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ORDER-THEORETIC COMMON FIXED POINT RESULTS FOR F-CONTRACTIONS

R. GUBRAN, M. IMDAD, I. A. KHAN AND W. M. ALFAQIEH

Abstract. By relaxing the conditions assumed on the auxiliary function F utilized in [Wardowski, Fixed Point theory and Appl., 94 (2012)], we establish order-theoretic results on coincidence and common fixed points. Our results generalize and extend many relevant results of the existing literature.

1. Introduction and Preliminaries

Even after more than nine decades of the appearance of Banach contraction principle, researchers still find novel ways to continue research on and around this principle. Recently, literature witnessed one such instants when Wardowski [26] coined the idea of F-contractions to generalize Banach contraction principle in a different but novel way.

The following definition of F-contraction is essentially due to Wardowski [26].

Definition 1.1. *Let $\mathcal{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:*
F1 : the function F is strictly increasing,
F2 : for all sequences $\{s_n\}$ in $(0, \infty)$, $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$,
F3 : there exists $k \in (0, 1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.
A self-mapping S on a metric space (M, d) is said to be an F-contraction if there exists $\tau > 0$ and $F \in \mathcal{F}$ such that (for all $u, v \in M$)

$$d(Su, Sv) > 0 \Rightarrow \tau + F(d(Su, Sv)) \leq F(d(u, v)). \tag{1.1}$$

Utilizing above type of auxiliary functions, Wardowski [26] proved that every F-contraction mapping on a complete metric space possesses a unique fixed point. Moreover, on varying the elements of $\mathcal{F}$ suitably, a variety of known contractions can be deduced. Some elements of $\mathcal{F}$ and their corresponding contractions can be found in [26] and subsequent researches thereafter.

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In recent years, the concept of F-contractions has attracted the attention of several researchers. By now, there exists a considerable literature in enriching this idea (e.g. [14, 21, 24, 27]). Out of all efforts attempted to investigate the possibility of weakening the conditions $F1 - F3$, we recall the following three: Piri and Kumam [19] replaced the condition $F3$ by the continuity of F which is essentially motivated by the fact that most of the utilized functions in the existing literature are continuous. In another effort, Vetro [25] widened the class $\mathcal{F}$ by replacing τ with a function whose infinitum is greater than zero. Recently, Secelean and Wardowski [22] noticed that it is enough to define the function F on (o, v) where v is grater than the diameter of the the underlying set.

On the other hand, proving fixed point results in the setting of ordered metric spaces continues to be an active direction of research (e.g., [3, 7–10, 15–18] and references therein). Our work here is on the same lines wherein we consider F-contractions on ordered metric spaces. Before chalking out our results, the following background materials are needed.

A triplet $(M, d, \preceq)$ is called an ordered metric space if (M, d) is a metric space and $(M, \preceq)$ is an ordered set. Moreover, two elements $u, v \in M$ are called comparable if either $u \preceq v$ or $u \succeq v$. Further, M is called totally or linearly ordered if every two elements of M are comparable.

Definition 1.2. *[4] Let (S, T) be a pair of self-mappings on an ordered set $(M, \preceq)$. Then, S is said to be T-increasing if $Tu \preceq Tv \Rightarrow Su \preceq Sv$ for every $u, v \in M$.*

Definition 1.3. *[20]. Let (S, T) be a pair of self-mappings on an ordered metric space $(M, d, \preceq)$. Then, S is said to be T-continuous at u if for any sequence $\{u_n\}$ in M,*

$$Tu_n \xrightarrow{d} Tu \Rightarrow Su_n \xrightarrow{d} Su.$$

Moreover, S is called T-continuous if it is T-continuous at every point of M.

Definition 1.4. *[12, 13] Let (S, T) be a pair of self-mappings on an ordered metric space $(M, d, \preceq)$. Then,*

(i) the pair (S, T) is said to be weakly compatible if S and T commute at their coincidence points,

(ii) the pair (S, T) is said to be compatible if $\lim\limits_{n\to\infty} d(TSu_n, STu_n) = 0$, whenever $\{u_n\}$ is a sequence in M such that $\lim\limits_{n\to\infty} Tu_n = \lim\limits_{n\to\infty} Su_n$.

Remark 1.1. *Every compatible pair of mappings is weakly compatible.*

Lemma 1.1. *[2] Let (S, T) be a pair of self-mappings defined on an ordered set $(M, \preceq)$. If S is T-monotone and $Tu = Tv$, then $Su = Sv$.*

Lemma 1.2. *[2] Let (S, T) be a pair of self-mappings defined on an ordered metric space $(M, \preceq, d)$. If the pair is weakly compatible, then every point of coincidence of the pair is also a coincidence point.*

The aim of this paper is to establish the existence of coincidence and common fixed point (of two self-mappings) on an ordered metric space. We relax conditions $F1$ and $F3$ and merely exercise one way implication of the condition $F2$ besides restricting F on its effective domain. Our results generalize, extend and improve some core results, especially those contained in [1, 6, 11, 14, 27] and some others. Some examples are also given to support the usability of newly proved results.

2. Main Results

In the sequel, $\mathbb{S}$ denotes the set of all the functions $\sigma : (0, \infty) \to (0, \infty)$ with $\liminf_{t \to s^+} \sigma(t) > 0$ for all $s \geq 0$. As the effective domain of the function F in (1.1) depends on the underlying metric space, we reformulate Definition 1.1 as under:

Definition 2.1. *Let (S, T) be a pair of self-mappings on an ordered metric space $(M, d, \preceq)$ and $\mathbb{E} =: (a, b)$ where $a = \inf_{u \prec v} d(u, v)$ and $b = \max_{u \prec v} d(u, v)$ where $u, v \in M$. Consider the mapping $F : \mathbb{E} \to \mathbb{R}$ with the following conditions:*

- F_1**:** *the function F is strictly increasing,*
- F_2**:** *for all sequence $\{s_n\}$ in $\mathbb{E}$, $\lim_{n \to \infty} F(s_n) = -\infty \Rightarrow \lim_{n \to \infty} s_n = 0$,*
- F_3**:** *there exists $k \in (0, 1)$ such that $\lim_{s \to 0^+} s^k F(s) = 0$.*

Let $\mathbb{F}_i$ be the family of all functions F satisfying condition Fi where $i \in \{1, 2, 3\}$. Then, the mapping S is said to be an F_T-weak contraction for a function $F \in \mathbb{F}_2$ and $\sigma \in \mathbb{S}$ if

$$d(Su, Sv) > 0 \Rightarrow \sigma(d(Tu, Tv)) + F(d(Su, Sv)) \leq F(M_{S,T}(u, v)), \tag{2.1}$$

for all $u, v \in M$ such that $Tu \preceq Tv$, where

$$M_{S,T}(u, v) = max \left\{ d(Tu, Tv), d(Tu, Su), d(Tv, Sv), \frac{d(Tu, Sv) + d(Tv, Su)}{2} \right\}.$$

Definition 2.2. *Let (S, T) be a pair of self-mappings on an ordered metric space $(M, d, \preceq)$. Then, M is said to be $\preceq_T$-regular if for every sequence $\{Tu_n\}$ in M with $Tu_n \to Tu$, there exists a subsequence $\{Tu_{n_k}\}$ (of $\{Tu_n\}$) and a positive integer k_0 such that $Tu_{n_k} \preceq Tu$ for all $k \geq k_0$.*

On choosing T to be the identity mapping, in Definition 2.1, S will be called 'F-weak contraction' while M in Definition (2.2) will be refereed '$\preceq$-regular'.

Now, we are ready to state and prove our main result as follows.

Theorem 2.1. *Let $(M, \preceq, d)$ be an ordered metric space and $S, T : M \to M$ be two mappings such that S is F_T-weak contraction wherein F is a continuous function. Suppose that the following conditions hold:*

(a) there exists $u_0 \in M$ such that $Tu_0 \preceq Su_0$,
(b) S is T-increasing,
(c) $S(M) \subseteq T(M)$,
(d) $(T(M), d)$ is complete.
(e) $(T(M), d)$ is $\preceq_T$-regular.

Then the set $C(S, T) = \{u \in M | Su = Tu\}$ is non-empty. Further, the set $C(S, T)$ is singleton if and only if it is totally ordered.

Proof. Choose $u_0 \in M$ such that $Tu_0 \preceq Su_0$. Using (b) and (c), one can define a sequence $\{u_n\}$ in M with $Su_n = Tu_{n+1}$ and $Tu_n \preceq Tu_{n+1}$ for all $n \in \mathbb{N}_0 =: \mathbb{N} \cup \{0\}$. Notice that, if $Tu_n = Tu_{n+1}$ for some $n \in \mathbb{N}_0$, then $u_n \in C(S, T)$ and we are done.

Otherwise, we assume that such equality does not occur for any $n \in \mathbb{N}_0$. On setting $u = u_{n-1}$ and $v = u_n$ in (2.1), we have

$$\sigma(d(Tu_{n-1}, Tu_n)) + F(d(Tu_n, Tu_{n+1})) \leq F(M_{S,T}(u_{n-1}, u_n)) \tag{2.2}$$

where $M_{S,T}(u_{n-1}, u_n) = max\left\{d(Tu_{n-1}, Tu_n), d(Tu_n, Tu_{n+1}), \frac{d(Tu_{n-1}, Tu_{n+1})}{2}\right\}$. Obviously, $max\{d(Tu_{n-1}, Tu_n), d(Tu_n, Tu_{n+1})\} \geq \frac{d(Tu_{n-1}, Tu_{n+1})}{2}$. If it is possible that $d(Tu_{n-1}, Tu_n) \leq d(Tu_n, Tu_{n+1})$ for some $n \in \mathbb{N}$, then (2.2) becomes

$$F(d(Tu_n, Tu_{n+1})) \leq F(d(Tu_n, Tu_{n+1})) - \sigma(d(Tu_{n-1}, Tu_n)),$$

a contradiction. Therefore, for all $n \in \mathbb{N}$, we have

$$F(d(Tu_n, Tu_{n+1})) \leq F(d(Tu_{n-1}, Tu_n) - \sigma(d(Tu_{n-1}, Tu_n)),$$

which, by induction on n, gives rise

$$F(d(Tu_n, Tu_{n+1})) \leq F(d(Tu_0, Tu_1) - n \inf_{1 \leq i \leq n} \sigma(d(Tu_{i-1}, Tu_i)). \tag{2.3}$$

On letting $n \to \infty$ in (2.3), we get $\lim_{n \to \infty} F(d(Tu_n, Tu_{n+1})) = -\infty$. Therefore, by F_2, we have

$$\lim_{n \to \infty} d(Tu_n, Tu_{n+1}) = 0. \tag{2.4}$$

We assert that $\{Tu_n\}$ is a Cauchy sequence . Let us assume that $\{Tu_n\}$ is not so. Then there exists $\epsilon > 0$ and two subsequences $\{Tu_{n_k}\}$ and $\{Tu_{m_k}\}$ of $\{u_n\}$ such that

$$n_k > m_k \geq k,\ d(Tu_{n_k}, Tu_{m_k}) \geq \epsilon \text{ and } d(Tu_{n_k-1}, Tu_{m_k}) < \epsilon \text{ for all } k \in \mathbb{N}.$$

Now, we have

$$\epsilon \leq d(Tu_{n_k}, Tu_{m_k}) \leq d(Tu_{n_k}, Tu_{n_k-1}) + d(Tu_{n_k-1}, Tu_{m_k}) \leq d(Tu_{n_k}, Tu_{n_k-1}) + \epsilon,$$

so that $\lim_{k \to \infty} d(Tu_{n_k}, Tu_{m_k}) = \lim_{k \to \infty} d(Tu_{n_k-1}, Tu_{m_k}) = \epsilon$. Again, we have

$$\begin{aligned}
\epsilon &\leq d(Tu_{n_k}, Tu_{m_k}) \\
&\leq d(Tu_{n_k}, Tu_{m_k-1}) + d(Tu_{m_k-1}, Tu_{m_k}) \\
&\leq d(Tu_{n_k}, Tu_{n_k-1}) + d(Tu_{n_k-1}, Tu_{m_k-1}) + d(Tu_{m_k-1}, Tu_{m_k}) \\
&\leq d(Tu_{n_k}, Tu_{n_k-1}) + d(Tu_{n_k-1}, Tu_{m_k}) + 2d(Tu_{m_k-1}, Tu_{m_k})
\end{aligned}$$

which on letting $k \to \infty$ gives rise

$$\lim_{k \to \infty} d(Tu_{n_k}, Tu_{m_k-1}) = \lim_{k \to \infty} d(Tu_{n_k-1}, Tu_{m_k-1}) = \epsilon.$$

It follows that there exists $l \in \mathbb{N}$ with $d(Tu_{n_k+1}, Tu_{m_k+1}) > 0$, $d(Tu_{n_k+1}, Tu_{m_k}) > 0$ and $d(Tu_{m_k+1}, Tu_{n_k}) > 0$ for all $k \geq l$. Then for all $k \geq l$, we have (on setting $u = u_{n_k}$ and $v = u_{m_k}$ in (2.1))

$$\sigma(d(Tu_{n_k-1}, Tu_{m_k-1})) + F(d(Tu_{n_k}, Tu_{m_k})) \quad \leq \quad F(M_{S,T}(u_{n_k-1}, u_{n_k-1})), \tag{2.5}$$

where

$$\begin{aligned}
M_{S,T}(u_{n_k-1}, u_{m_k-1}) \quad = \quad \max\Big\{&d(Tu_{n_k-1}, Tu_{m_k-1}), d(Tu_{n_k-1}, Tu_{n_k}), d(Tu_{m_k-1}, Tu_{m_k}), \\
&\frac{d(Tu_{n_k-1}, Tu_{m_k}) + d(Tu_{m_k-1}, Tu_{n_k})}{2}\Big\}
\end{aligned}$$

Letting $k \to \infty$ in (2.5) and making use of the definition of σ together with the continuity of F, we get

$$F(\epsilon) < \liminf_{k\to\infty} \sigma(d(Tu_{n_k-1}, Tu_{m_k-1})) + F(\epsilon) \quad \leq \quad F(\epsilon),$$

a contradiction so that $\{Tu_n\}$ is a Cauchy sequence having a limit $Tu \in T(M)$.

Next, we show that $u \in C(S,T)$. If $Tu_n = Su$ for infinitely many $n \in \mathbb{N}$, then there exists a subsequence of $\{Tu_n\}$ which converges to Su. Now, the uniqueness of the limit concludes the proof. Henceforth, we assume that $Tu_n \neq Su$ for all $n \in \mathbb{N}$. On using the $\preceq_T$-regularity of M, there exists a subsequence $\{Tu_{n_k}\}$ of $\{Tu_n\}$ and a positive integer k_0 such that $Tu_{n_k} \preceq Tu$ for all $n_k \geq k_0$. Now, for $n_k \geq k_0$, we can set $u = u_{n_k}$ and $v = u$ in (2.1) so that

$$\begin{aligned} \sigma(d(Tu_{n_k}, Tu)) + F(d(Su_{n_k}, Su)) \quad &\leq \quad F(M_{S,T}(u_{n_k}, u)) \\ &= \quad F\big(max\big\{d(Tu_{n_k}, Su), d(Tu_{n_k}, Tu_{n_k+1}), d(Tu, Su) \\ &\quad\quad , \frac{1}{2}[d(Tu_{n_k}, Su) + d(Tu, Tu_{n_k+1})]\big\}\big). \end{aligned}$$

If $d(Tu, Su) > 0$, then letting $n \to \infty$ implies that

$$\liminf_{d(u_n,Tu)\to 0^+} \sigma(d(Tu_n, Tu)) + F(d(Tu, Su)) \leq F(d(Tu, Su)),$$

a contradiction so that $u \in C(S,T)$.

Finally, let the set $C(S,T)$ be totally ordered. On a contrary, let v be another elements of $C(S,T)$ such that $d(Su, Sv) > 0$. Then, from (2.1), we have

$$\begin{aligned} \sigma(d(Tu, Tv)) + F(d(Su, Sv)) &\leq F\Bigg(max\Bigg\{d(Tu, Tv), d(Tu, Su), d(Tv, Sv), \\ &\qquad \frac{d(Tu, Sv) + d(Tv, Su)}{2}\Bigg\}\Bigg) \\ &= F(d(Su, Sv)), \end{aligned}$$

a contradiction so that $d(Su, Sv) = 0$. The converse implication is obvious. □

By setting $T = I$ in Theorem 2.1 , we deduce the following result for a single mapping S.

Corollary 2.1. *Let $(M, \preceq, d)$ be an ordered metric space, $S : M \to M$ be such that S is F-weak contraction with a continuous function $F \in \mathbb{F}_2$ and $\sigma \in \mathbb{S}$. If the following conditions hold:*

(a) there exists $u_0 \in M$ such that $u_0 \preceq Su_0$,
(b) S is increasing,
(c) (M, d) is complete.
(d) (M, d) is $\preceq$-regular.

Then S has a fixed point. Further, this fixed point is unique if the set of fixed points of S is totally ordered set.

The following result is another version of Theorem 2.1 where the auxiliary function F is not essentially continuous:

Theorem 2.2. *The conclusion of Theorem 2.1 remains true if $F \in \mathbb{F}_{2,3}$ and σ is replaced by a constant $\tau > 0$ provided the condition* (e) *is replaced by any of the following (besides retaining rest of the hypotheses):*

(e1) S is T-continuous.

(e2) Both S and T are continuous mappings and the pair (S,T) is compatible.

Proof. The proof runs on the lines of the proof of Theorem 2.1 up to (2.4). Thereafter, owing to F_3, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (d(Tu_n, Tu_{n+1}))^k F(d(Tu_n, Tu_{n+1})) = 0. \tag{2.6}$$

From (2.3), we have

$$d(Tu_n, Tu_{n+1})^k \left[F(d(Tu_n, Tu_{n+1})) - F(d(Tu_0, Tu_1))\right] \leq -n\tau d(Tu_n, Tu_{n+1})^k \leq 0$$

which on letting $n \to \infty$, gives rise (in view of 2.4), (2.6))

$$\lim_{n\to\infty} nd(Tu_n, Tu_{n+1})^k = 0.$$

Hence, there exists $N \in \mathbb{N}_0$ such that $nd(Tu_n, Tu_{n+1})^k \leq 1$ for all $n \geq N$, so that

$$d(Tu_n, Tu_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq N. \tag{2.7}$$

We assert that $\{Tu_n\}$ is a Cauchy sequence. Consider $n_1, n_2 \in \mathbb{N}_0$ with $n_2 > n_1 \geq N$. Using the triangle inequality and (2.7), we have

$$d(Tu_{n_2}, Tu_{n_1}) \leq \sum_{i=n_1}^{n_2-1} d(Tu_i, Tu_{i+1}) \leq \sum_{i=n_1}^{\infty} d(Tu_i, Tu_{i+1}) \leq \sum_{i=n_1}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

As $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent series, on letting $n_1, n_2 \to \infty$ one gets

$$\lim_{n_1,n_2\to\infty} d(Tu_{n_2}, Tu_{n_1}) = 0$$

so that the desired assertion is established. Hence, there exists $u \in T(M)$ such that

$$\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Su_n = Tu.$$

In view of condition $(e1)$, $\lim\limits_{n\to\infty} Tu_n = Tu$ implies that $\lim\limits_{n\to\infty} Su_n = Su$ which by the uniqueness of the limit implies $Tu = Su$. Now, assume that the condition $(e2)$ holds. Then, in view of the continuity of T, we have

$$\lim_{n\to\infty} T(Su_n) = Tw = \lim_{n\to\infty} T(Tu_n),$$

where $w \in M$ is such that $w = Tu$. Also, due to the compatibility, we have,

$$\lim_{n\to\infty} d(T(Su_n), S(Tu_n)) = 0,$$

so that

$$\lim_{n\to\infty} S(Tu_n) = Tw.$$

Now, the continuity of S implies that

$$\lim_{n\to\infty} S(Tu_n) = S(\lim_{n\to\infty} Tu_n) = Sw.$$

From the two preceding equations, we get $w \in C(S,T)$. Rest of the proof can be completed on the liens of the proof of Theorem 2.1. $\square$

Remark 2.1. *Theorems 2.1 and 2.2 reveal the fact that if the pair (S, T) has a common fixed point, then it is unique provided the set $C(S, T)$ is totally ordered.*

Example 2.1. *Consider the set $M = \{0, 1, 2, 3\}$ equipped with the usual metric and an order given by: $u \preceq v \Leftrightarrow$ either $u = v$ or ($u < v$ where $u, v \in \{0, 2, 3\}$). Define $S, T : M \to M$ by:*

$$Su = \begin{cases} u-1, & \text{for } u \neq 0, \\ 0, & \text{for } u = 0, \end{cases} \quad \text{and } Tu = \begin{cases} 0, & \text{for } u = 0, \\ 1, & \text{for } u = 3, \\ u+1, & \text{for } u \in \{1, 2\}. \end{cases}$$

Now, we verify the inequality (2.1) for $\sigma(s) = \tau$ where τ is such that $e^{-\tau} = 3/4$ and $F(t) = \ln t$. Due to the symmetricity of the metric, we distinguish the following cases:
Case I: If $u = 0, v = 2$, then

$$d(S0, S2) = 1 \leq \frac{9}{4} = \frac{3}{4}d(0, 3) = \frac{3}{4}d(T0, T2) = \frac{3}{4}M_{S,T}(0, 2).$$

Case II: If $u = 1, v = 2$, then

$$d(S1, S2) = 1 \leq \frac{3}{2} = \frac{3}{4}d(2, 0) = \frac{3}{4}d(T1, S1) = \frac{3}{4}M_{S,T}(1, 2).$$

Hence, in all cases, inequality (2.1) holds for all u, v such that $d(Su, Sv) > 0$ and $Tu \preceq Tv$. Obviously, the other conditions of Theorems 2.1 and 2.2 are satisfied ensuring the existence of a coincidence point of S and T.

Observe That, Theorem 2.2 is a proper generalization of Theorem 2.2 of [6]. Indeed, Theorem 2.2 of [6] is not applicable to present example as $d(S1, S2) \not\leq \frac{3}{4}d(T1, T2)$.

Example 2.2. *Let n be a given natural number and consider the set $M_n = \{0, 1, 2, 3, 4\} \cup \{4 + \frac{1}{i}\}_{i=1}^{n}$ equipped with the usual metric and an order given by: $u \preceq v \Leftrightarrow$ either $u = v$ or ($u < v$ where $u, v \in \{0, 1, 2, 3, 4\}$) . Define $S : M \to M$ by:*

$$Su = \begin{cases} 0, & \text{for } u = 4, \\ 3, & \text{otherwise}, \end{cases}$$

Observe that $F(t) = t$, $t \in \mathbb{E} = \{1, 2, 3, 4\}$ satisfies F_2 vacuously. Observe that there is no sequence $\{s_n\}$ in $\mathbb{E}$ such that $\lim_{n\to\infty} F(s_n) = -\infty$. Now, we verify the inequality (2.1) for $\sigma(s) = \frac{1}{2s}$ where T is the identity mapping on M. As $\max_{u=4, v\in\{0,1,2,3\}} \sigma(d(u, v)) = 1/2$, it is enough to show that

$$\frac{1}{2} + 3 \leq \max_{u=4, v\in\{0,1,2,3\}} \left\{ d(u, v), d(u, Su), d(v.Sv), \frac{d(u, Sv) + d(v, Su)}{2} \right\}$$

but this inequality is true as $d(4, S4) = 4$. Other conditions of Theorems 2.1 and 2.2 are satisfied ensuring the existence of a unique fixed point of S.

In the following lines, we highlight the superiority of our results over some core results of the existing literature:

(a) Theorems 2.1 and 2.2 respectively generalize Theorems 3.1 and 3.2 contained in [11].
(b) Theorems 2.1 and 2.2 (with $T = I_M$) are ordered-theoretic analogues of Theorem 2.2 of [14] and Theorem 2.4 of [27].

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Rqeeb Gubran
Department of Mathematics, A.M.U., Aligarh - 202002 U.P.India
E-mail address: rqeeeb@gmail.com

Mohammad Imdad
Department of Mathematics, A.M.U., Aligarh - 202002 U.P.India
E-mail address: mhimdad@gmail.com

Idrees A. Khan
Department of Mathematics Integral University, Lucknow 226026 U.P.India
E-mail address: idrees_maths@yahoo.com

Waleed M. Alfaqieh
Department of Mathematics, A.M.U., Aligarh - 202002 U.P.India
E-mail address: waleedmohd2016@gmail.com