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SOME PROPERTIES OF NEW MODIFIED SZÁSZ-MIRAKYAN OPERATORS IN POLYNOMIAL WEIGHT SPACES VIA POWER SUMMABILITY METHODS

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Abstract. In this paper we will prove the Korovkin type theorem for new modified Szász-Mirakyan operators via $A-$ statistical convergence and power summability method. Also we give the rate of the convergence related to the summability methods and in the last section we give a kind of Voronovskaya type theorem for $A-$ statistical convergence and Grüss-Voronovskaya type theorem.

1. Introduction

We shall denote the set of all natural numbers by $\mathbb{N}$. The sequence $x = (x_k)$ is said to be statistically convergent to L if, for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero [11], i.e. for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent but not conversely. In what follows we will use the definition of the $A-$ statistical convergence. Let $A = (a_{nj})$ be a summability matrix and $x = (x_j)$ be a sequence. If the series

$$(Ax)_n = \sum_j a_{nj} x_j,$$

converges for every $n \in \mathbb{N}$, then we say that $(Ax)_n$ is the $A-$ transform of the sequence $x = (x_n)$. If $(Ax)_n$ converges to a number L, we say that x is $A-$ summable to L. The summability matrix, A, is regular, whenever $\lim_j x_j = L$, then $\lim_n (Ax)_n = L$.

Let $A = (a_{nj})$ be a nonnegative regular summability matrix. The sequence $x = (x_j)$ is said to be A-statistically convergent, see [12], to real number a if for

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any $\epsilon > 0$

$$\lim_{n\to\infty} \sum_{j:|x_j-a|\geq \epsilon} a_{nj} = 0.$$

For this case we write $st_A - \lim x = a$.

The $A-$ statistical convergence is a generalization of the statistical convergence and it is proven in the Example given in [8]. The second summability method which is used in this paper is power summability method. Let (p_j) be real sequence with $p_0 > 0$ and $p_1, p_2, p_3, \cdots \geq 0$, and such that the corresponding power series $p(t) = \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If, for all $t \in (0, R)$,

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} x_j p_j t^j = L,$$

then we say that $x = (x_j)$ is convergent in the sense of power series method. (see [16], [19]) The power series method includes many known summability methods such as Abel and Borel. Both methods have in common that their definitions are based on power series and they are not matrix methods (see [2],[5] [21]). Matrix methods are more effective than ordinary methods as shown in an example in [20].

Note that the power series method is regular if and only if

$$\lim_{t\to R^-} \frac{p_j t^j}{p(t)} = 0,$$

holds for each $j \in \mathbb{N} \cup \{0\}$([4]). Throughout the paper we assume that power series method is regular.

In this paper we will prove the Korovkin type theorem for the new modified Szász-Mirakyan operators via $A-$statistical convergence and power summability method. In the second part we give the rate of convergence related to the summability methods and in the last section we give a Voronovskaya type theorem for $A-$ statistical convergence.

Define the class of new modified Szász-Mirakyan operators, [23], by

$$A_n(f, r, q, x) = \frac{1}{e^{(n^q x+1)^r}} \sum_{k=0}^{\infty} \frac{(n^q x + 1)^{rk}}{k!} f\left(\frac{k}{n^q (n^q x + 1)^{r-1}}\right),$$

for $x \in [0, \infty)$, $r \in [2, \infty)$, $n \in \mathbb{N}$, $q > 0$.

It is know that, see [23],

Lemma 1.1. *The first few moments for the modified Szász-Mirakyan operators, are:*

(1) $A_n(e_0, r, q, x) = 1$

(2) $A_n(e_1, r, q, x) = x + \frac{1}{n^q}$,

(3) $A_n(e_2, r, q, x) = \left(x + \frac{1}{n^q}\right)^2 \left[1 + \frac{1}{(n^q x+1)^r}\right]$

(4) $A_n(e_3, r, q, x) = \left(x + \frac{1}{n^q}\right)^3 \left[1 + \frac{3}{(n^q x+1)^r} + \frac{1}{(n^q x+1)^{2r}}\right]$.

Proof. The first fourth moments are proven in [23]. We will prove it for test function $e_4 = t^4$. After some calculation we get that

$$A_n(t^4, r, q, x) = \left(x + \frac{1}{n^q}\right) A_n(t^3, x, q, r) + 3\left(x + \frac{1}{n^q}\right) \frac{1}{(n^q(n^q x + 1)^{r-1})} A_n(t^2, x, q, r) +$$
$$3\left(x + \frac{1}{n^q}\right) \frac{1}{(n^q(n^q x + 1)^{r-1})^2} A_n(t, x, q, r) + \left(x + \frac{1}{n^q}\right) \frac{1}{(n^q(n^q x + 1)^{r-1})^3}.$$
□

The theory of Korovkin type theorems has been studied in several function spaces, and further details reader can be found in the following papers (see [1],[3, 5, 6, 7], [9], [10], [14], [15], [17],[18], [20], [21], [22]).

In what follows we define the following power series for sequence of operators A_n

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{n=0}^{\infty} A_n(f, r, q, x) p_n t^n = L(f, r, q, x),$$

and say that the sequence of operators, A_n, converges in the sense of power series, to L, for every $t \in (0, R)$, if this series converges.

2. MAIN RESULTS

In this section we obtain a Korovkin type theorem for the new modified Szász-Mirakyan operators and then $A-$ statistical convergence of the new modified Szász-Mirakyan operators to the identity operator.

In what follows we will prove the standard Korovkin type theorem for the new modified Szász-Mirakyan operators. With $C_p([0, \infty))$ we will denote the space of all real valued functions f continuous on $[0, \infty)$ and such that $\omega_p f$ is uniformly continuous and bounded on $[0, \infty)$, where $\omega_p(x) = (1 + x^p)^{-1}$, $p \geq 1$, and norm in C_p is defined by formula([5]):

$$||f||_p = \sup_{x \in [0, \infty)} \omega_p(x)|f(x)|.$$

With $B([0, R])$ we will denote the space of all bounded functions defined in $[0, R]$.

Theorem 2.1. *Let $f \in C_p([0, R])$, for any finite R and A_n sequence of positive linear operators from $C_p([0, R])$ into $B([0, R])$, such that for every $i \in \{0, 1, 2\}$*

$$\|A_n e_i - e_i\|_p = 0, \tag{2.1}$$

where $e_i = x^i$. Then for any $f \in C_p([0, R])$

$$\|A_n f - f\|_p = 0. \tag{2.2}$$

Proof of the theorem is similar to that given in [8] and for this reason we omit it.

The Korovkin type theorem for $A-$ statistical convergence was given in [10] as follows:

Theorem 2.2. *Let $A = (a_{nj})$ be a nonnegative regular summability matrix and let (B_j) be a sequence of positive linear operators on $C[0, 1]$, such that for $i = 0, 1, 2$,*

$$st_A - \lim_{n \to \infty} ||B_j e_i - e_i|| = 0.$$

Then for any function f, $f \in C[0,1]$,

$$st_A - \lim_{n\to\infty} ||B_j f - f|| = 0,$$

where $||f|| = \max_{0\le t\le 1} |f(t)|$.

Based on this theorem, we give the following result for the new modified Szász-Mirakyan operators.

Theorem 2.3. *Let $A = (a_{nj})$ be a nonnegative regular summability matrix and let (A_n) be a sequence of positive linear operators on $C[0,R]$, for any finite R, such that for $i = 1,2$*

$$st_A - \lim_{n\to\infty} ||A_n e_i - e_i||_p = 0,$$

where $e_i = x^i$. Then for any $f \in C[0,R]$

$$st_A - \lim_{n\to\infty} ||A_n f - f||_p = 0.$$

Proof. From Lemma 1.1 we have:

$$||A_n e_1 - e_1|| \le \left|\frac{1}{n^q}\right| \to 0, \quad \text{as} \quad n\to\infty,$$

and

$$||A_n e_2 - e_2|| \le \left\|\frac{x^2}{(n^q x+1)^r}\right\| + \left\|\left(\frac{2x}{n^q} + \frac{1}{n^{2q}}\right)\left(1 + \frac{1}{(n^q x+1)^r}\right)\right\| \to 0, \quad \text{as} \quad n\to\infty.$$

From these relations it follows that

$$st_A - \lim ||A_n e_2 - e_2||_p = 0.$$

$\square$

Following Example 1.1 in [8] we can prove that the following sequence is not statistically convergent but it is $A-$ statistically convergent.

Example 2.4. *Define the operators*

$$P_n(f,r,q,x) = (1+x_n)A_n(f,r,q,x),$$

where the sequence (x_n) is given as follows:

$$(x_k) = \begin{cases} \frac{1}{m^{2q+1}}; & k = m^2 - m, \cdots, m^2 - 1 \\ \frac{1}{m^{2q+2}}; & k = m^2; m \in \mathbb{N}\setminus\{1\} \\ 0; & \text{otherwise} \end{cases},$$

then the following relations are fulfilled

$$P_n(e_0,r,q,x) = 1,$$

$$P_n(e_1,r,q,x) = x + \frac{1}{n^1}$$

and

$$P_n(e_2,r,q,x) = \left(x + \frac{1}{n^1}\right)^2 \left[1 + \frac{1}{(n^1 x+1)^r}\right].$$

By Theorem 2.3 we obtain

$$st_A - \lim_n ||P_n f - f|| = 0,$$

but the operators $P_n(f,r,q,x)$, do not satisfy Theorem 2.1.

Remark. *Consider the case where $q=1$, then the sequence (x_n) does not converge statistically, nor converges. As an example consider the second order Cesáro matrix.*

$$A=(a_{nk})=\begin{cases} \frac{2(n+1-k)}{(n+1)(n+2)}; & 0\le k\le n \\ 0; & k>n \end{cases},$$

where

$$0\le \lim_n \sum_{k:|x_k-\alpha|\ge\epsilon} a_{nk} = \lim_n \sum_{\substack{k=m^2-m,\cdots,m^2-1\\ k=m^2;m\in\mathbb{N}\setminus\{1\}}} a_{nk} =$$

$$\lim_n \frac{2}{(n+1)(n+2)}\left[1+\cdots+n\right]\le \lim_n \frac{2}{(n+1)(n+2)}\cdot\frac{n(n+1)}{2}=1.$$

This proves that $x=(x_n)$ is $A-$ statistically convergent and the following relations are fulfilled

$$P_n(e_0,r,q,x)=1,$$

$$P_n(e_1,r,q,x)=x+\frac{1}{n^1},$$

$$P_n(e_2,r,q,x)=\left(x+\frac{1}{n^1}\right)^2\left[1+\frac{1}{(n^1x+1)^r}\right].$$

By Example 2.4 this shows that $P_n(f,r,q,x)$, does not satisfy Theorem 2.1.

Now we will prove the Korovkin type theorem for the new modified Szász-Mirakyan operators, by power series method. It is known that Korovkin type theorems are proved by the Abel summability method (see for example [18], [22]). $B[0,\infty)$ will denote the space of all bounded functions on the interval $[0,\infty)$ and with $C[0,\infty)$ we will denote the space of all continuous functions defined in the interval $[0,\infty)$.

Theorem 2.5. *Let (A_n), be a sequence of positive linear operators from $C[0,R]$ into $B[0,R]$, for any finite R, such that for every $i\in\{0,1,2\}$*

$$\lim_{t\to R^-}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_ne_i-e_i)p_nt^n\right\|=0, \tag{2.3}$$

where $e_i=x^i$. Then for any $f\in C[0,R]$

$$\lim_{t\to R^-}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_nf-f)p_nt^n\right\|=0. \tag{2.4}$$

Proof. It is obvious that (2.4) follows relation (2.3). Now we will prove the converse, that relation (2.3) is valid, and we will prove that relation (2.4) is valid, too. Let $f\in C[0,R]$ then there exists a constant $K>0$ such that $|f(t)|\le K$ for all $t\in[0,R]$. Therefore

$$|f(t)-f(x)|\le 2K, \qquad t\in[0,R]. \tag{2.5}$$

For every given $\epsilon>0$ there exist a $\delta>0$ such that

$$|f(t)-f(x)|\le\epsilon \tag{2.6}$$

whenever $|t-x|<\delta$ for all $t\in[0,R]$. Let ψ denote $\psi\equiv\psi(t,x)=(t-x)^2$. If $|t-x|\ge\delta$, then we have:

$$|f(t)-f(x)| \leq \frac{2K}{\delta^2}\psi(t,x). \tag{2.7}$$

Now from relations (2.5)-(2.7), we get

$$|f(t)-f(x)| < \epsilon + \frac{2K}{\delta^2}\psi(t,x).$$

Respectively,

$$-\epsilon - \frac{2K}{\delta^2}\psi(t,x) < f(t)-f(x) < \frac{2K}{\delta^2}\psi(t,x)+\epsilon.$$

Applying the operator $A_n(1,r,q,x)$ to this inequality, since $A_n(1,r,q,x)$ is monotone and linear, we obtain:

$$A_n(1,r,q,x)\left(-\epsilon-\frac{2K}{\delta^2}\psi\right) < A_n(1,r,q,x)\left(f(t)-f(x)\right) < A_n(1,r,q,x)\left(\frac{2K}{\delta^2}\psi+\epsilon\right) \Rightarrow$$

$$-\epsilon A_n(1,r,q,x) - \frac{2K}{\delta^2}A_n(\psi(t),r,q,x) < A_n(f,x)-f(x)A_n(1,r,q,x) < \frac{2K}{\delta^2}A_n(\psi(t),r,q,x)+\epsilon A_n(1,r,q,x). \tag{2.8}$$

On the other hand

$$A_n(f,r,q,x)-f(x) = A_n(f,r,q,x)-f(x)A_n(1,r,q,x)+f(x)[A_n(1,r,q,x)-1]. \tag{2.9}$$

From relations (2.8) and (2.9) we have:

$$A_n(f,r,q,x)-f(x) < \frac{2K}{\delta^2}A_n(\psi(t),r,q,x)+\epsilon A_n(1,r,q,x)+f(x)[A_n(1,r,q,x)-1]. \tag{2.10}$$

Let us now estimate the following expression:

$$\begin{aligned} A_n(\psi(t),r,q,x) &= A_n((x-t)^2,r,q,x) = A_n((x^2-2xt+t^2),r,q,x) \\ &= x^2A_n(1,r,q,x)-2xA_n(t,r,q,x)+A_n(t^2,r,q,x) \end{aligned}$$

Now, from the last relation and (2.10), we obtain

$$\begin{aligned} A_n(f,r,q,x)-f(x) &< \frac{2K}{\delta^2}\Big\{x^2[A_n(1,r,q,x)-1]-2x[A_n(t,r,q,x)-x] \\ &\quad +[A_n(t^2,r,q,x)-x^2]\Big\}+\epsilon A_n(1,r,q,x)+f(x)[A_n(1,r,q,x)-1] \\ &= \epsilon+\epsilon[A_n(1,r,q,x)-1]+f(x)[A_n(1,r,q,x)-1]+ \\ &\quad +\frac{2K}{\delta^2}\Big\{x^2[A_n(1,r,q,x)-1]-2x[A_n(t,r,q,x)-x]+[A_n(t^2,r,q,x)-x^2]\Big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} |A_n(f,r,q,x)-f(x)| &\leq \epsilon+\left(\epsilon+K+\frac{2KR^2}{\delta^2}\right)|A_n(1,r,q,x)-1| \\ &\quad +\frac{4KR}{\delta^2}|A_n(t,r,q,x)-x|+\frac{2K}{\delta^2}|A_n(t^2,r,q,x)-x^2|. \end{aligned}$$

Taking into consideration properties of the weight function, $\omega_p(x)$, and the above relations, we obtain:

$$\omega_p(x)|A_n(f,r,q,x)-f(x)| \leq \epsilon\cdot\omega_p(x)+\omega_p(x)\left(\epsilon+K+\frac{2KR^2}{\delta^2}\right)|A_n(1,r,q,x)-1|$$
$$+\omega_p(x)\frac{4KR}{\delta^2}|A_n(t,r,q,x)-x|+\omega_p(x)\frac{2K}{\delta^2}|A_n(t^2,r,q,x)-x^2|,$$

which leads to

$$\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_n(f,r,q,x)-f(x))p_nt^n\right\|\leq\epsilon+\left(\epsilon+K+\frac{2KR^2}{\delta^2}\right)\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_n(1,r,q,x)-1))p_nt^n\right\|+$$
$$+\frac{4KR}{\delta^2}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_n(t,r,q,x)-x))p_nt^n\right\|+\frac{2K}{\delta^2}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_n(t^2,r,q,x)-x^2))p_nt^n\right\|.$$

Now we get relation (2.4), from last relation and relation (2.3). From this we get (2.4) with use of (2.3) □

3. RATE OF CONVERGENCE

In this section, we study the rate of the $A-$statistical convergence for the new modified Szász-Mirakyan operators and power summability method. We begin by presenting the following facts:

The modulus of continuity, for a function $f(x)\in C([0,\infty))$, is defined as follows:

$$\omega(f,\delta)=\sup_{|h|<\delta}|f(x+h)-f(x)|.$$

It is known that for any value of the $|x-y|$,

$$|f(x)-f(y)|\leq\omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{3.1}$$

We have the following result:

Theorem 3.1. *Let $A=(a_{ij})$, be a nonnegative regular summability matrix with $f\in C[0,\infty)$. If (α_n) is a sequence of positive real numbers such that $\omega(f,\delta_n)=st_A-0\,(\alpha_n)$, then*

$$||A_nf-f||=st_A-0(\alpha_n),$$

where

$$\delta_n=\sup\left\{\left(\left(\frac{k}{n^q}\right)^2+x^2\right)-2x\frac{k}{n^q}A_n\left(\frac{1}{(n^qx+1)^{r-1}},r,q,x\right)\right\}^2,$$

for any positive integer $n\in\mathbb{N}$.

Proof. Let $f\in C[0,\infty)$. Taking into consideration the linearity and positivity of A_nf, and relation (3.1), we have

$$|A_n(f,r,q,x)-f|\leq A_n(|f(t)-f(x)|,r,q,x)\leq\frac{1}{e^{(n^qx+1)^r}}\sum_{k=0}^{\infty}\frac{(n^qx+1)^{rk}}{k!}\omega(\delta,f)\left(1+\frac{\left|\frac{k}{n^q(n^qx+1)^{r-1}}-x\right|}{\delta}\right),$$

and, by use of Lemma 1.1, we obtain

$$|A_n(f,r,q,x)-f|\leq\omega(f,\delta)\left[1+\frac{1}{\delta}\frac{1}{e^{(n^qx+1)^r}}\sum_{k=0}^{\infty}\frac{(n^qx+1)^{rk}}{k!}\left|\frac{k}{n^q(n^qx+1)^{r-1}}-x\right|\right]\leq$$

$$\omega(f,\delta)\left[1+\frac{1}{\delta}A_n\left(\left|\frac{k}{n^q(n^qx+1)^{r-1}}-x\right|,r,q,x\right)\right].$$

Applying the Cauchy-Schwarz inequality, to this expression, we get

$$|A_n(f,r,q,x)-f|\leq \omega(f,\delta)\left[1+\frac{1}{\delta}\left(A_n\left(\left|\frac{k}{n^q(n^qx+1)^{r-1}}-x\right|^2,r,q,x\right)\right)^{\frac{1}{2}}\right].$$

On the other hand, based on Lemma 1.1, and definition of the operators $A_n(f,r,q,x)$, we have this estimation

$$A_n\left(\left|\frac{k}{n^q(n^qx+1)^{r-1}}-x\right|^2,r,q,x\right)=\left(\frac{k}{n^q}\right)^2A_n\left(\frac{1}{(n^qx+1)^{2(r-1)}},r,q,x\right)-$$

$$\frac{2kx}{n^q}A_n\left(\frac{1}{(n^qx+1)^{r-1}},r,q,x\right)+x^2A_n(1,r,q,x)$$

$$\leq \sup\left\{\left(\left(\frac{k}{n^q}\right)^2+x^2\right)-2x\frac{k}{n^q}A_n\left(\frac{1}{(n^qx+1)^{r-1}},r,q,x\right)\right\}^2,$$

for every $x\in[0,\infty), r\in[2,\infty), q>0, k\in\mathbb{N}$. Taking

$$\delta_n=\sup\left\{\left(\left(\frac{k}{n^q}\right)^2+x^2\right)-2x\frac{k}{n^q}A_n\left(\frac{1}{(n^qx+1)^{r-1}},r,q,x\right)\right\}^2,$$

we get

$$||A_nf-f||\leq 2\cdot\omega(f,\delta_n).$$

Therefore, for every $\epsilon>0$, we get the relation

$$\frac{1}{\alpha_n}\sum_{||A_nf-f||\geq\epsilon}a_{nj}\leq\frac{1}{\alpha_n}\sum_{2\cdot\omega(f,\delta_n)\geq\epsilon}a_{nj}.$$

From conditions given in the theorem, we have

$$||A_nf-f||=st_A-0(\alpha_n).$$

□

In what follows we give the rate of convergence for power summability method.

Theorem 3.2. *Let $f\in C[0,\infty)$ and let ϕ be a positive real function defined on $(0,R)$. If $\omega(f,\psi)=0(\phi)$, as $t\to R^-$, then we have*

$$\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_ne_i-e_i)p_nt^n\right\|=0(\phi),$$

as $t\to R^-$, where the function $\psi:(0,R)\to\mathbb{R}$ is defined by relation

$$\psi=\sup\left\{A_n\left(\left(\frac{k}{n^q(n^qx+1)^{r-1}}-x\right)^2,r,q,x\right)\right\}.$$

Proof. Let $f \in C[0,\infty)$. For any $t \in (0,R)$, $x \in (0,\infty)$ and $\delta > 0$, we have

$$\left|\sum_{n=0}^{\infty}[A_n(f,r,q,x)-f(x)]p_nt^n\right| \leq \sum_{n=0}^{\infty}A_n(|f(t)-f(x)|,r,q,x)p_nt^n \leq$$

$$\sum_{n=0}^{\infty}A_n\left(\omega\left(f,\frac{\left|\frac{k}{n^q(n^qx+1)^{r-1}}-x\right|}{\delta}\delta\right),r,q,x\right)p_nt^n \leq$$

$$\sum_{n=0}^{\infty}A_n\left(\left(1+\left[\left|\frac{\frac{k}{n^q(n^qx+1)^{r-1}}-x}{\delta}\right|\right]\right)\omega(f,\delta),r,q,x\right)p_nt^n$$

$$\leq \omega(f,\delta)\sum_{n=0}^{\infty}A_n\left(1+\frac{\left(\frac{k}{n^q(n^qx+1)^{r-1}}-x\right)^2}{\delta^2},r,q,x\right)p_nt^n \leq \omega(f,\delta)\sum_{n=0}^{\infty}A_n(e_0(t),r,q,x)p_nt^n+$$

$$\frac{\omega(f,\delta)}{\delta^2}\sum_{n=0}^{\infty}A_n\left(\left(\frac{k}{n^q(n^qx+1)^{r-1}}-x\right)^2,r,q,x\right)p_nt^n = p(t)\omega(f,\delta)+$$

$$\frac{\omega(f,\delta)}{\delta^2}\sup\left\{A_n\left(\left(\frac{k}{n^q(n^qx+1)^{r-1}}-x\right)^2,r,q,x\right)\right\}\sum_{n=0}^{\infty}p_nt^n$$

$$= p(t)\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta^2}\sup\left\{A_n\left(\left(\frac{k}{n^q(n^qx+1)^{r-1}}-x\right)^2,r,q,x\right)\right\}p(t).$$

If we take $\delta = \psi$, from last inequality we obtain:

$$0 \leq \frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(A_nf-f)p_nt^n\right\| \leq 2\omega(f,\delta),$$

which proves the theorem. □

4. Voronovskaya theorem

In this section we will prove the Voronovskaya type theorem for new modified Szász-Mirakyan operators via $A-$ statistical convergence. First we give the following, see [23],

Lemma 4.1. *The first three central moments, for new Modified Szász-Mirakyan operators are:*

(1) $A_n((t-x),r,q,x) = \frac{1}{n^q}$,
(2) $A_n((t-x)^2,r,q,x) = \frac{1}{n^{2q}}\left[1+\frac{1}{(n^qx+1)^{r-2}}\right]$,
(3) $A_n((t-x)^3,r,q,x) = \frac{1}{n^{3q}}\left[1+\frac{3}{(n^qx+1)^{r-2}}+\frac{1}{(n^qx+1)^{2r-3}}\right]$,

and we have to find

$$A_n((t-x)^4,r,q,x).$$

Proof. Proof of the central moments till third order are given in [23]. We calculate the fourth order central moment, and after some calculation, we obtain

$$A_n((t-x)^4,r,q,x) = \left(x+\frac{1}{n^q}\right)^4\left[1+\frac{3}{(n^qx+1)^r}+\frac{1}{(n^qx+1)^{2r}}\right]+$$
$$3\left(x+\frac{1}{n^q}\right)^3\frac{1}{n^q(n^qx+1)^{r-1}}\left[1+\frac{1}{(n^qx+1)^r}\right]+$$
$$3\left(x+\frac{1}{n^q}\right)^2\frac{1}{(n^q(n^qx+1)^{r-1})^2}\left(x+\frac{1}{n^q}\right)+\left(x+\frac{1}{n^q}\right)\frac{1}{(n^q(n^qx+1)^{r-1})^3}-$$
$$3x\left(x+\frac{1}{n^q}\right)^3\left[1+\frac{3}{(n^qx+1)^r}+\frac{1}{(n^qx+1)^{2r}}\right]+$$
$$3x^2\left(x+\frac{1}{n^q}\right)^2\left[1+\frac{1}{(n^qx+1)^r}\right]-x^3\left(x+\frac{1}{n^q}\right)-$$
$$\frac{x}{n^{3q}}\left[1+\frac{3}{(n^qx+1)^{r-2}}+\frac{1}{(n^qx+1)^{2r-3}}\right].$$

□

In [23] Walczak proved the Voronovskaya type theorem for the new modified Szász-Mirakyan operators which is stated as:

Theorem 4.2. *Let $f \in C_1^p$ and let $r \in [2,\infty)$ be fixed number. Then*
$$\lim_{n\to\infty} n^q[A_n(f,r,q,x)-f(x)] = f^{'}(x),$$
for every $x > 0$.

In what follows we will show that the Voronovskaya type theorem can be extended to $A-$ statistical summability method for the new modified Szász-Mirakyan operators. Let us consider operators P_n from Example 2.4. We first prove the following.

Lemma 4.3. *For every $x \in [0,R]$, for any finite R, we have*
$$n^{2q}P_n(\Phi^4) \sim 8x^4(st_A) \quad on \quad [0,\infty),$$
where $\Phi_x(y) = (y-x)$.

Proof. Proof of the Lemma follows directly from Lemma 4.1 and Example 2.4, for this reason we omit it. □

Theorem 4.4. *Let $f \in C[0,R]$, for any finite R, such that $f^{'}, f^{''} \in C[0,R]$ and $x \in [0,R]$. Then*
$$n^q[P_nf-f] \sim f^{'}(x)(st_A),$$
on $[0,R]$.

Proof. Let us suppose that $f^{'}, f^{''} \in C[0,R]$ and $x \in [0,R]$. By Taylor's expansion we have:
$$f(y) = f(x)+(y-x)f^{'}(x)+\frac{1}{2}(y-x)^2f^{''}(x)+(y-x)^2\psi(y-x), \tag{4.1}$$
where $\psi(y-x) \to 0$, as $y-x \to 0$. From Lemma 4.1, we obtain
$$P_n(f,r,x) = (1+x_n)f(x)+(1+x_n)f^{'}(x)A_n((y-x),r,q,x)+(1+x_n)\frac{f^{''}(x)}{2}A_n((y-x)^2,r,q,x)$$

$$+(1+x_n)A_n((y-x)^2\psi(y-x),r,q,x).$$

This yields

$$n^qP_n(f)=n^q(1+x_n)f(x)+n^q(1+x_n)f^{'}(x)\left(\frac{1}{n^q}\right)+n^q(1+x_n)\frac{f^{''}(x)}{2}\frac{1}{n^{2q}}\left[1+\frac{1}{(n^qx+1)^{r-2}}\right]$$
$$+n^q(1+x_n)A_n((y-x)^2\psi(y-x),r,q,x),$$

as well as

$$\left|n^q[P_n(f,r,q,x)-f(x)]-f^{'}(x)\right|\leq n^qx_nM+x_nM_1+\frac{(1+x_n)}{2n^q}M_2\cdot\left(1+\frac{1}{(n^qx+1)^{r-2}}\right)+$$
$$n^q(1+x_n)A_n((y-x)^2\psi(y-x),r,q,x), \tag{4.2}$$

where $M=||f||$, $M_1=||f^{'}||$ and $M_2=||f^{''}||$.

Now we have to prove that

$$\lim_{n\to\infty}n^qA_n(\Phi^2\psi(y-x),r,q,x)=0.$$

Applying the Cauchy-Schwarz inequality to $n^qA_n(\Phi^2\psi(y-x),r,q,x)$ yields

$$n^qA_n(\Phi^2\psi(y-x),r,q,x)\leq\left[n^{2q}A_n(\Phi^4,r,q,x)\right]^{\frac{1}{2}}\cdot\left[A_n(\psi^2,r,q,x)\right]^{\frac{1}{2}}. \tag{4.3}$$

Also, by putting $\eta_x(y)=(\psi(y-x))^2$, we see that $\eta_x(x)=0$ and $\eta_x(\cdot)\in C[0,R]$. Clearly it follows that

$$A_n(\eta_x)\to 0(st_A)\quad\text{on}\quad[0,R]. \tag{4.4}$$

Now from relations (4.2), (4.3), (4.4) and Lemma 4.3 , we obtain

$$n^qA_n(\Phi^2\psi(y-x),r,q,x)\to 0(st_A)\quad\text{on}\quad[0,R]. \tag{4.5}$$

From the definition of the sequence (x_n), it follows that

$$n^{2q}x_n\to 0(st_A)\quad\text{on}\quad[0,R].$$

For a given $\epsilon>0$, we define the following sets:

$$A = \left|\left\{n:\left|n^q[P_n(f,r,q,x)-f(x)]-f^{'}(x)\right|\geq\epsilon\right\}\right|, \tag{4.6}$$
$$A_1 = \left|\left\{n:|n^qx_n|\geq\frac{\epsilon}{3M}\right\}\right|, \tag{4.7}$$
$$A_2 = \left|\left\{n:|n^qx_nA_n((y-x)^2\psi(y-x),r,q,x)|\geq\frac{\epsilon}{3}\right\}\right|, \tag{4.8}$$
$$A_2 = \left|\left\{n:|n^qA_n((y-x)^2\psi(y-x),r,q,x)|\geq\frac{\epsilon}{3}\right\}\right|. \tag{4.9}$$

From these relations we obtain:

$$A\leq A_1+A_2+A_3. \tag{4.10}$$

The desired result follows from relations (4.2), (4.6)-(4.9) and (4.10). □

Remark. *Since the sequence $x=(x_n)$, given in Example 2.4, is not statistically convergent we conclude that the operators (P_n), defined in Example 2.4, do not satisfy Voronovskaya type theorem given in Theorem 4.2.*

5. Grüss-Voronovskaya theorems

In this section we will show some kind of Grüss-Voronovskaya type theorem for the new modified Szász-Mirakyan operators. This kind of Theorem, was first seen in [13]. In what follows we give the Theorem in the usual sense and then in $A-$ statistical sense.

Theorem 5.1. *Let* $f^{'}(x), g^{'}(x), (fg)^{'}(x) \in C[0,\infty)$. *Then*

$$\lim_{n\to\infty} n^q[A_n(fg,r,q,x) - A_n(f,r,q,x)\cdot A_n(g,r,q,x)] = f^{'}g^{'}.$$

Proof. By simple calculation we see that

$$n^q\{A_n(fg,r,q,x) - A_n(f,r,q,x)A_n(g,r,q,x)\} = n^q\{A_n(fg,r,q,x) - fg - (fg)^{'} -$$

$$g(x)[A_n(f,r,q,x)-f-f^{'}]-A_n(f,r,q,x)[A_n(g,r,q,x)-g-g^{'}]+g^{'}[f(x)-A_n(f,r,q,x)]\}.$$

The reminder of the Theorem follows from Theorem 4.2. □

Now we will prove this theorem in the sense of $A-$ statistical convergence for sequence $P_n(f,r,q,x)$ defined in the Example 2.4

Theorem 5.2. *Let* $f^{'}(x), g^{'}(x), (fg)^{'}(x) \in C[0,R]$, *for any finite* R. *Then*

$$\lim_{n\to\infty} n^q[P_n(fg,r,q,x) - P_n(f,r,q,x)\cdot P_n(g,r,q,x)] \sim f^{'}g^{'}(st_A).$$

Proof. After some calculation we have:

$$n^q[P_n(fg,r,q,x)-P_n(f,r,q,x)\cdot P_n(g,r,q,x)] = n^q\left[P_n(fg,r,q,x) - fg(x) - (fg)^{'}(x)\right] -$$

$$n^q g(x)\left[P_n(f,r,q,x) - f(x) - f^{'}(x)\right] - n^q P_n(f,r,q,x)\left[P_n(g,r,q,x) - g(x) - g^{'}(x)\right] +$$

$$g^{'}(x)\left[f(x) - P_n(f,r,q,x)\right].$$

From Theorem 4.4, we get

$$n^q\left[P_n(fg,r,q,x) - fg(x) - (fg)^{'}(x)\right] \sim 0(st_A) \tag{5.1}$$

$$n^q g(x)\left[P_n(f,r,q,x) - f(x) - f^{'}(x)\right] \sim 0(st_A) \tag{5.2}$$

$$n^q P_n(f,r,q,x)\left[P_n(g,r,q,x) - g(x) - g^{'}(x)\right] \sim 0(st_A) \tag{5.3}$$

as $n\to\infty$, for $x\in[0,R]$.

On the other hand,

$$n^q[P_n(f,r,q,x) - f(x)] \sim f^{'}, \quad \text{as} \quad n\to\infty. \tag{5.4}$$

The reminder of the poof of Theorem follows from relations (5.1)-(5.4). □

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