Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://bmathaa.org
Volume 10 Issue 2(2018), Pages 53-67

SET VALUED CONTRACTION OF SUZUKI-EDELSTEIN-WARDOWSKI TYPE AND BEST PROXIMITY POINT RESULTS

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Abstract. The aim of this paper is to show the existence of best proximity points for multivalued Suzuki-Edelstein-Wardowski type α-proximal contractions in the setting of complete metric spaces and partially ordered metric spaces. We give examples to illustrate the main results. Our results improve and extend the corresponding results in the literature.

1. Introduction

Let A and B be two nonempty subsets of a metric space (X, d) and $F : A \to CB(B)$. A point $x^* \in A$ is called a best proximity point of F if

$$D(x^*, Fx^*) \quad = \quad \inf\{d(x^*, y) : y \in Fx^*\} = dist(A, B),$$

where

$$dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

If $A \cap B \neq \phi$, then x^* is a fixed point of F. If $A \cap B = \phi$, then $D(x, Fx) > 0$ for all $x \in A$ and F has no fixed point.

Consider the following optimization problem:

$$\min\{D(x, Fx) : x \in A\}. \tag{1.1}$$

It is then important to study necessary conditions so that the above minimization problem has at least one solution.

Since

$$d(A, B) \leq D(x, Fx) \tag{1.2}$$

for all $x \in A$. Hence the optimal solution to the problem

$$\min\{D(x, Fx) : x \in A\} \tag{1.3}$$

for which the value $d(A, B)$ is attained is indeed a best proximity point of multivalued mapping F. Many authors has explored the existence and convergence of best proximity points under different contractive conditions in certain distance spaces (see e.g. [1, 2, 4, 5, 13, 14, 19, 20, 22, 26, 27, 31, 33] and references therein).

2000 *Mathematics Subject Classification.* 55M20, 47H10.
Key words and phrases. Best proximity point, weak P-property, α-admissible mapping.

Submitted May 9, 2018. Published May 29, 2018.
Communicated by E. Karapinar.

Recently, Samet et al. [38] introduced the notion of α-ψ-contraction and proved some fixed point theorems for such mappings in the context of complete metric spaces. Some interesting multivalued generalizations of α-ψ-contractive type mappings are given in [3, 6, 7, 8, 10, 15, 16, 32, 35].

In 1962, Edelstein [21] obtained the following result:

Theorem 1.1. *Let (X, d) be a compact metric space and T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.*

In 2008, Suzuki [40] introduced a new type of mapping and presented generalization of Banach contraction principle in which the completeness can be characterized by the existence of a fixed point of these mappings.

Theorem 1.2. *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & if \quad 0 \leq r \leq \sqrt{5} - 12 \\ \frac{1-r}{r^2} & if \quad \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}} \\ \frac{1}{1+r} & if \quad \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Inspired by Theorem (1.2), Suzuki [40] proved a generalization of Edelstein's fixed point theorem.

Theorem 1.3. [41] *Let (X, d) be a compact metric space, and let T be mapping on X. Assume that $(\frac{1}{2})d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Then T has a unique fixed point.*

On the other hand, a generalized version of contraction mapping introduced by Wardowski [43] called $\mathcal{F}$-contraction, i.e. a mapping $T : X \to X$ satisfying

$$\tau + \mathcal{F}(d(Tx, Ty)) \leq \mathcal{F}(d(x, y))$$

for all $x, y \in X$ with $Tx \neq Ty$, where $\tau > 0$ and $\mathcal{F} : (0, \infty) \to \mathbb{R}$ satisfy the following conditions:

($\mathcal{F}_1$) $\mathcal{F}$ is strictly increasing;
($\mathcal{F}_2$) for all sequence $\{a_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} \mathcal{F}(a_n) = -\infty$;
($\mathcal{F}_3$) there exist $0 < k < 1$ such that $\lim_{a\to 0^+} a^k \mathcal{F}(a) = 0$.

It was also proved that every $\mathcal{F}$-contraction on a complete metric space has a unique fixed point. In 2014, Piri and Kumam [37] combined the concept of $\mathcal{F}$-contraction with Suzuki as

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + \mathcal{F}(d(Tx, Ty)) \leq \mathcal{F}(d(x, y)).$$

Gopal *et al.* [23] introduced the concept of α-type-$\mathcal{F}$-contraction by using function $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$,

$$\tau + \alpha(x, y)\mathcal{F}(d(Tx, Ty)) \leq \mathcal{F}(d(x, y))$$

and proved some fixed point results. Recently, Turinici in [42] relaxed the condition $(\mathcal{F}_2)$ by

$(\mathcal{F}_2')$ for all sequence $\{a_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} \mathcal{F}(a_n) = -\infty$.

Then the implication

$(\mathcal{F}_2'')$ $\mathcal{F}(a_n) \to -\infty$ implies $a_n \to \infty$

can be derived from $(\mathcal{F}_1)$. Recently, Wardowski [44] consider the class of $\mathcal{F}$-contractions in a generalized way by taking constant τ as a function φ on $\mathbb{R}^+$ to itself and defined $(\varphi, \mathcal{F})$-contraction (nonlinear contraction) on a metric space (X, d) satisfying

$(\mathcal{H}_1)$ $\mathcal{F}$ satisfies $(\mathcal{F}_1)$ and $(\mathcal{F}_2')$;
$(\mathcal{H}_2)$ $\liminf_{s\to t^+} \varphi(s) > 0$ for all $t \geq 0$;
$(\mathcal{H}_3)$ $\varphi(d(x,y)) + \mathcal{F}(d(Tx,Ty)) \leq \mathcal{F}(d(x,y))$ for all $x, y \in X$ such that $Tx \neq Ty$.

Wardowski [44] proved a fixed point result for such nonlinear contraction by omitting $(\mathcal{F}_3)$. For more work concerning F-contraction, we refer to [12, 16, 29, 30] and references therein.

The purpose of this paper is to consider the condition

$(\mathcal{H}_3')$ for all $x, y \in A$,

$$\phi((D(x,Tx)) - \alpha(x,y)D(A,B) \leq \alpha(x,y)d(x,y)$$

implies

$$\varphi(d(x,y)) + \alpha(x,y)\mathcal{F}(D(y,Ty)) \leq \mathcal{F}(M(x,y))$$

where

$$\begin{aligned} M(x,y) \quad = \quad & \max\left\{ d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(x,Ty)}{2}, \right. \\ & \left. \frac{D(y,Ty)[1+D(x,Tx)]}{1+d(x,y)}, \frac{D(y,Tx)[1+D(x,Ty)]}{1+d(x,y)} \right\}, \qquad (1.4) \end{aligned}$$

where A and B are subsets of X, $\phi : [0,\infty) \to [0,\infty)$ a nondecreasing functions such that $\lim_{n\to\infty} \phi^n(t) = 0$, $\phi(t) < t$ for all $t > 0$ [34] and $T : A \to C(B)$ and to prove multivalued and single valued best proximity point results for such nonlinear contractions omitting $(\mathcal{F}_3)$ in the framework of complete metric spaces and partially ordered complete metric spaces. Some fixed point results for such mappings will be obtained as an applications of our results.

2. **Preliminaries**

In the sequel, (X, d) a metric space. $C(X)$, $CB(X)$ and $K(X)$ by the families of all nonempty closed subsets, nonempty closed and bounded subsets, nonempty compact subsets of (X, d) and Φ the set of all functions ϕ respectively. For any $A, B \in CB(X)$ and $x \in X$, define

$$\begin{aligned} A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } d(a,b) = D(A,B)\} \\ B_0 &= \{a \in A : \text{there exists some } a \in A \textit{ suchthat } d(a,b) = D(A,B)\} \\ D(x,A) &= \inf\{d(x,a) : a \in A\}. \end{aligned}$$

We present now the necessary definitions and results which will be useful in the sequel.

Definition 2.1. *Let A and B be nonempty subsets of a metric space (X, d). A point x is called a best proximity point of mapping $T : A \to B$ if*

$$D(x, Tx) = D(A, B).$$

Definition 2.2. [5] *Let (A, B) be a pair of nonempty subsets of a metric space (X, d). Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,*

$$\left.\begin{array}{rcl} d(x_1, y_1) & = & D(A, B) \\ d(x_2, y_2) & = & D(A, B) \end{array}\right\} \text{ implies } d(x_1, x_2) \leq d(y_1, y_2).$$

Definition 2.3. [5] *Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to C(B)$ is called α-proximal admissible if there exists a mapping $\alpha : A \times A \to [0, \infty)$ such that*

$$\left.\begin{array}{rcl} \alpha(x_1, x_2) & \geq & 1 \\ d(u_1, y_1) & = & D(A, B) \\ d(u_2, y_2) & = & D(A, B) \end{array}\right\} \text{ implies } \alpha(u_1, u_2) \geq 1,$$

where $x_1, x_2, u_1, u_2 \in A$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$.

Definition 2.4. *Let A and B be two nonempty subsets of a metric space (X, d). Let $\alpha : A \times A \to [0, \infty)$ and $T : A \to C(B)$. We say that T is an α_*-continuous mapping on (X, d), if for given given $x \in X$ and sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x \in A$ such that $x_n \to x^*$ as $n \to \infty$ then $Tx_n \xrightarrow{H} Tx^*$.*

3. Best Proximity Point Results in Metric Spaces

Throughout the paper, we denote $\mathfrak{F}$ the class of all functions $\mathcal{F}$ satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3')$. First we introduce the notion of Suzuki-Edelstein-Wardowski type α-proximal contraction as follows:

Definition 3.1. *Suppose A and B are two non-empty subsets of a metric space (X, d). A multivalued mapping $T : A \to C(B)$ is said to be Suzuki-Edelstein-Wardowski type α-proximal contraction if there exist functions $\alpha : A \times A \to [0, \infty)$, $\phi \in \Phi$ along with conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3')$.*

Example 3.2. *Let $X = \mathbb{R}$ with usual metric d. Let $A = \{1 + \frac{1}{p^{n-1}} : p, n \in \mathbb{N}, p > 1\}$ and $B = \{\frac{1}{p^{n-1}} : p, n \in \mathbb{N}, p > 1\}$ are subsets of $\mathbb{R}$. Then $D(A, B) = 1$. Define $T : A \to C(B)$ by*

$$Tx = \begin{cases} \{1, \frac{1}{p^n}\} \; if & x = 1 + \frac{1}{p^{n-1}} \\ \{0, \frac{1}{p}\} \; if & x = 2, \end{cases}$$

$\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} p^{n+1} \; if & x = 1 + \frac{1}{p^{n-1}}, y = 1 + \frac{1}{p^n} \\ p \; if & x = y = 1 + \frac{1}{p^{n-1}} \\ 0 \; if & x = 2, \end{cases}$$

$\varphi : (0, \infty) \to (0, \infty)$ by $\varphi(t) = \frac{1}{t} + \frac{1}{p}$, $\phi \in \Phi$ by $\phi(t) = \frac{t}{p}$ for all t and $\mathcal{F} \in \mathfrak{F}$ by $\mathcal{F}(t) = ln(t)$. Then

$$D(x, Tx) = \begin{cases} \frac{1}{p^n} \; if & x = 1 + \frac{1}{p^{n-1}} \\ 0 \; if & x = 1, 2. \end{cases}$$

Now for $x = 1 + \frac{1}{p^{n-1}}, n > 1$, $Tx = \{1, \frac{1}{p^n}\}$. *Thus for* $y = 1 + \frac{1}{p^n}$, *we have*

$$\phi(D(x,Tx)) - \alpha(x,y)D(A,B) = \left(\frac{1}{p}\right)\left(\frac{1}{p^n}\right) - p^{n+1}(1) = \frac{1}{p^{n+1}} - p^{n+1} \tag{3.1}$$

and

$$\alpha(x,y)d(x,y) = (p^{n+1})\left(\frac{1}{p^n}\right) = p. \tag{3.2}$$

From (3.1) *and* (3.2) *we have*

$$\phi(D(x,Tx)) - \alpha(x,y)D(A,B) \leq \alpha(x,y)d(x,y). \tag{3.3}$$

Now

$$\begin{aligned} M(x,y) &= \max\Big\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(x,Ty)}{2}, \\ &\qquad \frac{D(y,Ty)[1+D(x,Tx)]}{1+d(x,y)}, \frac{D(y,Tx)[1+D(x,Ty)]}{1+d(x,y)}\Big\} \\ &= \max\left\{\frac{1}{p^n}, \frac{1}{p^n}, \frac{1}{p^{n+1}}, 0\frac{1}{p^{n+1}}, 0\right\} = \frac{1}{p^n}. \end{aligned}$$

So

$$\begin{aligned} \varphi(d(x,y)) + \alpha(x,y)\mathcal{F}(D(y,Ty)) &= \varphi\left(\frac{1}{p^n}\right) + p^{n+1}\mathcal{F}\left(\frac{1}{p^{n+1}}\right) \\ &= p^n + \frac{1}{p} + p^{n+1} ln\left(\frac{1}{p^{n+1}}\right) \\ &< -ln(p^n) = ln\left(\frac{1}{p^n}\right) \\ &= \mathcal{F}(M(x,y)). \end{aligned}$$

This implies

$$\varphi(d(x,y)) + \alpha(x,y)\mathcal{F}(D(y,Ty)) \leq \mathcal{F}(M(x,y)). \tag{3.4}$$

Thus T *is Suzuki-Edelstein-Wardowski type* α*-proximal contraction.*

Theorem 3.3. *Let* A *and* B *be two nonempty closed subsets of a complete metric space* (X,d) *such that* A_0 *is non-empty and* $T : A \to K(B)$ *be continuous multivalued mapping satisfying the following assertions:*

(i) T *is Suzuki-Edelstein-Wardowski type* α*-proximal contraction;*
(ii) $Tx \subseteq B_0$ *for each* $x \in A_0$ *and* (A,B) *satisfies the weak P-property;*
(iii) T *is* α*-proximal admissible;*
(iv) *there exists* $x_0, x_1 \in A_0$ *with* $\alpha(x_0,x_1) \geq 1$ *and* $y_1 \in Tx_0$ *such that*

$$d(x_1,y_1) = D(A,B).$$

Then the mapping T *has a best proximity point.*

Proof. By hypothesis (iv), there exists $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1,y_1) = D(A,B),\ \alpha(x_0,x_1) \geq 1. \tag{3.5}$$

If $y_1 \in Tx_1$, we obtain

$$D(A,B) \leq D(x_1,Tx_1) \leq d(x_1,y_1) = D(A,B)$$

and so x_1 is best proximity point of T and the proof is complete. So we suppose that $y_1 \notin Tx_1$. Since $Tx \subseteq B_0$ for all $x \in A_0$, then there exist $x_2 \in A_0$ and $y_2 \in Tx_1$ such that

$$d(x_2, y_2) = D(A, B). \tag{3.6}$$

Now we have

$$\left.\begin{array}{rcl} \alpha(x_0, x_1) & \geq & 1 \\ d(x_1, y_1) & = & \mathrm{D}(A, B) \\ d(x_2, y_2) & = & \mathrm{D}(A, B) \end{array}\right\}. \tag{3.7}$$

Since T is α-proximal admissible so,

$$\alpha(x_1, x_2) \geq 1.$$

Thus, we have

$$d(x_2, y_2) = D(A, B) \text{ and } \alpha(x_1, x_2) \geq 1. \tag{3.8}$$

If $y_2 \in Tx_2$, we obtain

$$D(A, B) \leq D(x_2, Tx_2) \leq d(x_2, y_2) = D(A, B)$$

and x_2 is best proximity point of T. Thus we suppose that $y_2 \notin Tx_2$. Again since $Tx \subseteq B_0$ for all $x \in A_0$, then there exist $x_3 \in A_0$ and $y_3 \in Tx_2$ such that

$$d(x_3, y_3) = D(A, B). \tag{3.9}$$

Now we have

$$\left.\begin{array}{rcl} \alpha(x_1, x_2) & \geq & 1 \\ D(x_2, y_2) & = & \mathrm{D}(A, B) \\ D(x_3, y_3) & = & \mathrm{D}(A, B) \end{array}\right\}. \tag{3.10}$$

Since T is α-proximal admissible. This implies

$$\alpha(x_2, x_3) \geq 1.$$

Thus we have

$$d(x_3, y_3) = D(A, B) \text{ and } \alpha(x_2, x_3) \geq 1. \tag{3.11}$$

Continuing this process we construct sequences $\{x_n\} \subseteq A_0$ and $\{y_n\} \subseteq B_0$ such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \neq x_{n+1}$, $y_n \in Tx_{n-1}$ and $y_n \notin Tx_n$ and

$$d(x_n, y_n) = D(A, B). \tag{3.12}$$

Since (A, B) satisfies weak P-property, we have

$$d(x_{n-1}, x_n) \leq d(y_{n-1}, y_n) \tag{3.13}$$

for all $n \in \mathbb{N}$.
Now

$$\begin{array}{rcl} \phi(D(x_{n-1}, Tx_{n-1})) & \leq & D(x_{n-1}, Tx_{n-1}) \\ & \leq & \alpha(x_{n-1}, x_n) D(x_{n-1}, Tx_{n-1}) \\ & \leq & \alpha(x_{n-1}, x_n)(d(x_{n-1}, x_n) + D(x_n, Tx_{n-1})) \\ & \leq & \alpha(x_{n-1}, x_n)(d(x_{n-1}, x_n) + D(A, B) \\ & \leq & \alpha(x_{n-1}, x_n)(d(x_{n-1}, x_n)) + \alpha(x_{n-1}, x_n) D(A, B), \end{array}$$

we have

$$\phi(D(x_{n-1}, Tx_{n-1})) - \alpha(x_{n-1}, x_n) D(A, B) \leq \alpha(x_{n-1}, x_n) d(x_{n-1}, x_n). \tag{3.14}$$

Since T is Suzuki-Edelstein-Wardowski type α-proximal multivalued contraction, we get that

$$\varphi(d(x_{n-1},x_n))+\alpha(x_{n-1},x_n)\mathcal{F}(D(x_n,Tx_n))\leq\mathcal{F}(M(x_{n-1},x_n)) \tag{3.15}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n),D(x_{n-1},Tx_{n-1}),D(x_n,Tx_n),\frac{D(x_{n-1},Tx_n)+D(x_n,Tx_{n-1})}{2},\right.\\ &\left.\frac{D(x_n,Tx_n)[1+D(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)},\frac{D(x_n,Tx_{n-1})[1+D(x_{n-1},Tx_n)]}{1+d(x_{n-1},x_n)}\right\}.\end{aligned}$$

Since Tx_{n-1} and Tx_n are compact, we have

$$\begin{aligned}M(x_{n-1},x_n) &= max\left\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2},\right.\\ &\left.\frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)},\frac{d(x_n,x_n)[1+d(x_{n-1},x_{n+1})]}{1+d(x_{n-1},x_n)}\right\}\\ &= \max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_{n+1})}{2}\right\}.\end{aligned}$$

Since

$$\frac{d(x_{n-1},x_{n+1})}{2}\leq\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\leq max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\},$$

it follows that

$$M(x_{n-1},x_n)\leq max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}. \tag{3.16}$$

Suppose that $d(x_{n-1},x_n)<d(x_n,x_{n+1})$, then (3.15) implies that

$$\begin{aligned}\varphi(d(x_{n-1},x_n))+\mathcal{F}(d(x_n,x_{n+1})) &\leq \varphi(d(x_{n-1},x_n))+\alpha(x_{n-1},x_n)\mathcal{F}(d(x_n,x_{n+1}))\\ &\leq \mathcal{F}(d(x_n,x_{n+1})),\end{aligned}$$

a contradiction. Hence $M(x_{n-1},x_n)\leq d(x_{n-1},x_n)$, therefore (3.15) implies that

$$\varphi(d(x_{n-1},x_n))+\alpha(x_{n-1},x_n)\mathcal{F}(D(x_n,Tx_n))\leq\mathcal{F}(d(x_{n-1},x_n)). \tag{3.17}$$

Since Tx_n is compact, we have

$$\begin{aligned}\varphi(d(x_{n-1},x_n))+\mathcal{F}(d(x_n,x_{n+1})) &\leq \varphi(d(x_{n-1},x_n))+\alpha(x_{n-1},x_n)\mathcal{F}(d(x_n,x_{n+1}))\\ &\leq\mathcal{F}(d(x_{n-1},x_n)).\end{aligned}$$

This implies that

$$\varphi(d(x_{n-1},x_n))+\mathcal{F}(d(x_n,x_{n+1}))\leq\mathcal{F}(d(x_{n-1},x_n))$$

and hence

$$\mathcal{F}(d(x_n,x_{n+1}))\leq\mathcal{F}(y_n,y_{n+1})\leq\mathcal{F}(d(x_{n-1},x_n))-\varphi(d(x_{n-1},x_n)) \tag{3.18}$$

for all $n\geq 1$. Let $\beta_n=d(x_{n-1},x_n)$ for all $n\geq 0$. Without loss of generality we can assume that $\beta_n>0$ for all $n\in\mathbb{N}$. From (3.18), there exists $c>0$, we have

$$\mathcal{F}(\beta_{n+1})\leq\mathcal{F}(\beta_n)-\varphi(\beta_n)\text{for all } n\in\mathbb{N}.$$

From $\mathcal{F}_1$ we get that (β_n) is decreasing, and hence, $\beta_n \searrow t,\ t \geq 0$. From $(\mathcal{H}_2)$ there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that $\varphi(\beta_n) > 0$ for all $n \geq n_0$.
In consequence, we have

$$\begin{aligned}\mathcal{F}(\beta_n) &\leq \mathcal{F}(\beta_{n-1}) - \varphi(\beta_{n-1}) \leq \cdots \mathcal{F}(\beta_1) - \sum_{i=1}^{n-1} \varphi(\beta_i) \\ &= \mathcal{F}(\beta_1) - \sum_{i=1}^{n_0-1} \varphi(\beta_i) - \sum_{i=n_0}^{n-1} \varphi(\beta_i) < \mathcal{F}(\beta_1) - (n-n_0)c,\ n > n_0\end{aligned}$$

Tending with $n \to \infty$ we get $\mathcal{F}(\beta_n) \to \infty$ and, by $(\mathcal{F}_2''), \beta_n \to 0$.
To show that (x_n) is the Cauchy sequence. Suppose on contrary that (x_n) is not Cauchy. From $(\mathcal{F}_1)$ the set ∇ of all discontinuity points of $\mathcal{F}$ is at most countable. There exists $\gamma > 0,\ \gamma \notin \nabla$ such that for every $k \geq 0$ one can find $m_k, n_k \in \mathbb{N}$ satisfying

$$k \leq m_k < n_k \text{ and } d(x_{m_k}, x_{n_k}) > \gamma. \tag{3.19}$$

Denote by $\overline{m}_k$ the least of m_k satisfying (3.19) and by $\overline{n}_k$ the least of n_k such that $\overline{m}_k < n_k$ and $d(x_{\overline{m}_k}, x_{n_k}) > \gamma$. Naturally

$$d(x_{\overline{m}_k}, x_{\overline{n}_k}) > \gamma \tag{3.20}$$

Observe that taking $k_0 \in \mathbb{N}$ such that $\beta_k < \gamma$ for all $k \geq k_0$, we have

$$\gamma < d(x_{\overline{m}_k}, x_{\overline{n}_k}) \leq d(x_{\overline{m}_k}, x_{\overline{n}_k-1}) + d(x_{\overline{n}_k-1}, x_{\overline{n}_k}) \leq \gamma + \beta_{\overline{n}_k},\ \text{for all } k \geq k_0.$$

We have

$$\lim_{k\to\infty} d(x_{\overline{m}_k}, x_{\overline{n}_k}) \to \gamma. \tag{3.21}$$

For all $k \geq 0$ we observe that

$$\begin{aligned} d(x_{\overline{m}_k}, x_{\overline{n}_k}) - \beta_{\overline{m}_k+1} - \beta_{\overline{n}_k+1} &\leq d(x_{\overline{m}_k+1}, x_{\overline{n}_k+1}) \\ &\leq \beta_{\overline{m}_k+1} + d(x_{\overline{m}_k}, x_{\overline{n}_k}) + \beta_{\overline{n}_k+1}.\end{aligned}$$

Again, we have

$$\lim_{k\to\infty} d(x_{\overline{m}_k+1}, x_{\overline{n}_k+1}) \to \gamma. \tag{3.22}$$

From (3.18), we get

$$\varphi(d(x_{\overline{m}_k}, x_{\overline{n}_k})) \leq \mathcal{F}(d(x_{\overline{m}_k}, x_{\overline{n}_k})) - \mathcal{F}(d(x_{\overline{m}_k+1}, x_{\overline{n}_k}+1)),\ k \geq 0.$$

Now, using (3.20)-(3.22) and from above inequality and by the continuity of $\mathcal{F}$ at γ we get

$$\liminf_{s\to\gamma^+} \varphi(s) \leq \liminf_{k\to\infty} \varphi(d(x_{\overline{m}_k}, x_{\overline{n}_k})) \leq \lim_{k\to\infty} (\mathcal{F}(d(x_{\overline{m}_k}, x_{\overline{n}_k}))) - \mathcal{F}(d(x_{\overline{m}_k+1}, x_{\overline{n}_k}+1)) = 0,$$

which is contradiction to $(\mathcal{H}_2)$. Therefore (x_n) is Cauchy. Similarly, we can show that $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of a complete metric space (X, d), there exist $x^* \in A$ and $y^* \in B$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, respectively. Since $d(x_n, y_n) \to D(A, B)$ for all $n \in \mathbb{N}$, we conclude that

$$\lim_{n\to\infty} d(x_n, y_n) = d(x^*, y^*) = D(A, B).$$

Since T is continuous, we have $\lim\limits_{n\to\infty} H(Tx_n, Tx^*) = 0$. On the other hand, since $y_{n+1} \in Tx_n$, we have

$$D(y^*, Tx^*) \leq d(y^*, y_{n+1}) + D(y_{n+1}, Tx^*) \leq d(y^*, y_{n+1}) + H(Tx_n, Tx^*).$$

Letting $n \to \infty$, we obtain

$$D(y^*, Tx^*) \leq 0,$$

which leads to $y^* \in Tx^*$. Furthermore, one has

$$D(A, B) \leq D(x^*, Tx^*) \leq d(x^*, y^*) = D(A, B)$$

which gives

$$D(A, B) \leq D(x^*, Tx^*) \leq D(A, B).$$

Hence

$$D(x^*, Tx^*) = D(A, B).$$

Therefore, x^* is a best proximity point of T. This completes the proof. □

Example 3.4. *Let $X = \mathbb{R}$ with usual metric d and let $A = \{1, 3, 5, ..., 2n+1\}$ and $B = \{0, 2, 4, ..., 2n\}, n \geq 1$ be subsets of $\mathbb{R}$. Then $D(A, B) = 1$. Define $T : A \to K(B)$, $\alpha : A \times A \to [0, \infty)$, $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$, $\varphi : (0, \infty) \to (0, \infty)$ and $\phi \in \Phi$ by*

$$Tx = \begin{cases} \{0\} & if\ x = 1 \\ \{0, 2, 4, ..., x-1\} & if\ x \geq 3, \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y, \end{cases}$$

$\mathcal{F}(x) = ln(x) + x$, $\varphi(t) = \frac{1}{t}$ and $\phi(t) = \frac{t}{2}$ for all t respectively. Notice that $A_0 = A, B_0 = B$, and $Tx \subseteq B_0$ for each $x \in A_0$. Also the pair (A, B) satisfies weak $P-$property. Let $x_0, x_1 \in A_0$ with $\alpha(x_0, x_1) \geq 1$, then Tx_0, $Tx_1 \subseteq B_0$. Consider $y_1 \in Tx_0$, $y_2 \in Tx_1$, and $u_1, u_2 \in A$ such that $d(u_1, y_1) = D(A, B)$ and $d(u_2, y_2) = D(A, B)$. Then we have $\alpha(u_1, u_2) \geq 1$. Hence T is α-proximal admissible. For $x_0 = 3 \in A_0$ and $y_1 = 2 \in Tx_0 \subseteq B_0$, we have $x_1 = 1 \in A_0$ such that $d(x_1, y_1) = 1 = D(A, B)$ and $\alpha(x_0, x_1) = 1$. Now let $x = 2m+1$ and $y = 2m+3$ where $1 \leq m \leq n$, then $Tx = \{0, 2, 4, ..., 2m\}$ and $Ty = \{0, 2, 4, ..., 2m+2\}$. So, we have

$$\phi(D(x, Tx)) - \alpha(x, y)D(A, B) = \phi(1) - 1 = \frac{1}{2} - 1 = -\frac{1}{2} \tag{3.23}$$

and

$$\alpha(x, y)d(x, y) = 2. \tag{3.24}$$

From (3.23) *and* (3.24) *we have*

$$\phi(D(x, Tx)) - \alpha(x, y)D(A, B) \leq \alpha(x, y)d(x, y). \tag{3.25}$$

Now

$$\begin{aligned} M(x, y) &= \max\Big\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(x, Ty)}{2}, \\ &\quad \frac{D(y, Ty)[1 + D(x, Tx)]}{1 + d(x, y)}, \frac{D(y, Tx)[1 + D(x, Ty)]}{1 + d(x, y)}\Big\} \\ &= \max\{2, 1, 1, \frac{3+1}{2}, \frac{1(1+1)}{1+2}, \frac{3(1+1)}{1+2}\} \\ &= \max\{2, 1, 1, 2, \frac{2}{3}, 2\} = 2. \end{aligned}$$

Thus

$$\begin{aligned}\varphi(d(x,y))+\alpha(x,y)\mathcal{F}(D(y,Ty)) &= \varphi(2)+\mathcal{F}(1)\\ &= \frac{1}{2}+(\ln(1)+1)\\ &= \frac{1}{2}+(0+1)\\ &= \frac{1}{2}+1\leq \ln(2)+2=\mathcal{F}(M(x,y)).\end{aligned}$$

This implies that

$$\varphi(d(x,y))+\alpha(x,y)\mathcal{F}(D(y,Ty))\leq \mathcal{F}(M(x,y)). \tag{3.26}$$

Hence T is Suzuki-Edelstein-Wardowski type α-proximal multivalued contraction. Therefore all conditions of Theorem 3.3 hold. Hence T has a best proximity point.

Remark. *If we remove the condition of continuity of T in Theorem 3.3 and replace it with α_*-continuity of T, then we have following result:*

Theorem 3.5. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is non-empty and $T: A\to K(B)$ be a multivalued mapping satisfying the following assertions:*

- (i) *T is Suzuki-Edelstein-Wardowski type α-proximal multivalued contraction;*
- (ii) *$Tx\subseteq B_0$ for each $x\in A_0$ and (A,B) satisfies the weak P-property;*
- (iii) *T is α-proximal admissible;*
- (iv) *there exists $x_0,x_1\in A_0$ and $y_1\in Tx_0$ such that*

$$d(x_1,y_1)=D(A,B) \text{ and } \alpha(x_0,x_1)\geq 1;$$

Then the mapping T has a best proximity point.

Proof. Following the proof of Theorem 3.3, since T is α_*-continuous, we get that $Tx_n\xrightarrow{H} Tx^*$ as $n\to\infty$. On the other hand, since $y_{n+1}\in Tx_n$, we have

$$D(y^*,Tx^*)\leq d(y^*,y_{n+1})+D(y_{n+1},Tx^*)\leq d(y^*,y_{n+1})+H(Tx_n,Tx^*).$$

Letting $n\to\infty$, we obtain

$$D(y^*,Tx^*)\leq 0$$

which leads to $y^*\in Tx^*$. Furthermore, one has

$$\begin{aligned}D(A,B) &\leq D(x^*,Tx^*)\leq d(x^*,y^*)=D(A,B)\\ D(A,B) &\leq D(x^*,Tx^*)\leq D(A,B).\end{aligned}$$

This implies

$$D(x^*,Tx^*)=D(A,B).$$

Therefore, x^* is a best proximity point of T. This completes the proof. □

Remark. *If we replace $K(B)$ by $CB(B)$ in Theorem 3.3, then we have the following problem:*

Does T has a proximity point?

We give answer to this question in following way:

Theorem 3.6. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is non-empty and $T: A\to CB(B)$ be continuous multivalued mapping satisfying the following assertions:*

(i) *T is Suzuki-Edelstein-Wardoski type α-proximal contraction;*
(ii) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*
(iii) *T is α-proximal admissible;*
(iv) *there exists $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1, y_1) = D(A, B) \text{ and } \alpha(x_0, x_1) \geq 1;$$

(v) *$\mathcal{F}(\inf M) = \inf \mathcal{F}(M)$ for all $M \subset (0, \infty)$.*

Then the mapping T has a best proximity point.

Proof. Following the proof of Theorem 3.3, there is a sequence $x_n \in A$, since $Tx \in CB(B)$ for every $x \in A$ and $\mathcal{F} \in \mathfrak{F}$ with $\mathcal{F}(\inf M) = \inf \mathcal{F}(M)$ for all $M \subset (0, \infty)$. Then there exist $x_{n+1} \in A$ and by hypothesis $\alpha(x_n, x_{n+1}) \geq 1$. Assume that $x_{n+1} \notin Tx_{n+1}$. Since Tx_{n+1} is closed, $D(x_{n+1}, Tx_{n+1}) > 0$, we have

$$\varphi(d(x_n, x_{n+1})) + \alpha(x_n, x_{n+1})\mathcal{F}(D(x_{n+1}, Tx_{n+1})) \leq \mathcal{F}(M(x_n, x_{n+1})) \qquad (3.27)$$

with

$$M(x_n, x_{n+1}) = max\left\{d(x_n, x_{n+1}), D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1}), \frac{D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)}{2}\right.$$
$$\left.\frac{D(x_{n+1}, Tx_{n+1})[1 + D(x_n, Tx_n)]}{1 + d(x_n, x_{n+1})}, \frac{D(x_{n+1}, Tx_n)[1 + D(x_n, Tx_{n+1})]}{1 + d(x_n, x_{n+1})}\right\}.$$

The rest of proof is similar to the proof of Theorem 3.3 by considering the Tx as closed for all $x \in A$. $\square$

If we take $\alpha(x, y) = 1$ in Theorem 3.3 then we have following result.

Corollary 3.7. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is non-empty and $T : A \to K(B)$ be continuous multivalued mapping and for all $x \in A$ with $D(x, Tx) > 0$, there exist $y \in A$ with $D(y, Ty) > 0$ satisfying (H_1), (H_2) and*

$$\phi(D(x, Tx)) \leq d(x, y) + D(A, B) \Rightarrow \varphi(d(x, y)) + \mathcal{F}(D(y, Ty)) \leq \mathcal{F}(M(x, y))$$

where $M(x, y)$ is given in (1.4) with

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*
(ii) *there exists $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1, x_0) = D(A, B).$$

Then the mapping T has a best proximity point.

If we take $\alpha(x, y) = 1$ in Theorem 3.6 then we have following result:

Corollary 3.8. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is non-empty and $T : A \to K(B)$ be continuous multivalued mapping and for all $x \in A$ with $D(x, Tx) > 0$, there exist $y \in A$ with $D(y, Ty) > 0$ satisfying (H_1), (H_2) and*

$$\phi((D(x, Tx)) - D(A, B) \leq d(x, y) \Rightarrow \varphi(d(x, y)) + \mathcal{F}(D(y, Ty)) \leq \mathcal{F}(M(x, y))$$

where $M(x, y)$ is given in (1.4) with

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*
(ii) *there exists $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1, x_0) = D(A, B);$$

(iii) $\mathcal{F}(\inf M) = \inf \mathcal{F}(M)$ *for all* $M \subset (0, \infty)$.

Then the mapping T *has a best proximity point.*

4. Best Proximity Point Results in Partially Ordered Metric Spaces

Let $(X, d, \preceq)$ be a partially ordered metric space and $T : X \to CB(X)$ be a multivalued mapping. For $A, B \in CB(X), A \preceq B$ implies that $a \preceq b$ for all $a \in A$ and $b \in B$.

Definition 4.1. [25] *Let* A *and* B *be tow non empty subsets of a partially ordered metric space* $(X, d, \preceq)$. *A mapping* $T : A \to B$ *is said to be proximal nondreasing if*

$$\left.\begin{array}{rcl} x_1 \preceq x_2 & & \\ d(u_1, y_1) & = & D(A, B) \\ d(u_2, y_2) & = & D(A, B) \end{array}\right\} \text{ implies } u_1 \preceq u_2$$

where $x_0, x_1, u_1, u_2 \in A$ *and* $y_1 \in Tx_0, y_2 \in Tx_1$.

In this section, we derive some new results in partially ordered metric spaces from our main results.

Definition 4.2. *Suppose* A *and* B *are two non-empty subsets of a partially ordered metric space* $(X, d, \preceq)$. *A multivalued mapping* $T : A \to C(B)$ *is said to be Suzuki-Edelstein-Wardowski type* α*-proximal contraction if for all* $x, y \in A$ *with* $x \preceq y$ *there exist functions* $\alpha : A \times A \to [0, \infty)$, $\phi \in \Phi$ *along with conditions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3')$.

Theorem 4.3. *Let* A *and* B *be two nonempty closed subsets of a complete partially ordered metric space* $(X, d, \preceq)$ *such that* A_0 *is non-empty,* $T : A \to K(B)$ *be continuous multivalued mapping and for all* $x \in A$ *with* $D(x, Tx) > 0$, *there exist* $y \in A$ *with* $D(y, Ty) > 0$ *and* $x \preceq y$ *satisfying* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and*

$$\phi((D(x, Tx)) - D(A, B) \leq d(x, y)$$

implies

$$\varphi(d(x, y)) + \mathcal{F}(D(y, Ty)) \leq \mathcal{F}(M(x, y))$$

where $M(x, y)$ *is same as in* (1.4), *with* T *satisfying the following assertions:*

(i) $Tx \subseteq B_0$ *for each* $x \in A_0$ *and* (A, B) *satisfies the weak P-property;*
(ii) T *is proximal nondecreasing;*
(iii) *there exist* $x_0, x_1 \in A$ *and* $y_1 \in A$ *such that* $d(x_1, y_1) = D(A, B)$, *and* $x_0 \preceq x_1$.

Then the mapping T *has a best proximity point.*

Proof. Consider a function $\alpha : A \times A \to [0, \infty)$ such that

$$\alpha(x, y) = \begin{cases} 1, & if\ x \preceq y \\ 0, & otherwise. \end{cases} \tag{4.1}$$

Now we show that T is α-proximal admissible. By hypothesis (iii), there exist $x_0, x_1 \in A$ and $y_1 \in A$ such that

$$d(x_1, y_1) = D(A, B) \text{ and } x_0 \preceq x_1. \tag{4.2}$$

Since $x_0 \preceq x_1$, we obtain $\alpha(x_0, x_1) \geq 1$. Also by hypothesis (i), we have $Tx_1 \subset B_0$ and so there exist $x_2 \in A_0$ such that

$$d(x_2, y_2) = D(A, B). \tag{4.3}$$

From (4.2) and (4.3), we have

$$\left.\begin{array}{lcl} x_0 \preceq x_1 & & \\ d(x_1, y_1) & = & \mathrm{D}(A, B) \\ d(x_2, y_2) & = & \mathrm{D}(A, B) \end{array}\right\},$$

since T is proximal decreasing, we get $x_1 \preceq x_2$. Thus $\alpha(x_1, x_2) \geq 1$. Hence T is α-proximal admissible. Finally, suppose that

$$\phi((D(x, Tx)) - \alpha(x, y)D(A, B) \leq \alpha(x, y)d(x, y).$$

For all $x, y \in A, x \preceq y$, we have $\alpha(x, y) \geq 1$ and hence we have

$$\varphi(d(x, y)) + \alpha(x, y)\mathcal{F}(D(y, Ty)) \leq \mathcal{F}(M(x, y)).$$

That is T is Suzuki-Edelstein-Wardowski α-proximal contraction. Thus all the conditions of Theorem 3.3 holds and T has a best proximity point. □

Corollary 4.4. *Let A and B be two nonempty closed subsets of a complete partially ordered metric space $(X, d, \preceq)$ such that A_0 is non-empty, $T : A \to K(B)$ be continuous multivalued mapping and for all $x \in A$ with $D(x, Tx) > 0$, there exist $y \in A$ with $D(y, Ty) > 0$ and $x \preceq y$ satisfying ($\mathcal{H}_1$), ($\mathcal{H}_2$) and*

$$\phi((D(x, Tx)) - D(A, B) \leq d(x, y)$$

implies

$$\varphi(d(x, y)) + \mathcal{F}(D(y, Ty)) \leq \mathcal{F}(d(x, y))$$

where $M(x, y)$ is same as in (1.4)*, with T satisfying the following assertions:*

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*
(ii) *T is proximal nondecreasing;*
(iii) *there exist $x_0, x_1 \in A$ and $y_1 \in A$ such that $d(x_1, y_1) = D(A, B)$, and $x_0 \preceq x_1$.*

Then the mapping T has a best proximity point.

Proof. If we take $M(x, y) = d(x, y)$ in Theorem 4.3, then we get the proof. □

Acknowledgments. This paper was funded by the University of Sargodha, Sargodha funded research project No. UOS/ORIC/2016/54. The first, second and fourth author, therefore, acknowledges with thanks UOS for financial support. The authors would like to thank the anonymous reviewers for their valuable comments.

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