=1}^{m}A_k^*\mathcal{Q}(H)A_k\succ 0$.*

Then the matrix equation (5.1) has a solution. Moreover, the iteration $X_n=H+\sum_{k=1}^{n}A_k^*\mathcal{Q}(X_{n-1})A_k$ converges in the sense of trace norm $\|.\|_{tr}$ to the solution of the matrix equation (5.1), where $X_0\in\mathcal{H}(n)$ such that $X_0\preceq\sum_{k=1}^{m}A_k^*\mathcal{Q}(X_0)A_k$.

Proof. Define a mapping $T:\mathcal{H}(n)\to\mathcal{H}(n)$ by:

$$T(X)=H+\sum_{k=1}^{n}A_k^*\mathcal{Q}(X)A_k,\quad\text{for all } X\in\mathcal{H}(n).\tag{5.2}$$

Observe that T is well defined, continuous, $\preceq$ is T-closed and X is a fixed point of T if and only if it is a solution of the matrix equation (5.1). To accomplish this, we need to show that T is $(F,\mathcal{R})$-contraction with respect to τ, $\mathcal{R}$ $(=\preceq)$ wherein the mapping $F:(0,\infty)\to\mathbb{R}$ given by: $F(\beta)=\frac{-1}{\beta}$ for all $\beta\in(0,\infty)$.
Let $X,Y\in\mathcal{H}(n)$ be such that $X\preceq Y$ and $\mathcal{Q}(X)\neq\mathcal{Q}(Y)$. Then, $X\prec Y$ and since $\mathcal{Q}$ is an order preserving mapping, therefore we obtain $\mathcal{Q}(X)\prec\mathcal{Q}(Y)$. Hence, we

[1] $\mathcal{Q}$ is order preserving if $A,B\in\mathcal{H}(n)$ such that $A\preceq B$ implies that $\mathcal{Q}(A)\preceq\mathcal{Q}(B)$.

have

$$\begin{aligned}
\|T(Y)-T(X)\|_{tr} &= tr\big(T(Y)-T(X)\big)\\
&= tr\Big(\sum_{k=1}^{m} A_k^*\big(\mathcal{Q}(Y)-\mathcal{Q}(X)\big)A_k\Big)\\
&= \sum_{k=1}^{m} tr\big(A_k^*\big(\mathcal{Q}(Y)-\mathcal{Q}(X)\big)A_k\big)\\
&= \sum_{k=1}^{m} tr\big(A_k^*A_k\big(\mathcal{Q}(Y)-\mathcal{Q}(X)\big)\big)\\
&= tr\Big(\Big(\sum_{k=1}^{m} A_k^*A_k\Big)\big(\mathcal{Q}(Y)-\mathcal{Q}(X)\big)\Big)\\
&\leq \Big\|\sum_{k=1}^{m} A_k^*A_k\Big\|\|\mathcal{Q}(Y)-\mathcal{Q}(X)\|_{tr}\\
&\leq \frac{1}{c}\Big\|\sum_{k=1}^{m} A_k^*A_k\Big\|\Big(\frac{\|Y-X\|_{tr}}{1+\tau\|Y-X\|_{tr}}\Big)\\
&< \frac{\|Y-X\|_{tr}}{1+\tau\|Y-X\|_{tr}},
\end{aligned}$$

so that

$$\frac{1+\tau\|Y-X\|_{tr}}{\|Y-X\|_{tr}} \leq \frac{1}{\|T(Y)-T(X)\|_{tr}},$$

which implies that

$$\tau - \frac{1}{\|T(Y)-T(X)\|_{tr}} \leq -\frac{1}{\|Y-X\|_{tr}}.$$

This yields that

$$\tau + F\big(\|T(Y)-T(X)\|_{tr}\big) \leq F\big(\|Y-X\|_{tr}\big),$$

which shows that T is an $(F,\preceq)$-contraction. Since $\sum_{k=1}^{m} A_k^*\mathcal{Q}(H)A_k \succ 0$, therefore $H \preceq T(H)$. So that, there exists a $(T,\preceq)$-Picard sequence in $\mathcal{H}(n)$ (in view of Proposition 3.9). Thus, all the hypotheses of Theorem 4.1 are satisfied. Hence there exists $X \in \mathcal{H}(n)$ such that $T(X) = X$, i.e., the matrix equation (5.1) has a solution in $\mathcal{H}(n)$. □

Theorem 5.4. *Under the assumptions of Theorem 5.3, equation (5.1) has a unique solution.*

Proof. In view of Theorem 5.3, the set $Fix(T)$ is nonempty. According to [20] there always exist a greatest lower bound as well as a least upper bound for each $X, Y \in \mathcal{H}(n)$, so that $Fix(T)$ is $\preceq^s$-connected. Therefore using Theorem 4.3 we conclude that T has a unique solution, i.e., equation (5.1) has a unique solution in $\mathcal{H}(n)$. □

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